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Compact AC-operators

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Abstract. We prove that compact AC-operators have a representation as a combination of disjoint projections which mirrors that for compact normal operators. We also show that unlike arbitrary AC-operators, compact AC-operators admit a unique splitting into real and imaginary parts, and that these parts must necessarily be compact.

1. Introduction. One of the first major results that students of operator theory meet is the spectral theorem for compact self-adjoint or compact normal operators on a Hilbert space. This says that if T is such an operator, then there exist a sequence of disjoint orthogonal projections P_j on the Hilbert space $\mathcal{H}$ such that

$$(*)\qquad T = \sum_{j=1}^{\infty} \lambda_j P_j.$$

Here $\{\lambda_j\}$ are the non-zero eigenvalues of T. The sum converges in the norm topology of $B(\mathcal{H})$. This theorem has a direct analogue for compact scalar-type spectral operators acting on a Banach space X. Unfortunately, many of the important operators in analysis, whilst being normal on L^2, fail to be scalar-type spectral on the other L^p spaces. What often causes the problem here is that, although the operator still admits eigenfunction expansions, these expansions only converge conditionally. To provide a theory which covers such operators, Smart [Sm] and Ringrose [R] introduced the class of well-bounded operators. A restriction with well-bounded operators is that their spectra must be real. It was not until Berkson and Gillespie introduced AC-operators [BG] that a suitable analogue of normal operators existed in this context. These are the operators which can be written in the form $A+iB$ where A and B are commuting well-bounded operators.

In [CD], Cheng and Doust showed that compact well-bounded opera-

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tors have a spectral representation theory similar to that for compact self-adjoint operators given in (*). The main difference is that the sum need only converge conditionally in the norm topology of $B(X)$. The aim of the present paper is to prove that a similar representation holds for compact AC-operators.

The main complication here is in fixing an order in which to take the sum (*). As will be shown in Section 4, the most natural order turns out to be difficult to work with because of the special nature of the norm on the Banach algebra of absolutely continuous functions on a rectangle in the plane.

A consequence of this work is that if $T = A + iB$ is a compact AC-operator, then A and B must also be compact. Furthermore, there does not exist a different pair of commuting well-bounded operators C and D such that $T = C + iD$, a situation which may occur for general AC-operators.

2. Background and notation. In this section we shall give some of the basic definitions regarding well-bounded and AC-operators. The theory of well-bounded operators is given in more detail in [Dow] or [DQ].

Throughout, X will denote a complex Banach space with dual space X^*. The Banach algebra of all bounded linear operators on X will be denoted by $B(X)$.

An operator $T \in B(X)$ is said to be *well-bounded* if there exists a compact interval $[a,b] \subset \mathbb{R}$ and a constant K such that

$$\|g(T)\| \le K\Big\{|g(a)| + \int_a^b |g'(t)|\,dt\Big\} \equiv K\|g\|_{BV},$$

for all polynomials g. Since the polynomials are dense in the Banach algebra of absolutely continuous functions on $[a,b]$, this is equivalent to the statement that there is a unital Banach algebra homomorphism $\Phi : AC[a,b] \to B(X)$ such that if $e_n(x) = x^n$, then $\Phi(e_n) = T^n$ $(n = 0,1,\ldots)$. The spectral theorem for well-bounded operators states that there exists a family of projections $\{E(\lambda)\}_{\lambda\in\mathbb{R}} \subset B(X^*)$, known as a decomposition of the identity, such that

$$\langle Tx, x^*\rangle = b\langle x, x^*\rangle - \int_a^b \langle x, E(\lambda)x^*\rangle\,d\lambda, \qquad x \in X,\ x^* \in X^*.$$

We shall not need the properties of decompositions of the identity, so we shall refer the reader to [Dow] or [DQ] for the details of the spectral theorem.

One of the major complications one encounters when trying to extend this theory to operators with complex spectra is deciding upon the correct concept of absolutely continuous functions of two variables to use. In the discussion that follows we shall identify subsets of $\mathbb{R}^2$ with subsets of $\mathbb{C}$ in the usual way. Let $J = [a,b]$ and $K = [c,d]$ be two compact intervals in $\mathbb{R}$. Let $\mathcal{P}$ be a rectangular partition of $J \times K$:

$$a = s_0 < s_1 < \ldots < s_n = b, \qquad c = t_0 < t_1 < \ldots < t_m = d.$$

For a function $f : J \times K \to \mathbb{C}$, define

$$V_{\mathcal{P}} = \sum_{i=1}^{n}\sum_{j=1}^{m} |f(s_i,t_j) - f(s_i,t_{j-1}) - f(s_{i-1},t_j) + f(s_{i-1},t_{j-1})|.$$

The *variation* of f is defined to be

$$\operatorname*{var}_{J\times K} f = \sup\{V_{\mathcal{P}} : \mathcal{P} \text{ is a rectangular partition of } J \times K\}.$$

We shall say that the function f is of *bounded variation* if $\operatorname{var}_{J\times K} f$, $\operatorname{var}_J f(\cdot,d)$, and $\operatorname{var}_K f(b,\cdot)$ are all finite. The set $BV(J\times K)$ of all functions $f : J \times K \to \mathbb{C}$ of bounded variation is a Banach algebra under the norm

$$\|f\|_{BV} = |f(b,d)| + \operatorname*{var}_J f(\cdot,d) + \operatorname*{var}_K f(b,\cdot) + \operatorname*{var}_{J\times K} f.$$

As with functions of one variable, there is the concept of an absolutely continuous function. Let m denote Lebesgue measure on $\mathbb{R}^2$. A function $f : J \times K \to \mathbb{C}$ is said to be *absolutely continuous* if

(1) for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{R\in\mathcal{R}} \operatorname*{var}_R f < \varepsilon$$

whenever $\mathcal{R}$ is a finite collection of non-overlapping subrectangles of $J \times K$ with $\sum_{R\in\mathcal{R}} m(R) < \delta$;

(2) the marginal functions $f(\cdot,d)$ and $f(b,\cdot)$ are absolutely continuous functions on J and K respectively.

The set $AC(J\times K)$ of all absolutely continuous functions $f : J\times K \to \mathbb{C}$ is a Banach subalgebra of $BV(J\times K)$, and is the closure in $BV(J\times K)$ of the polynomials in two real variables on $J\times K$. Equivalently, one can consider $AC(J\times K)$ to be the closure of the polynomial functions $p(z,\overline{z})$ on $J\times K \subset \mathbb{C}$.

Define the functions $u, v, e \in AC(J\times K)$ by $u(x,y) = x$, $v(x,y) = y$ and $e = u + iv$. An operator $T \in B(X)$ is said to be an *AC-operator* if there exists a Banach algebra homomorphism $\theta : AC(J\times K) \to B(X)$ for which $\theta(e) = T$. Berkson and Gillespie [BG] proved that this is equivalent to the condition that T can be written as $T = A + iB$, where A and B are commuting well-bounded operators on X. In what follows we shall assume that $T = A + iB$ is a compact AC-operator, and that we have fixed the algebra homomorphism θ to be the one consistent with the given splitting. That is, we shall assume that $\theta(u) = A$ and $\theta(v) = B$. (We shall see in

Section 5 that it would not in fact be possible to choose θ so that this is not the case!)

3. Eigenspaces. The first step is to look at the restriction of T to an eigenspace.

LEMMA 3.1. *Suppose that T is as above, and that $\lambda \in \sigma(T) \setminus \{0\}$. Let E_λ denote the Riesz projection corresponding to the spectral set $\{\lambda\}$, and let $X_\lambda = E_\lambda X$. Then X_λ is an invariant subspace for T, A, and B.*

Remark. In general this does not hold. Example 3.1 of [BDG] provides an AC-operator $T = A + iB$ and a reducing subspace $Y \subset X$ of T which is not even an invariant subspace of A. It should be noted, however, that this operator did have a different splitting into real and imaginary parts $T = A' + iB'$ such that every invariant subspace for T was also one for A' and B'.

Proof of Lemma 3.1. That X_λ is invariant under T is standard. Since T is compact, X_λ is a generalized eigenspace for T. That is, $x \in X_\lambda$ if and only if $(T - \lambda I)^n x = 0$ for some $n \in \mathbb{N}$. Suppose then that $x \in X_\lambda$. Since A commutes with T,

$$(T - \lambda I)^n Ax = A(T - \lambda I)^n x = 0,$$

and so $Ax \in X_\lambda$. Similarly, X_λ is an invariant subspace for B.

LEMMA 3.2. *Suppose that T, λ, and X_λ are as in the previous lemma. Let $\lambda = \alpha + i\beta$ where $\alpha, \beta \in \mathbb{R}$. Then $T_\lambda = T|X_\lambda = \lambda I$, $A_\lambda = A|X_\lambda = \alpha I$ and $B_\lambda = B|X_\lambda = \beta I$.*

Proof. Again, general spectral theory shows that $\sigma(T_\lambda) = \{\lambda\}$. Note that since the restriction of a well-bounded operator to an invariant subspace is again well-bounded, both A_λ and B_λ are well-bounded. Thus $T_\lambda - \lambda I = (A_\lambda - \alpha I) + i(B_\lambda - \beta I)$ is a quasinilpotent AC-operator. Since $T_\lambda - \lambda I$ and $A_\lambda - \alpha I$ commute,

$$\sigma(A_\lambda - \alpha I) = \sigma((A_\lambda - \alpha I) - (T_\lambda - \lambda I)) = \sigma(-i(B_\lambda - \beta I)).$$

Clearly then $\sigma(A_\lambda - \alpha I) = \sigma(B_\lambda - \beta I) = \{0\}$. But the only quasinilpotent well-bounded operator is 0, so $A_\lambda = \alpha I$ and $B_\lambda = \beta I$.

LEMMA 3.3. *Let T, λ, E_λ and X_λ be as in the previous lemmas. Then E_λ commutes with T, A, and B.*

Proof. By the previous lemmas we infer that with respect to the splitting $X = X_\lambda \oplus (I - E_\lambda)X$,

$$T = \begin{pmatrix} \lambda & 0 \\ 0 & T_{22} \end{pmatrix}, \quad A = \begin{pmatrix} \alpha & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} \beta & iA_{12} \\ 0 & B_{22} \end{pmatrix}.$$

The condition that A and B commute allows one to deduce that

$$i\alpha A_{12} + A_{12}B_{22} - \beta A_{12} - iA_{12}A_{22} = iA_{12}(\lambda I - T_{22}) = 0.$$

Now $\lambda \notin \sigma(T_{22})$, so this implies that $A_{12} = 0$. The result clearly follows.

LEMMA 3.4. *Let T, λ and E_λ be as in the previous lemmas. If $f \in AC(J \times K)$ then E_λ commutes with $\theta(f)$.*

Proof. Suppose that $p(s,t)$ is a polynomial in two variables. Then $\theta(p) = p(A, B)$, so it is clear by the previous lemma that $\theta(p)$ commutes with E_λ. The result follows from the usual density arguments.

COROLLARY 3.5. *$T|X_\lambda$ is an AC-operator on X_λ with (unique) splitting $T|X_\lambda = A|X_\lambda + iB|X_\lambda$. Similarly, if $X'_\lambda = (I - E_\lambda)X$, then $T|X'_\lambda$ is an AC-operator with splitting $T|X'_\lambda = A|X'_\lambda + iB|X'_\lambda$.*

To save notational inconvenience later, we shall dispose at this point of the case where $\sigma(T)$ is a finite set $\{\lambda_j\}_{j=1}^n \cup \{0\}$.

THEOREM 3.6. *Suppose that T is a compact AC-operator with $\sigma(T) = \{\lambda_j\}_{j=1}^n \cup \{0\}$. Then there exist disjoint projections $\{P_j\}_{j=1}^n$ such that*

$$T = \sum_{j=1}^n \lambda_j P_j.$$

Proof. For $j = 1, \ldots, n$ let $P_j = E_{\lambda_j}$. Let $P = P_1 + \ldots + P_n$. Then $I = \sum_{j=1}^n P_j + (I - P)$, so

$$T = \sum_{j=1}^n TP_j + T(I - P).$$

By Lemma 3.2, $TP_j = \lambda_j P_j$. The only remaining thing to check is that $T|(I - P)X = 0$. It is, however, easily seen that $T|(I - P)X$ is both quasinilpotent and an AC-operator. As in the proof of Lemma 3.2, this implies that $T|(I - P)X = 0$ and so we are done.

4. The spectral theorem for compact AC-operators. As before, we shall assume that $T \in B(X)$ is compact. From now on we shall also assume that $\sigma(T) = \{\lambda_j\}_{j=1}^\infty \cup \{0\}$ is infinite. In proving a representation of the form $(*)$ the order in which the points $\{\lambda_j\}$ are taken is of course vital. The order which we take is perhaps not the most obvious. We shall remark after Theorem 4.5 about the problems one encounters if one chooses a different order.

For a complex number $\lambda = x + iy$ with $x, y \in \mathbb{R}$, let $|\lambda|_\infty = \max\{x, y\}$. We shall now define an order $\prec$ on $\mathbb{C}$ by setting $\lambda \prec \mu$ if

(i) $|\lambda|_\infty < |\mu|_\infty$, or

(ii) if $|\lambda|_\infty = |\mu|_\infty = \alpha$ and μ lies on the part of the square $|z|_\infty = \alpha$ between $-\alpha + i\alpha$ and λ going from $-\alpha + i\alpha$ in a clockwise direction.

Assume now that $\{\lambda_j\}$ has been ordered so that

$$\lambda_1 \succ \lambda_2 \succ \dots \tag{4-1}$$

The most technical part of the proof of the spectral theorem is contained in Lemma 4.4. The analogous result for when $\{\lambda_j\} \subset \mathbb{R}$ is much simpler. Before proceeding it is convenient to introduce some terminology.

Definition 4.1. A polygon is called *Cartesian* if each of its sides is parallel to one of the axes. A region $\Gamma \subset \mathbb{C}$ is said to be *L-shaped* if $\Gamma = R\setminus\overline{Q}$, where Q and R are the interiors of two Cartesian rectangles which share a vertex and satisfy $Q \subset R$. For such a region we shall let $s(\Gamma)$ denote the length of the shortest side.

Definition 4.2. Let $\Gamma \subset \mathbb{C}$ and let $\varepsilon > 0$. The *ε-dilation* of Γ, written Γ^ε, is the set $\{z \in \mathbb{C} : |z - w|_\infty < \varepsilon \text{ for some } w \in \Gamma\}$.

Lemma 4.3. *Define* $\phi : [0,2] \times [0,2] \to \mathbb{R}$ *by*

$$f(x,y) = \begin{cases} x^2y + xy^2 - x^2y^2 & \text{if } x \le 1,\ y \le 1, \\ x & \text{if } x \le 1,\ y > 1, \\ y & \text{if } x > 1,\ y \le 1, \\ 1 & \text{if } x > 1,\ y > 1. \end{cases}$$

Then $\phi \in AC([0,2] \times [0,2])$ *and* $\|\phi\|_{BV} = 4$.

Proof. One can prove this directly from the definition, but it is perhaps easier to note that $\phi(x,y) = \int_0^x \int_0^y F(u,v)\,dv\,du$, where F is the function

$$F(x,y) = \begin{cases} 2x + 2y - 4xy & \text{if } x \le 1,\ y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $F \in L^1([0,2] \times [0,2])$ it now follows from [BG, Theorem 4] that $\phi \in AC([0,2] \times [0,2])$. We shall leave it to the reader to verify that

$$\|\phi\|_{BV} = |\phi(2,2)| + \operatorname*{var}_J \phi(\cdot,2) + \operatorname*{var}_K \phi(2,\cdot) + \operatorname*{var}_{[0,2]\times[0,2]} \phi = 1 + 1 + 1 + 1 = 4.$$

The main point to be noted from this lemma is that it is possible to define AC functions by gluing together AC functions defined on smaller subrectangles.

Suppose now that we are given an L-shaped region Γ and that $\varepsilon < s(\Gamma)/2$. Our aim is to explain how to define an absolutely continuous function ψ (with reasonable norm) which takes the value 0 on Γ and the value 1 outside of Γ^ε. Referring to Figure 1, we take $\psi(x,y)$ to be $\operatorname{dist}((x,y),\partial\Gamma)/\varepsilon$ on the lightly shaded regions. Note that ψ will thus be linear there.

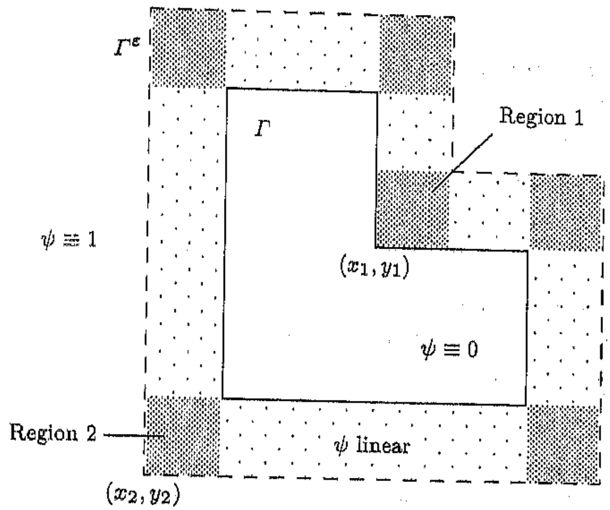

Fig. 1

At the corners (i.e. the darkly shaded regions in Figure 1), we want to make ψ behave like the function ϕ from the previous lemma. Let $\widetilde{\phi}$ be the restriction of ϕ to $[0,1] \times [0,1]$. At the "bend" of Γ (e.g., Region 1 in Figure 1), we shall define ψ by rescaling and translating $\widetilde{\phi}$. In the example, for Region 1, we would take

$$\psi(x,y) = \widetilde{\phi}((x - x_1)/\varepsilon, (y - y_1)/\varepsilon).$$

On the other corners, set ψ to be a rescaling and translation of $1 - \widetilde{\phi}$. For example, for Region 2 in Figure 1, we would take

$$\psi(x,y) = 1 - \widetilde{\phi}((x - x_2)/\varepsilon, (y - y_2)/\varepsilon).$$

It is clear that any translation or rescaling of ϕ or $1-\phi$ remains an absolutely continuous function on the appropriate domain, as does a rotation of (the domain of) ϕ by a multiple of $\pi/2$. What is important here is that the norm of ψ does not depend on the steepness of the slide from the region where $\psi = 1$ to the one where $\psi = 0$; i.e. it does not depend on the size of ε. If ψ is defined in this way, the analogue of the calculation in the proof of Lemma 4.3 shows that if $\Gamma^\varepsilon \subset J \times K$ then $\operatorname{var}_{J\times K} \phi \le 6$. (The norm will of course be smaller than 6 if Γ is a rectangle, because there are then fewer corners.) Consequently, $\|\psi\|_{BV} \le 7$.

For technical reasons we shall henceforth assume that each of the points λ_j lies in the interior of $J \times K$. This does not affect our final result since if J' and K' are intervals containing J and K respectively, and T has an $AC(J \times K)$ functional calculus, then T also has an $AC(J' \times K')$ functional calculus.

LEMMA 4.4. *Suppose that* $\{\lambda_j\} \subset J \times K$ *is a sequence of complex numbers which converge to* 0 *and which satisfy* (4-1). *Then there exists a constant* M *and a sequence of functions* $f_n \in AC(J \times K)$ *such that*

(1) $\|f_n\| \le M$ *for all* $n \in \mathbb{N}$;
(2) $f_n \to e$ *in* $AC(J \times K)$;
(3) $f_n(\lambda_k) = \lambda_k$ *for all* $k \le n$;
(4) $f_n(z) = 0$ *in a neighbourhood of* λ_k *for all* $k > n$.

Proof. Given the ordering of $\{\lambda_j\}$ it is clear that given n, there exists $\varepsilon_n > 0$ and an L-shaped region Γ_n such that

(i) for all $k \ge n$, $\lambda_k \in \Gamma_n$, and
(ii) for all $k < n$, $\lambda_k \notin \Gamma_n^{\varepsilon_n}$.

For reasons later in the proof, we shall assume that $|\varepsilon_n| < |\lambda_n|$ for all n. The choice of Γ_n and ε_n is perhaps best illustrated by a diagram like that in Figure 2, which shows Γ_4 and $\Gamma_4^{\varepsilon_4}$ for a particular sequence of λ_j:

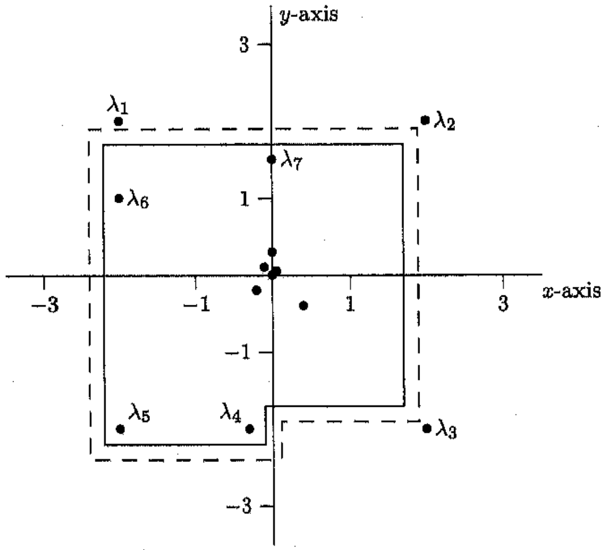

Fig. 2

For each n define the function $\psi_n : \mathbb{C} \to \mathbb{R}$ as in the discussion preceding the lemma. Let $f_n(z) = z\psi_n(z)$. It is clear then that $f_n \in AC(J \times K)$ and that

(i) $\|f_n\| \le 7\|e\|_{BV}$ for all $n \in \mathbb{N}$;
(ii) $f_n(\lambda_k) = \lambda_k$ for all $k \le n$;
(iii) $f_n(z) = 0$ in a neighbourhood of λ_k for all $k > n$.

It now just remains to show that $f_n \to e$ in $AC(J \times K)$. Let $\alpha_n = |\lambda_n|_\infty + \varepsilon_n$ and let $J_n = K_n = [-\alpha_n, \alpha_n]$. Then, as $f_n - e = 0$ outside of $J_n \times K_n$, we have

$$\begin{aligned}\|f_n - e\|_{BV} &= \operatorname*{var}_{J\times K}(f_n - e) = \operatorname*{var}_{J_n \times K_n}(e\psi_n - e) \\ &= \|e\psi_n - e\|_{BV(J_n\times K_n)} \\ &\le \|e\|_{BV(J_n\times K_n)}\|\psi_n - 1\|_{BV(J_n\times K_n)} \\ &\le 8(\sqrt{2}+4)\alpha_n \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

THEOREM 4.5. *Suppose that* T *is a compact* AC-*operator with spectrum* $\{0\} \cup \{\lambda_j\}_{j=1}^\infty$ *and that* $\{\lambda_j\}$ *satisfies* 4-1. *Then there exists a uniformly bounded sequence of disjoint projections* $E_j \in B(X)$ *such that*

$$T = \sum_{j=1}^{\infty} \lambda_j E_j,$$

where the sum converges in the norm topology of $B(X)$.

Proof. The proof mimics that for compact well-bounded operators. Without loss of generality, $\sigma(T) \subset J \times K$. Choose a sequence of functions $\{f_n\}$ satisfying (1)–(4) of Lemma 4.4. Let E_j denote the Riesz projection associated with the spectral set $\{\lambda_j\}$, and let $X_j = E_jX$. Let $\sigma_n = \sigma(T) \setminus \{\lambda_1, \ldots, \lambda_n\}$. We shall let E_{σ_n} denote the Riesz projection associated with the spectral set σ_n and let $X_{\sigma_n} = E_{\sigma_n}X$.

We claim that

$$\theta(f_n) = \sum_{j=1}^{n} \lambda_j E_j.$$

We know that $X = X_1 \oplus \ldots \oplus X_n \oplus X_{\sigma_n}$, and that each of these subspaces is invariant under $\theta(f)$ for all $f \in AC(J \times K)$. Let θ_j denote the AC-functional calculus for $T|X_j$, and θ_{σ_n} the AC-functional calculus for $T|X_{\sigma_n}$. Then, for all $f \in AC(J \times K)$,

$$\theta(f) = \theta_1(f) \oplus \ldots \oplus \theta_n(f) \oplus \theta_{\sigma_n}(f).$$

By Lemma 3.2, $T|X_j = \lambda_j I$ and so $\theta_j(f_n) = f_n(\lambda_j)I = \lambda_j I$ (on X_j). Also, $f_n \equiv 0$ on an open neighbourhood of $\sigma(T|X_{\sigma_n})$. Thus $\theta_{\sigma_n}(f_n) = 0$. Hence

$$\theta(f_n) = \sum_{j=1}^{n} \lambda_j E_j.$$

Now the projections E_j are clearly disjoint. Showing that these projections are uniformly bounded is similar to the proof for well-bounded operators. Simple rearrangement shows that

$$E_n = \frac{1}{\lambda_n}\theta(f_n - f_{n-1}).$$

A calculation similar to that at the end of the proof of Lemma 4.4 then shows that

$$\|E_n\| \le \frac{|\lambda_n|_\infty + \varepsilon_n}{|\lambda_n|} 14(\sqrt{2}+4)\|\theta\|,$$

which is bounded independently of n. The fact that $f_n \to e$ shows that

$$T = \theta(e) = \lim_{n\to\infty} \theta(f_n) = \lim_{n\to\infty} \sum_{j=1}^{n} \lambda_j E_j.$$

5. Orderings of the eigenvalues. A direct analogue of the spectral theorem for compact well-bounded operators would allow us to take the eigenvalues of T ordered so that

$$|\lambda_1| \ge |\lambda_2| \ge \dots \tag{5-1}$$

We have been unable to show that one can use this ordering in Theorem 4.5 rather than the ordering given by (4-1). We might note here that (5-1) potentially allows a large number of rearrangements whilst (4-1) dictates the order uniquely. The problem with using an order which satisfies (5-1) relates to the nature of the AC norm. One would like to construct AC functions which are zero inside a circular region Ω, and equal to e outside Ω^ε. It does not appear to be possible to do this in a way which does not depend on ε.

In certain circumstances you can of course use an ordering which satisfies (5-1). In what follows, E_j will, as usual, denote the Riesz projection associated with the eigenvalue λ_j of T. If there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $|\pi(n) - n|$ is uniformly bounded (by M say) and $\{\lambda_{\pi(j)}\}$ satisfies (4-1), then $T = \sum_{j=1}^{\infty} \lambda_j E_j$ as before. To see this note that for N large,

$$\sum_{j=1}^{N} \lambda_j E_j = \sum_{j=1}^{N-M} \lambda_{\pi(j)} E_{\pi(j)} + \sum_{j\in I_N} \lambda_j E_j,$$

where $I_N = \{j \in \{1,\dots,N\} : \pi(j) \notin \{1,\dots,N-M\}\}$. The first sum on the right hand side converges to T by Theorem 4.5, whereas the second sum (which contains only M terms) converges to 0.

If $\{\lambda_j\} \in \ell^1$ then the uniform bounds on E_j show that $\sum_{j=1}^{\infty} \lambda_j E_j$ converges absolutely (and hence unconditionally) in the norm topology of $B(X)$.

If $\sigma(T)$ lies on a finite number of lines through the origin, we can drop the above conditions on $\{\lambda_j\}$. In particular, this allows orderings which are "unbounded" permutations of (4-1).

THEOREM 5.1. *Suppose that T is a compact AC-operator with spectrum $\{0\} \cup \{\lambda_j\}_{j=1}^{\infty}$. Suppose also that $\sigma(T)$ lies on a finite number of lines through the origin and that $\{\lambda_j\}$ satisfies* (5-1). *Then there exists a uniformly bounded sequence of disjoint projections $E_j \in B(X)$ such that*

$$T = \sum_{j=1}^{\infty} \lambda_j E_j,$$

where the sum converges in the norm topology of $B(X)$.

Proof. The proof requires that we find an analogue for Lemma 4.4. Suppose that $\sigma(T)$ can be covered by N lines through the origin. It is clear that for all n one can find a Cartesian polygon, say $P_n \subset J \times K$, and $\varepsilon_n > 0$ such that

(i) for all $k \ge n$, λ_k is in the interior of P_n, and
(ii) for all $k < n$, λ_k is in the exterior of $P_n^{\varepsilon_n}$.

A little more work will show that there exists a constant $C(N)$ so that we can always choose P_n in such a way that it has $C(N)$ or fewer corners. (It is easy to see that $C(N) \le 8N$, for example.) One can then follow the method of the proof of Lemma 4.4 to form functions $f_n \in AC(J \times K)$ so that

(i) $f_n(\lambda_k) = \lambda_k$ for all $k \le n$;
(ii) $f_n(z) = 0$ in a neighbourhood of λ_k for all $k > n$;
(iii) $f_n \to e$ in $AC(J \times K)$.

Each corner carries a cost of 1 in the norm of f_n, so we also deduce that the functions f_n are uniformly bounded by $(8N+1)\|e\|_{BV}$. The remainder of the proof is more or less identical to the proof of Theorem 4.5.

6. Properties of the splitting $T = A + iB$. In general it is possible to construct examples of compact operators T of the form $T = A + iB$ where A and B are commuting operators with real spectrum, but where A and B are not compact. Looking just at AC-operators one can find examples where T has two distinct splittings into "real and imaginary parts", i.e. as a combination of commuting well-bounded operators. Our present setting rules out both of these undesirable behaviours.

THEOREM 6.1. *Let $T = A + iB \in B(X)$ be an AC-operator. Then T is compact if and only if A and B are both compact. If this is the case then the splitting of T into real and imaginary parts is unique.*

Proof. It is clear that if A and B are compact, then so is T. Suppose then that T is compact. Consider the set $S = \{\mathrm{Re}(\lambda) : \lambda \in \sigma(T)\}$. Then S is at most countable, and has no accumulation point apart from 0. Enumerate S as $\{0\} \cup \{\mu_j\}_{j=1}^{\infty}$, where $|\mu_1| \ge |\mu_2| \ge \dots$ If E_λ denotes the Riesz projection corresponding to $\lambda \in \sigma(T)$, then for $j \ge 1$ let

$$P_j = \sum_{\mathrm{Re}(\lambda)=\mu_j} E_\lambda.$$

Since no μ_j is zero, all such sums contain only finitely many terms. Thus P_j is a finite rank projection for each j. Working exactly as in [CD, Proof of Theorem 3.4], one can find AC functions f_n (depending only on x) such that $f_n \to u$ in $AC(J \times K)$ and $\theta(f_n) = \sum_{j=1}^{n} \mu_j P_j$. It follows that

$$\theta(u) = A = \lim_{n\to\infty} \sum_{j=1}^{n} \mu_j P_j.$$

That is, A is the limit of finite rank operators and hence A is compact. The proof for B is similar. The fact that the splitting into real and imaginary parts is unique follows immediately from this formula for A.

7. An example. The following example is useful in clarifying some of the above results. Suppose that $A = \sum_{j=1}^{\infty} \alpha_j Q_j$ and $B = \sum_{j=1}^{\infty} \beta_j R_j$ are commuting compact well-bounded operators with representations given as in [CD]; that is, $|\alpha_1| \ge |\alpha_2| \ge \ldots$ and $|\beta_1| \ge |\beta_2| \ge \ldots$ Let $T = A + iB = \sum_{j=1}^{\infty} \lambda_j P_j$ be the corresponding AC-operator with representation given according to Theorem 4.5. As is shown below, the ordering of the eigenvalues may differ markedly in these representations. This shows, in particular, that it is not straightforward to go from the representations of A and B to a representation of T.

EXAMPLE 7.1. Let $X = bv$, the Banach algebra of sequences of bounded variation under the norm

$$\|(x_n)\|_{bv} = |x_1| + \sum_{j=1}^{\infty} |x_{j+1} - x_j|.$$

For $k = 1, 2, \ldots$, let $t_k = \sum_{j=1}^{k} j = k(k+1)/2$. Define elements $a = (a_n)$ and $b = (b_n)$ in bv by

$$a_{t_k+j} = \frac{1}{k} - \frac{j}{k^2(k+1)}, \qquad b_{t_k+j} = \frac{1}{k},$$

where $j = 0, 1, \ldots, k$. Noting that both a and b are decreasing sequences, it is easy to see that $\|a\|_{bv} = \|b\|_{bv} = 2$. It follows that the operators defined by $A((x_n)) = (a_n x_n)$ and $B((x_n)) = (b_n x_n)$ are commuting well-bounded operators on X (for more details, see the proof of Theorem 4.4 in [DdL]). Indeed, since they are limits of finite rank operators, both A and B are compact and so $T = A + iB$ is a compact AC-operator on X. Clearly $T((x_n)) = ((a_n + ib_n)x_n)$. That is,

$$T = \sum_{n=1}^{\infty} (a_n + ib_n) P_n,$$

where P_n is the nth coordinate projection. Let $\lambda_n = a_n + ib_n$. This ordering of the eigenvalues of T does not satisfy (4-1) (although it does satisfy (5-1)). Indeed, if π is the permutation of $\mathbb{N}$ such that $\{\lambda_{\pi(n)}\}$ does satisfy (4-1), then π is an unbounded permutation since $\pi(t_{k+1} - 1) = t_k$.

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