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Submultiplicative properties of the φ_K-distortion function

by

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Abstract. Some inequalities related to the submultiplicative properties of the distortion function $\varphi_K(r)$ are derived.

1. Introduction. The φ_K-distortion function defined for $0 < K < \infty$, $0 < r < 1$ and $r' = \sqrt{1-r^2}$ by

$$\varphi_K(r) = \mu^{-1}(\mu(r)/K), \quad \mu(r) = \frac{\pi}{2} \cdot \frac{\mathcal{K}(r')}{\mathcal{K}(r)},$$

$$\mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}},$$

plays a very important role in the theory of quasiconformal and quasiregular mappings (cf. [LV, Vu1], for example). During the past several years many properties of this and related functions were proved in [AVV1-AVV5], [P1-P2], [Z]. Very recently it was observed in [Vu2] that for small prime numbers p Ramanujan's work on modular equations yields several elegant algebraic identities for the function $\varphi_p(r)$. Some number theoretic aspects of the theory of modular equations are studied in [B] and [BB].

In [AVV1, VV] the following submultiplicative properties of $\varphi_K(r)$ were proved for $K > 1$ and $r, t \in (0,1)$:

$$\varphi_K(r)\varphi_K(t) \le 4^{1-1/K}\varphi_K(rt) \tag{1.1}$$

and

$$\varphi_K(rt) \le \varphi_K(r)\varphi_K(t) \le \varphi_{K^2}(rt). \tag{1.2}$$

Some function-theoretic applications of the submultiplicative property (1.2) were given in [AVV5, QVV1]. For related results see [Z] where it is shown

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for instance that for all $r, t \in (0,1)$ and $K > 1$,

$$\varphi_K(r)\varphi_K(t) \le \varphi_{K'}(rt), \quad K' = \begin{cases} K\left(1 + \frac{K-1}{\log_4 \frac{31}{33}}\right)^{-1} & \text{for } 1 < K \le K_0, \\ 2K & \text{for } K > K_0, \end{cases}$$

with $K_0 = 1 + \log_{16} \frac{33}{31}$.

The main purpose of this paper is to sharpen and generalize the existing inequalities concerning the submultiplicative property of $\varphi_K(r)$, such as (1.1) and (1.2), and to derive some monotonicity results for some functions defined in terms of $\varphi_K(r)$, from which new functional inequalities follow.

We now state some of our main results.

1.3. THEOREM. *For all $r, t \in (0,1)$ and $K \in [1, \infty)$,*

$$\varphi_K(r)\varphi_K(t) \le \varphi_{K^c}(rt) \tag{1.4}$$

and

$$\varphi_{1/K}(r)\varphi_{1/K}(t) \ge \varphi_{1/K^c}(rt), \tag{1.5}$$

where $c = 2/m \in (1, 4/3)$ and

$$m = \inf_{0<r<1} (1+r^2)\mathcal{K}(r^2)\mathcal{K}'(r^2)/\{\mathcal{K}(r)\mathcal{K}'(r)\} \approx 1.5324.$$

In each of (1.4) *and* (1.5), *equality holds if and only if $K = 1$.*

We were originally led to Theorem 1.3 by computer experiments which provided some evidence of the validity of (1.4) with $c = 1.5$.

In applications, we sometimes need that the upper bound in (1.4) and the lower bound in (1.5) involve the function φ_K with the same subindex K as the left sides. For this, we establish the following two theorems which improve (1.1).

1.6. THEOREM. *Let $a(r,t) = 2^{r'^{4/3} + t'^{4/3}}$, with $r' = \sqrt{1-r^2}$. Then the function*

$$f(r,t,K) = \varphi_K(r)\varphi_K(t)a(r,t)^{1/K}/\varphi_K(rt)$$

is strictly decreasing in r and t from $(0,1]$ onto

$$[(1+t')^{1/K}, \varphi_K(t)[a(0,t)/t]^{1/K}) \quad \text{and} \quad [(1+r')^{1/K}, \varphi_K(r)[a(r,0)/r]^{1/K}),$$

respectively, and strictly decreasing in K from $[1,\infty)$ onto $(1, a(r,t)]$. In particular, for $K \in [1,\infty)$ and $r, t \in (0,1)$,

$$\varphi_K(r)\varphi_K(t) \le a(r,t)^{1-1/K}\varphi_K(rt),$$

with equality if and only if $K = 1$.

1.7. THEOREM. *For fixed $r, t \in (0,1)$, define the function g on $[1,\infty)$ by*

$$g(K) = \varphi_{1/K}(r)\varphi_{1/K}(t)b(r,t)^K/\varphi_{1/K}(rt),$$

where

$$b(r,t) = \frac{2(1+\sqrt{r'})(1+\sqrt{t'})\sqrt{(1+r')(1+t')}}{(1+\sqrt[4]{x'})\sqrt{(1+\sqrt{x'})(1+x')}}$$
$$= 2\exp(\operatorname{arth}\sqrt{r'} + \operatorname{arth}\sqrt{t'} - \operatorname{arth}\sqrt[4]{x'}), \quad x = rt.$$

Then $g(K)$ is strictly increasing on $[1,\infty)$. In particular, for $K \in [1,\infty)$ and $r, t \in (0,1)$,

$$\varphi_{1/K}(r)\varphi_{1/K}(t) \ge b(r,t)^{1-K}\varphi_{1/K}(rt) \ge 4^{1-K}\varphi_{1/K}(rt),$$

with equalities if and only if $K = 1$.

2. Preliminary results.

We often use the notation $\mathcal{K}'(r) = \mathcal{K}(r')$, $r' = \sqrt{1-r^2}$, and

$$\mathcal{E}(r) = \int_0^1 \sqrt{\frac{1-r^2t^2}{1-t^2}}\,dt \quad \text{and} \quad \mathcal{E}'(r) = \mathcal{E}(r') \tag{2.1}$$

for complete elliptic integrals of the first and second kinds, respectively, and refer the reader to the standard references [BB], [Bo], [C], [WW] for the basic properties of elliptic integrals. Two such properties, useful for the sequel, are:

$$\mathcal{K}'(r) = \log\frac{4}{r} + O(r^2\log r) \tag{2.2}$$

as r tends to 0 [Bo] and the formula ([He], [AVV2, Lemma 2.1])

$$\mu'(r) = -\frac{\pi^2}{4}\cdot\frac{1}{rr'^2\mathcal{K}(r)^2}, \quad 0 < r < 1. \tag{2.3}$$

2.4. LEMMA. *The function $f_1(r) = (1+r^2)\mathcal{K}'(r^2) - 2\mathcal{K}'(r)$ is strictly increasing from $(0,1)$ onto $(-\log 4, 0)$.*

Proof. By differentiation,

$$\frac{rr'^2}{2}f_1'(r) = r^2[\mathcal{K}'(r^2) - \mathcal{K}'(r)] + \mathcal{E}'(r) - \mathcal{E}'(r^2),$$

which is positive for $r \in (0,1)$, since $\mathcal{K}'(k)$ and $\mathcal{E}'(k)$ are strictly decreasing and increasing on $(0,1)$, respectively.

Next, $f_1(1) = 0$, and

$$f_1(0) = \lim_{r\to0} f_1(r) = \lim_{r\to0}\left\{\log\frac{4}{r^2} + r^2\log\frac{4}{r^2} + O(r^4\log r^2) + O(r^6\log r^2) - 2\log\frac{4}{r} - O(r^2\log r)\right\} = -\log 4$$

by (2.2). ∎

2.5. COROLLARY. *For each* $r \in (0,1)$,

$$2 - \frac{2}{\pi}\log 4 < \frac{(1+r^2)\mathcal{K}'(r^2)}{\mathcal{K}'(r)} < 2. \tag{2.6}$$

The upper bound is sharp.

Proof. By Lemma 2.4, we have

$$2 - \frac{\log 4}{\mathcal{K}'(r)} < \frac{(1+r^2)\mathcal{K}'(r^2)}{\mathcal{K}'(r)} < 2$$

for $r \in (0,1)$, from which the result follows since the left side of the above double inequality is strictly decreasing from $(0,1)$ onto $\left(2 - \frac{2}{\pi}\log 4, 2\right)$. ∎

2.7. LEMMA. *The function* $f_2(r) = \mu(r^2)/\mu(r)$ *is strictly decreasing from* $(0,1)$ *onto* $(1,2)$.

Proof. By logarithmic differentiation and simplification,

$$\frac{2}{\pi} r(1-r^4)\mathcal{K}(r^2)^2\mathcal{K}'(r)^2 f_2'(r) = (1+r^2)\mathcal{K}(r^2)\mathcal{K}'(r^2) - 2\mathcal{K}(r)\mathcal{K}'(r). \tag{2.8}$$

Here, we have used (2.3).

Since $\mathcal{K}(r^2) < \mathcal{K}(r)$, it follows from (2.8) that

$$\frac{2}{\pi} r(1-r^4)\mathcal{K}(r^2)^2\mathcal{K}'(r)^2 f_2'(r) < \mathcal{K}(r) f_1(r),$$

where $f_1(r)$ is as in Lemma 2.4. This yields the monotonicity of f_2 by Lemma 2.4.

Finally, by l'Hospital's Rule, one can obtain the limiting values: $f_2(0) = 2$ and $f_2(1) = 1$. ∎

2.9. COROLLARY. *The function* $\mu(r)\mu(1-r^2)$ *is strictly increasing from* $(0,1)$ *onto* $(\pi^2/4, \pi^2/2)$.

Proof. Since $\mu(r)\mu(r') = \pi^2/4$ [LV, p. 60], the result follows from Lemma 2.7. ∎

In the proof of the next results we use the following self-evident proposition.

2.10. PROPOSITION. *Let* $x_0 = 0, x_k < x_{k+1}$ *for all* $k = 0, 1, \ldots, m-1$, $x_m = 1$, *and let* $f : [0,1] \to (0,\infty)$ *be increasing, and* $g : [0,1] \to (0,\infty)$ *be decreasing. Then for all* $x \in [0,1]$,

$$f(x)g(x) \ge C, \quad C = \min\{f(x_k)g(x_{k+1}) : 0 \le k \le m-1\}.$$

2.11. PROPOSITION. *For each* $r \in (0,1)$,

$$1.5008765 < \frac{(1+r^2)\mathcal{K}(r^2)\mathcal{K}'(r^2)}{\mathcal{K}(r)\mathcal{K}'(r)} < 2. \tag{2.12}$$

The upper bound is sharp.

Proof. The upper bound and its sharpness follow from Corollary 2.5 since $\mathcal{K}(r^2) < \mathcal{K}(r)$ and $\lim_{r\to 1}\mathcal{K}(r^2)/\mathcal{K}(r) = 1$. Next, let

$$f_3(r) = \frac{(1+r^2)\mathcal{K}(r^2)\mathcal{K}'(r^2)}{\mathcal{K}(r)\mathcal{K}'(r)}, \quad 0 < r < 1.$$

Clearly,

$$f_3(r) = \frac{(1+r^2)\mathcal{K}(r^2)^2}{\mathcal{K}(r)^2} \cdot \frac{\mu(r^2)}{\mu(r)} = g_1(r)f_2(r),$$

where $g_1(r) = (1+r^2)\mathcal{K}(r^2)^2/\mathcal{K}(r)^2$ and f_2 is as in Lemma 2.7. Since g_1 is strictly increasing from $(0,1)$ onto $(1,2)$ [QV, Theorem 1.5], and f_2 is decreasing,

$$f_3(r) > g_1(a)f_2(b) = C_{ab} \tag{2.13}$$

for each $r \in [a,b]$ (or $[a,b)$, (a,b)), an arbitrary subinterval of $[0,1]$.

By computation, we can use (2.13) to get the following estimates of C_{ab} in some subintervals $[a,b]$ or $[a,b)$ of $[0,1]$:

$[a,b]$ (or $[a,b)$)	lower bounds of C_{ab}	$[a,b]$ (or $[a,b)$)	lower bounds of C_{ab}
$[\sin 85°, 1]$	1.5687525	$[\sin 29°, \sin 30°)$	1.5044843
$[\sin 70°, \sin 85°)$	1.6124875	$[\sin 28°, \sin 29°)$	1.5059447
$[\sin 60°, \sin 70°)$	1.5583325	$[\sin 27°, \sin 28°)$	1.5093703
$[\sin 50°, \sin 60°)$	1.523167	$[\sin 25°, \sin 27°)$	1.5091234
$[\sin 45°, \sin 50°)$	1.552789	$[\sin 24°, \sin 25°)$	1.5165843
$[\sin 40°, \sin 45°)$	1.5276828	$[\sin 23°, \sin 24°)$	1.5169185
$[\sin 37°, \sin 40°)$	1.5146115	$[\sin 22°, \sin 23°)$	1.5124104
$[\sin 35°, \sin 37°)$	1.5101042	$[\sin 21°, \sin 22°)$	1.5034801
$[\sin 33°, \sin 35°)$	1.5008765	$[\sin 20°, \sin 21°)$	1.5207334
$[\sin 32°, \sin 33°)$	1.5057146	$[\sin 10°, \sin 20°)$	1.6600081
$[\sin 31°, \sin 32°)$	1.5041397	$[0, \sin 10°)$	1.515003
$[\sin 30°, \sin 31°)$	1.503706		

From this table and Proposition 2.10, we see that $f_3(r) > 1.5008765$ for $0 < r < 1$. ∎

2.14. Remark. In getting estimates for C_{ab} in the above table we have used the following estimate ([K], [Q2], [QV, Theorem 1.9]):

$$\mathcal{K}(r) > \frac{9}{8+r^2}\log\frac{4}{r'} \quad \text{for } r \in (0,1),$$

so as to improve the precision of the estimates. Numerical computation gives an improved lower bound in (2.12) (see Theorem 1.3). In Figure 1 we have graphed the function $\mathcal{K}(r) - \frac{9}{8+r^2}\log\frac{4}{r'}$ to illuminate the sharpness of the above inequality.

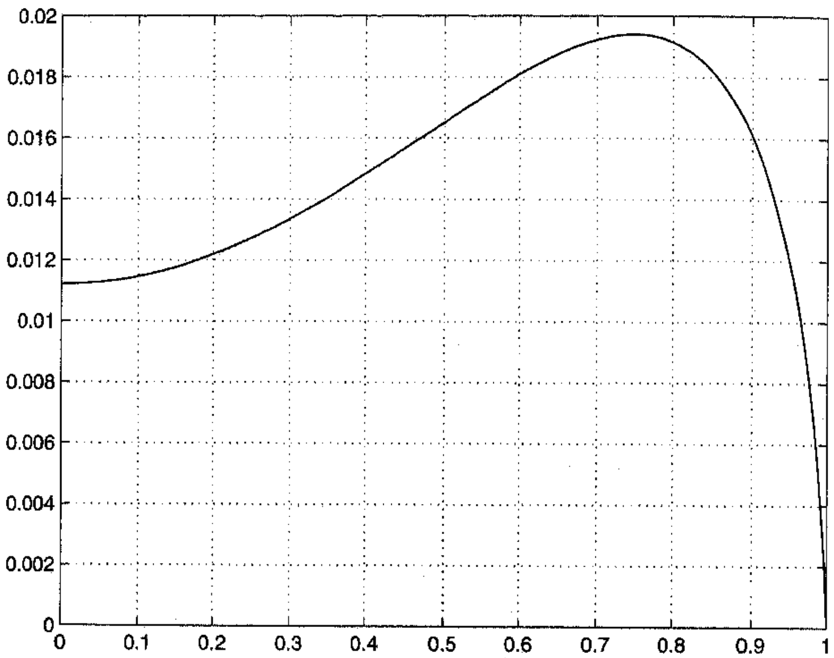

Fig. 1

2.15. LEMMA. *For $r, t \in (0,1)$, let $s = \varphi_K(r)$, $x = rt$ and $y = \varphi_K(x)$. Then the function $f_4(K) = s'^2\mathcal{K}(s)^2/[y'^2\mathcal{K}(y)^2]$ is strictly decreasing on $[1,\infty)$.*

Proof. Making use of the formula ([He], [AVV3, Lemma 2.1])

$$\frac{\partial s}{\partial K} = \frac{2}{\pi K} s s'^2 \mathcal{K}(s)\mathcal{K}'(s), \tag{2.16}$$

we get, by logarithmic differentiation,

$$\frac{\pi K}{4 f_4(K)} f_4'(K) = h(y) - h(s), \tag{2.17}$$

where $h(k) = \mathcal{K}'(k)[\mathcal{K}(k) - \mathcal{E}(k)]$. The function $h(k)$ is strictly increasing from $(0,1)$ onto $(0,\infty)$ by [AVV3, Theorem 2.2(3)] and [AVV4, Theorem 2.1(6)]. Hence it follows from (2.17) that

$$f_4'(K) < 0 \quad \text{for all } K \in (1,\infty),$$

since $y < s$, and the result follows. ■

2.18. LEMMA. *For $r, t \in (0,1)$, let $v = \varphi_{1/K}(r)$, $x = rt$ and $u = \varphi_{1/K}(x)$. Then the function $f_5(K) = v'^2\mathcal{K}(v)^2/[u'^2\mathcal{K}(u)^2]$ is strictly increasing on $[1,\infty)$.*

Proof. Note that by (2.16) we have the formula

$$\frac{\partial v}{\partial K} = -\frac{2}{\pi K} v v'^2 \mathcal{K}(v)\mathcal{K}'(v), \quad K > 1,\ 0 < r < 1. \tag{2.19}$$

By logarithmic differentiation,

$$\frac{\pi K}{4 f_5(K)} f_5'(K) = h(v) - h(u) > 0,$$

where $h(k)$ is as in the proof of Lemma 2.15, since $v > u$, and hence the result follows. ■

2.20. LEMMA. *For $K \in (1,\infty)$ and $a, r, t \in (0,1)$, define the function f_a on $(0,1)\times(0,1)$ by $f_a(r,t) = \varphi_{K^a}(r)\varphi_{K^{1-a}}(t)/\varphi_{K^a}(rt)$. Then $f_a(r,t)$ is strictly decreasing in r from $(0,1)$ onto $(\varphi_{K^{1-a}}(t)/\varphi_{K^a}(t), \varphi_{K^{1-a}}(t)/t^{1/K^a})$. Moreover, we have:*

(1) *For all $a \in (0,1/2)$, $f_a(r,t)$ is strictly decreasing in t from $(0,1)$ onto $(1,\infty)$. In particular, for all $K \in [1,\infty)$, $r, t \in (0,1)$ and $a \in (0,1/2)$,*

$$\varphi_{K^a}(r)\varphi_{K^{1-a}}(t) \ge \varphi_{K^a}(rt), \tag{2.21}$$

with equality if and only if $K = 1$. The inequality is reversed if $K \in (0,1)$.

(2) *For $a = 1/2$, f_a is strictly decreasing in t from $(0,1)$ onto $(1, \varphi_L(r)/r^{1/L})$, where $L = \sqrt{K}$. In particular, for all $L \in [1,\infty)$ and $r, t \in (0,1)$,*

$$\varphi_L(rt) \le \varphi_L(r)\varphi_L(t) \le \min\{4^{r'^{4/3}(1-1/L)}, 4^{t'^{4/3}(1-1/L)}\}\varphi_L(rt), \tag{2.22}$$

with equalities if and only if $K = 1$.

(3) *For $a \in (1/2,1]$, $r, t \in (0,1)$ and $K \in [1,\infty)$,*

$$\varphi_{K^a}(r)\varphi_{K^{1-a}}(t) \le 4^{t'^{4/3}(1-K^{a-1})} t^{K^{a-1}(1-K^{1-2a})} \varphi_{K^a}(rt), \tag{2.23}$$

with equality if and only if $K = 1$. However, for each $a \in (1/2,1)$, $K > 1$, $\varphi_{K^a}(r)\varphi_{K^{1-a}}(t)$ and $\varphi_{K^a}(rt)$ are not comparable.

Proof. Let $s = \varphi_{K^a}(r)$, $u = \varphi_{K^{1-a}}(t)$, $x = rt$ and $y = \varphi_{K^a}(x)$. Then $f_a(r,t) = su/y$. By logarithmic differentiation,

$$\frac{r}{f_a(r,t)} \cdot \frac{\partial f_a}{\partial r} = \psi(r, K^a) - \psi(x, K^a), \tag{2.24}$$

$$\frac{t}{f_a(r,t)} \cdot \frac{\partial f_a}{\partial t} = \psi(t, K^{1-a}) - \psi(x, K^a), \tag{2.25}$$

where $\psi(k, L) = v'^2\mathcal{K}(v)\mathcal{K}'(v)/[k'^2\mathcal{K}(k)\mathcal{K}'(k)]$ with $v = \varphi_L(k)$. Now the monotonicity of f_a in r follows from (2.24), since $\psi(k, L)$ is strictly decreasing in k on $(0,1)$ by [AVV1, Theorem 3.27] and [Q1, Lemma 1]. The limiting values are clear.

(1) If $a \in (0,1/2)$, then $K^{1-a} \geq K^a$ and $\partial f_a/\partial t < 0$ by (2.25) since $\psi(k,L)$ is also strictly decreasing in L on $[1,\infty)$ by [AVV3, Theorem 2.2(3)] and $t > x$. Hence f_a is strictly decreasing in t. The limiting values and (2.21) are clear.

If $K \in (0,1)$, then, by (2.21),

$$\varphi_{1/K^a}(\varphi_{K^a}(r)\varphi_{K^{1-a}}(t)) \leq \varphi_{1/K^a}(\varphi_{K^a}(r))\varphi_{K^{a-1}}(\varphi_{K^{1-a}}(t)) = rt$$

and hence (2.21) is reversed.

(2) For $a = 1/2$, $K^{1-a} = K^a$ and then $\partial f_a/\partial t < 0$ by (2.25) since $t > x$. The limiting values are clear. Hence

$$\varphi_L(rt) \leq \varphi_L(r)\varphi_L(t) \leq 4^{r'^{4/3}(1-1/L)}\varphi_L(rt)$$

by the inequality [QVV2, Theorem 1.12 (1)]

(2.26) $$\varphi_K(r) \leq 4^{r'^{4/3}(1-1/K)} r^{1/K}$$

for all $r \in (0,1)$ and all $K \in [1,\infty)$. Similarly,

$$\varphi_L(r)\varphi_L(t) \leq 4^{t'^{4/3}(1-1/L)}\varphi_L(rt)$$

and hence the second inequality in (2.22) follows.

(3) If $a \in (1/2,1]$, then, by the monotonicity of f_a in r,

$$\varphi_{K^{1-a}}(t)/\varphi_{K^a}(t) \leq f_a(r,t) \leq \varphi_{K^{1-a}}(t)/t^{1/K^a}$$

and (2.23) follows from (2.26).

For the last conclusion, we consider the function $F_1(t) = f_a(0,t) = u/t^{1/K^a}$. From its derivative

$$\frac{tK^a F_1'(t)}{F_1(t)} = K^{2a-1}\psi(t,K^{1-a}) - 1,$$

which is strictly decreasing in t from $(0,1)$ onto $(-1, K^{2a-1}-1)$, we see that there exists a $t_0 \in (0,1)$ such that $F_1(t)$ is increasing on $(0,t_0)$ and decreasing on $(t_0,1)$. Since $F_1(1) = 1$, $F_1(t) > 1$ for all $t \in (t_0,1)$. Therefore if r is sufficiently close to 0 and $t \in (t_0,1)$, then $f_a(r,t) > 1$, and hence it is not true that

$$\varphi_{K^a}(r)\varphi_{K^{1-a}}(t) \leq \varphi_{K^a}(rt)$$

for all $K \in (1,\infty)$ and $r,t \in (0,1)$.

On the other hand, since

$$\lim_{t\to 0} f_a(1,t) = \lim_{t\to 0} f_a(0,t) = 0,$$

we also see that the inequality

$$\varphi_{K^a}(r)\varphi_{K^{1-a}}(t) \geq \varphi_{K^a}(rt)$$

cannot hold for all values of r and t. Therefore, for each $a \in (1/2,1)$ and $K \in (1,\infty)$, $r,t \in (0,1)$, $\varphi_{K^a}(r)\varphi_{K^{1-a}}(t)$ and $\varphi_{K^a}(rt)$ are not comparable. ∎

2.27. Remark. The monotonicity of $f_a(r,t)$ in r and the first inequality in (2.22) have been proved in [AVV1, Theorem 3.13]. The second inequality in (2.22), however, improves some inequalities known to us, such as (1.1) and

$$\varphi_K(r)^p \leq 4^{(p-1)(1-1/K)}\varphi_K(r^p)$$

for each positive integer p [AVV1, Lemma 3.19], and (1.1).

To end this section, we recall the following inequalities [QVV2, Theorem 1.9(4)]:

(2.28) $$0 < \frac{2}{\pi}r'^2\mathcal{K}(r)\mathcal{K}'(r) + \log r < r'^{4/3}\log 4$$

for all $r \in (0,1)$, which will be frequently employed in the proof of the main theorems.

3. Proofs.

In this section, we prove the theorems stated in Section 1.

3.1. Proof of Theorem 1.3. To prove the inequality (1.4), it suffices to prove that

(3.2) $$\varphi_K(r)\varphi_K(t) \leq \varphi_{K^{c+\varepsilon}}(rt)$$

for any $\varepsilon > 0$. For this purpose, we define the function F on $D = (0,1)\times(0,1)\times(1,\infty)$ by

$$F(r,t,K) = \frac{\varphi_K(r)\varphi_K(t)}{\varphi_{K^b}(rt)},$$

where $b = c+\varepsilon$. We shall show that

(3.3) $$\sup_{(r,t,K)\in D} F(r,t,K) \leq 1.$$

Set $s = \varphi_K(r)$, $T = \varphi_K(t)$, $x = rt$ and $y = \varphi_{K^b}(x)$. Then

$$F(r,t,K) = \frac{s}{r^{1/K}}\cdot\frac{T}{t^{1/K}}\cdot\frac{x^{1/K^b}}{y}\cdot x^{(1/K)(1-K^{1-b})},$$

from which we get

(3.4) $$\begin{cases} F(0,t,K) = F(r,0,K) = 0,\\ F(1,t,K) = T/\varphi_{K^b}(t) \leq 1,\\ F(r,1,K) = s/\varphi_{K^b}(r) \leq 1,\\ F(r,t,1) = F(r,t,\infty) = 1,\end{cases}$$

since $1+\varepsilon < b < 2/1.5008765+\varepsilon$ by Lemma 2.11.

Next, by (2.16) and the formula [He]

(3.5) $$\frac{\partial s}{\partial r} = \frac{ss'^2\mathcal{K}(s)^2}{Krr'^2\mathcal{K}(r)^2},$$

we have

$$\frac{\partial y}{\partial r} = yy'^2\mathcal{K}(y)^2/[rx'^2\mathcal{K}(x)^2K^b],$$
$$\frac{\partial y}{\partial t} = yy'^2\mathcal{K}(y)^2/[tx'^2\mathcal{K}(x)^2K^b],$$
$$\frac{\partial y}{\partial K} = \frac{2b}{\pi K}yy'^2\mathcal{K}(y)\mathcal{K}'(y).$$

Employing these formulas for partial derivatives and logarithmic differentiation, we get

(3.6) $$\frac{r}{F(r,t,K)}\cdot\frac{\partial F}{\partial r} = \psi(r,K) - \psi(x,K^b),$$

(3.7) $$\frac{t}{F(r,t,K)}\cdot\frac{\partial F}{\partial t} = \psi(t,K) - \psi(x,K^b)$$

and

(3.8) $$\frac{\pi K}{2F(r,t,K)}\cdot\frac{\partial F}{\partial K} = s'^2\mathcal{K}(s)\mathcal{K}'(s) + T'^2\mathcal{K}(T)\mathcal{K}'(T) - by'^2\mathcal{K}(y)\mathcal{K}'(y),$$

where $\psi(k,L)$ is as in the proof of Lemma 2.20.

Letting $\partial F/\partial r = \partial F/\partial t = \partial F/\partial K = 0$, from (3.6)–(3.8) we obtain

(3.9) $$r = t \quad\text{and}\quad \frac{(1+r^2)\mathcal{K}(r^2)\mathcal{K}'(r^2)}{\mathcal{K}(r)\mathcal{K}'(r)} = \frac{2}{b},$$

by the monotonicity of ψ.

Since $2/b = 2m/(2+m\varepsilon) < m$, the second equality in (3.9) contradicts the definition of m. This shows that $F(r,t,K)$ has no extreme points in D, and hence (3.3) holds by (3.4) and so does (3.2).

By the above discussion, the condition for equality in (1.4) is clear.

Finally, since by (1.4),

$$\varphi_{K^c}(\varphi_{1/K}(r)\varphi_{1/K}(t)) \geq \varphi_K(\varphi_{1/K}(r))\varphi_K(\varphi_{1/K}(t)) = rt,$$

(1.5) follows from the relation

(3.10) $$\varphi_K^{-1}(r) = \varphi_{1/K}(r)$$

and completes the proof. ∎

3.11. Corollary. *For each $r \in (0,1)$ and $K \in [1,\infty)$,*

(3.12) $$\varphi_K(r)^2 \leq \varphi_{K^c}(r^2),$$

where c is as in Theorem 1.3. Equality holds if and only if $K = 1$. The inequality is reversed if $K \in (0,1)$.

This corollary is the special case of Theorem 1.3 when $r = t$. However, we can prove that Theorem 1.3 is equivalent to Corollary 3.11 without using Lemma 2.11.

3.13. Corollary. *The following two conditions are equivalent:*

(1) *For any $r,t \in (0,1)$ and $K \in [1,\infty)$,*

$$\varphi_K(r)\varphi_K(t) \leq \varphi_{K^c}(rt).$$

(2) *For any $r \in (0,1)$ and $K \in [1,\infty)$,*

$$\varphi_K(r)^2 \leq \varphi_{K^c}(r^2).$$

Proof. It is enough to show that (1) is true if (2) holds. For this purpose, define the function G on $D = (0,1)\times(0,1)\times(1,\infty)$ by

$$G(r,t,K) = sT/y,$$

where $s = \varphi_K(r), T = \varphi_K(t)$, $y = \varphi_{K^c}(x)$ and $x = rt$. Then one can show that

$$G|_{\partial D}(r,t,K) \leq 1,$$

similarly to the proof of Theorem 1.3, and that if $G(r,t,K)$ has an extreme point $(r_0,t_0,K_0) \in D$, then $r_0 = t_0$ and

$$G(r_0,t_0,K_0) = \varphi_K(r_0)^2/\varphi_{K^c}(r_0^2),$$

which is less than or equal to 1 by assumption. Hence $G(r,t,K) \leq 1$ for all $(r,t,K) \in D$, that is, (1) is true. ∎

3.14. Corollary. *For each $K \in [2^{1/(c-1)},\infty)$, where c is as in Theorem 1.3, the function $f(r) = \varphi_K(r)^2/\varphi_{K^c}(r^2)$ is strictly increasing from $(0,1)$ onto itself.*

Proof. Since $K \geq 2^{1/(c-1)}$,

(3.15) $$\varphi_{K^c}(r^2) = \varphi_K(\varphi_{K^{c-1}}(r^2)) > \varphi_K(r^{2/K^{c-1}}) \geq \varphi_K(r)$$

by the well-known estimate ([LV], [He])

(3.16) $$\varphi_K(r) > r^{1/K}$$

for all $K \in (1,\infty)$ and $r \in (0,1)$, and by the monotonicity of $\varphi_K(r)$. Set $s = \varphi_K(r)$, $x = r^2$ and $y = \varphi_{K^c}(x)$. Then similarly to (3.6), we have

$$\frac{r}{2f(r)}\cdot\frac{df}{dr} = \psi(r,K) - \psi(x,K^c),$$

where $\psi(k,L)$ is as in the proof of Lemma 2.20. Since $y > s$ by (3.15), it follows that

$$\frac{r}{2f(r)}f'(r) > \frac{\psi(r,K)}{x'^2\mathcal{K}(x)\mathcal{K}'(x)}\{x'^2\mathcal{K}(x)\mathcal{K}'(x) - r'^2\mathcal{K}(r)\mathcal{K}'(r)\} > 0,$$

by [AVV3, Theorem 2.2(3)]. The limiting values are clear. ∎

3.17. Proof of Theorem 1.6. From [AVV1, Proof of Theorem 3.13], we see immediately that $f(r,t,K)$ is strictly decreasing in r and t on $(0,1)$.

Next, let $s=\varphi_K(r)$, $u=\varphi_K(t)$, $x=rt$ and $y=\varphi_K(x)$. By logarithmic differentiation,

$$\frac{K^2}{y'^2\mathcal{K}(y)^2f(r,t,K)}\cdot\frac{\partial f}{\partial K} \tag{3.18}$$
$$=\frac{2}{\pi}\left\{F_1(K)\frac{\mathcal{K}'(r)}{\mathcal{K}(r)}+F_2(K)\frac{\mathcal{K}'(t)}{\mathcal{K}(t)}-\frac{\mathcal{K}'(x)}{\mathcal{K}(x)}\right\}-\frac{\log a(r,t)}{y'^2\mathcal{K}(y)^2},$$

where $F_1(K)=s'^2\mathcal{K}(s)^2/[y'^2\mathcal{K}(y)^2]$ and $F_2(K)=u'^2\mathcal{K}(u)^2/[y'^2\mathcal{K}(y)^2]$. By Lemma 2.15, $F_1(K)$ and $F_2(K)$ are both strictly decreasing on $[1,\infty)$. Now, it follows from (3.18) and (2.28) that

$$\frac{K^2x'^2\mathcal{K}(x)^2}{y'^2\mathcal{K}(y)^2f(r,t,K)}\cdot\frac{\partial f}{\partial K}$$
$$\le\frac{2}{\pi}r'^2\mathcal{K}(r)\mathcal{K}'(r)+\frac{2}{\pi}t'^2\mathcal{K}(t)\mathcal{K}'(t)$$
$$-\frac{2}{\pi}x'^2\mathcal{K}(x)\mathcal{K}'(x)-\log a(r,t)$$
$$=\left\{\frac{2}{\pi}r'^2\mathcal{K}(r)\mathcal{K}'(r)+\log r\right\}+\left\{\frac{2}{\pi}t'^2\mathcal{K}(t)\mathcal{K}'(t)+\log t\right\}$$
$$-\left\{\frac{2}{\pi}x'^2\mathcal{K}(x)\mathcal{K}'(x)+\log x\right\}-\log a(r,t)$$
$$<g(r)+g(t)-\frac{1}{2}\{g(r)+g(t)\}-\log a(r,t)$$
$$=\frac{1}{2}\{[g(r)-r'^{4/3}\log 4]+[g(t)-t'^{4/3}\log 4]\}<0,$$

where $g(k)=\frac{2}{\pi}k'^2\mathcal{K}(k)\mathcal{K}'(k)+\log k$, since $g(k)$ is strictly decreasing on $(0,1)$ ([W], [AVV3, Lemma 4.2]) and $x<r$, $x<t$. This yields the monotonicity of f in K.

Clearly, $f(r,t,1)=a(r,t)$ and $f(r,t,\infty)=1$ for $r,t\in(0,1)$. The last conclusion is trivial. ■

Taking $t=r$ in Theorem 1.6, we get the following result.

3.19. Corollary. *For each $r\in(0,1)$, $f(K)=\varphi_K(r)^2 4^{[r'^{4/3}/K]}/\varphi_K(r^2)$ is strictly decreasing from $[1,\infty)$ onto $(1,4^{r'^{4/3}}]$. In particular, for all $K\in[1,\infty)$ and $r\in(0,1)$,*

$$\varphi_K(r)^2/\varphi_K(r^2)\le 4^{r'^{4/3}(1-1/K)},$$

with equality if and only if $K=1$.

3.20. Remark. It was conjectured in [AVV5] that, for $K\ge 1$ and $r\in(0,1)$,

$$\varphi_K(\sqrt{r})^2/\varphi_K(r)\le(1+r')^{2(1-1/K)}.$$

Corollary 3.19 gives a stronger inequality.

3.21. Proof of Theorem 1.7. Set $v=\varphi_{1/K}(r)$, $T=\varphi_{1/K}(t)$, $x=rt$ and $y=\varphi_{1/K}(x)$. Then, by logarithmic differentiation,

$$\frac{g'(K)}{y'^2\mathcal{K}(y)^2g(K)}=-\frac{2}{\pi}\left\{g_1(K)\frac{\mathcal{K}'(r)}{\mathcal{K}(r)}+g_2(K)\frac{\mathcal{K}'(t)}{\mathcal{K}(t)}-\frac{\mathcal{K}'(x)}{\mathcal{K}(x)}\right\} \tag{3.22}$$
$$+\frac{1}{y'^2\mathcal{K}(y)^2}\log b(r,t),$$

where $g_1(K)=v'^2\mathcal{K}(v)^2/[y'^2\mathcal{K}(y)^2]$ and $g_2(K)=T'^2\mathcal{K}(T)^2/[y'^2\mathcal{K}(y)^2]$.

By Lemma 2.18, $g_1(K)$ and $g_2(K)$ are both strictly increasing on $[1,\infty)$. On the other hand, $y'^2\mathcal{K}(y)^2$ is increasing in K on $[1,\infty)$ by [AVV3, Theorem 2.2], and $1<b(r,t)<4$ for $0<r,t<1$. Hence it follows from (3.22) that

$$\frac{\pi^2g'(K)}{4y'^2\mathcal{K}(y)^2g(K)}>-\{\mu(r)+\mu(t)-\mu(x)-\log b(r,t)\}$$
$$=\left[\log\frac{(1+\sqrt{r'})\sqrt{2(1+r')}}{r}-\mu(r)\right]$$
$$+\left[\log\frac{(1+\sqrt{t'})\sqrt{2(1+t')}}{t}-\mu(t)\right]$$
$$+\left[\mu(x)-\log\frac{(1+\sqrt[4]{x'})\sqrt{(1+\sqrt{x'})(1+x')}}{x}\right]>0$$

by the inequalities ([AQV, Theorem 3.1], [AVV, Theorem 4.9])

$$\operatorname{arth}(\sqrt[4]{k'})<\mu(k)<\log\frac{(1+\sqrt{k'})\sqrt{2(1+k')}}{k}, \tag{3.23}$$

$0<k<1$. This yields the monotonicity of g.

The remaining conclusions are clear. ■

Taking $t=r$ in Theorem 1.12, we get

3.24. Corollary. *For each $r\in(0,1)$ and $x=r^2$,*

$$b(r)=2(1+\sqrt{r'})^2(1+r')/[(1+\sqrt[4]{x'})\sqrt{(1+\sqrt[4]{x'})(1+x')}]$$

the function $g_3(K)=\varphi_{1/K}(r)^2b(r)^K/\varphi_{1/K}(r^2)$ is strictly increasing on $[1,\infty)$. In particular, for $K\in[1,\infty)$ and $r\in(0,1)$,

$$\varphi_{1/K}(r)^2\ge b(r)^{1-K}\varphi_{1/K}(r^2)\ge 4^{1-K}\varphi_{1/K}(r^2),$$

with equalities if and only if $K=1$.

4. Generalizations. In this section, we generalize some results in the theorems proved in the previous section, and (1.1), (1.2) to the case of different parameters K. First, we have

4.1. THEOREM. *For $r,t \in (0,1)$, $K_1,K_2 \in [1,\infty)$, let $a(r) = 2^{r'^{4/3}}$, $s=\varphi_{K_1}(r)$, $u=\varphi_{K_2}(t)$, $x=rt$ and $y=\varphi_K(x)$, where $K=\max\{K_1,K_2\}$. Then the function $F(K_1,K_2)=sua(r)^{1/K_1}a(t)^{1/K_2}/y$ is strictly decreasing in K_1 and K_2 on $[K_2,\infty)$ and $[K_1,\infty)$, respectively. In particular, for all $K_1,K_2\in[1,\infty)$ and $r,t\in(0,1)$,*

$$\varphi_{K_1}(r)\varphi_{K_2}(t) \le a(r)^{1-1/K_1}a(t)^{1-1/K_2}\varphi_K(rt),$$

with equality if and only if $K_1=K_2=1$.

Proof. Firstly, we may assume $K_1 \ge K_2$. Then $K=K_1$. By logarithmic differentiation,

$$\begin{aligned}(4.2)\quad \frac{K_1^2x'^2\mathcal{K}(x)^2}{y'^2\mathcal{K}(y)^2F(K_1,K_2)}\cdot\frac{\partial F}{\partial K_1} &= A(K_1)\\ &= \frac{2}{\pi}f_4(K_1)x'^2\mathcal{K}(x)^2\frac{\mathcal{K}'(r)}{\mathcal{K}(r)}\\ &\quad -\frac{2}{\pi}x'^2\mathcal{K}(x)\mathcal{K}'(x)-\frac{x'^2\mathcal{K}(x)^2}{y'^2\mathcal{K}(y)^2}\log a(r),\end{aligned}$$

where f_4 is defined in Lemma 2.15. By Lemma 2.15, $A(K_1)$ is strictly decreasing on $[1,\infty)$. Hence

$$\begin{aligned}A(K_1)\le A(K_2)\le A(1) &= \frac{2}{\pi}r'^2\mathcal{K}(r)\mathcal{K}'(r)-\frac{2}{\pi}x'^2\mathcal{K}(x)\mathcal{K}'(x)-\log a(r)\\ &< -\log a(r) < 0\end{aligned}$$

since $r > x$ and $k'^2\mathcal{K}(k)\mathcal{K}'(k)$ is decreasing on $(0,1)$. This yields that $\partial F/\partial K_1 < 0$ by (4.2), showing that F is strictly decreasing in K_1 on $[K_2,\infty)$.

Similarly, F is strictly decreasing in K_2 on $[K_1,\infty)$.

Next, by the monotonicity of F, we have

$$F(K_1,K_2)\le F(K_0,K_0)=\varphi_{K_0}(r)\varphi_{K_0}(t)[a(r)a(t)]^{1/K_0}/\varphi_{K_0}(rt),$$

where $K_0=\min\{K_1,K_2\}$, which is strictly decreasing in K_0 from $[1,\infty)$ onto $(1,a(r)a(t)]$ by Theorem 1.6, and hence $F(K_1,K_2)\le a(r)a(t)$, or equivalently,

$$\varphi_{K_1}(r)\varphi_{K_2}(t)\le a(r)^{1-1/K_1}a(t)^{1-1/K_2}\varphi_K(rt).$$

The condition for the equality is clear. ■

4.3. THEOREM. (1) *For $K_1,K_2\in[1,\infty)$ and $r,t\in(0,1)$,*

$$(4.4)\qquad \varphi_{K_1}(r)\varphi_{K_2}(t)\le\varphi_{K_1K_2}(rt),$$

and

$$(4.5)\qquad \varphi_{1/K_1}(r)\varphi_{1/K_2}(t)\ge\varphi_{1/(K_1K_2)}(rt),$$

with equalities if and only if $K_1=K_2=1$.

(2) *Let $K_1,K_2\in[1,\infty)$, $r,t\in(0,1)$, $a\in(0,1]$ and $K=K_1K_2$. Then we have the following results:*

1° *If $K_j\le K^a, j=1,2$, then $a\in[1/2,1]$ and for all $r,t\in(0,1)$,*

$$(4.6)\qquad \varphi_{K_1}(r)\varphi_{K_2}(t)\le\min\{\varphi_{K^{ac}}(rt),\ B^{1-1/K^a}\varphi_{K^a}(rt)\},$$

where c is as in Theorem 1.3, and $B=\min\{4^{r'^{4/3}},4^{t'^{4/3}},2^{r'^{4/3}+t'^{4/3}}\}$. Equality holds if and only if $K_1=K_2=1$.

2° *If $K_j\ge K^a$, $j=1,2$, then $a\in(0,1/2]$ and for all $r,t\in(0,1)$,*

$$\varphi_{K_1}(r)\varphi_{K_2}(t)\ge\varphi_{K^a}(rt),\qquad \varphi_{1/K_1}(r)\varphi_{1/K_2}(t)\le\varphi_{1/K^a}(rt),$$

with equalities if and only if $K_1=K_2=1$.

3° *If $K_1>K^a$ and $K_2<K^a$ (or $K_1<K^a$ and $K_2>K^a$), with $a\in(0,1)$, then $\varphi_{K_1}(r)\varphi_{K_2}(t)$ and $\varphi_{K^a}(rt)$ are not comparable.*

Proof. Let $s=\varphi_{K_1}(r)$, $u=\varphi_{K_2}(t)$, $x=rt$ and $y=\varphi_{K^a}(x)$, for convenience, where $K=K_1K_2$ and $a>0$ is a constant. Define the function H on $D=(0,1)\times(0,1)\times(1,\infty)\times(1,\infty)$ by

$$H(r,t,K_1,K_2)=su/y.$$

Then, by logarithmic differentiation,

$$(4.7)\qquad \frac{r}{H(r,t,K_1,K_2)}\cdot\frac{\partial H}{\partial r}=\psi(r,K_1)-\psi(x,K^a),$$

$$(4.8)\qquad \frac{t}{H(r,t,K_1,K_2)}\cdot\frac{\partial H}{\partial t}=\psi(t,K_2)-\psi(x,K^a),$$

$$(4.9)\qquad \frac{\pi K_1}{2H(r,t,K_1,K_2)}\cdot\frac{\partial H}{\partial K_1}=H_1(s)-aH_1(y),$$

$$(4.10)\qquad \frac{\pi K_2}{2H(r,t,K_1,K_2)}\cdot\frac{\partial H}{\partial K_2}=H_1(u)-aH_1(y),$$

where $\psi(k,K)$ is as in the proof of Lemma 2.20 and $H_1(k)=k'^2\mathcal{K}(k)\mathcal{K}'(k)$, $0<k<1$.

Letting $\partial H/\partial r=\partial H/\partial t=\partial H/\partial K_1=\partial H/\partial K_2=0$, we get

$$(4.11)\qquad r=t,\ \ K_1=K_2\quad\text{and}\quad f_3(r)=1/a,$$

by the monotonicity of ψ and H_1, where f_3 is as in the proof of Lemma 2.11.

On the other hand, we have

$$\begin{aligned}(4.12)\quad & H(r,t,K_1,K_2)\\ &=\frac{s}{r^{1/K_1}}\cdot\frac{u}{t^{1/K_2}}\cdot\frac{x^{1/K^a}}{y}r^{(1/K_1)(1-K_1/K^a)}t^{(1/K_2)(1-K_2/K^a)},\end{aligned}$$

and hence, noting that $K_1 = K^a$ (resp. $K_2 = K^a$) implies that $K_2 = K^{1-a}$ (resp. $K_1 = K^{1-a}$),

$$(4.13)\qquad H(0,t,K_1,K_2) = \begin{cases} 0 & \text{if } K_1 < K^a, \\ \varphi_{K^{1-a}}(t)/t^{1/K^a} & \text{if } K_1 = K^a, \\ \infty & \text{if } K_1 > K^a, \end{cases}$$

$$(4.14)\qquad H(r,0,K_1,K_2) = \begin{cases} 0 & \text{if } K_2 < K^a, \\ \varphi_{K^{1-a}}(r)r^{-1/K^a} & \text{if } K_2 = K^a, \\ \infty & \text{if } K_2 > K^a, \end{cases}$$

$$(4.15)\qquad H(1,t,K_1,K_2) = \varphi_{K_2}(t)/\varphi_{K^a}(t),$$

$$(4.16)\qquad H(r,1,K_1,K_2) = \varphi_{K_1}(r)/\varphi_{K^a}(r),$$

$$(4.17)\qquad H(r,t,1,K_2) = r\varphi_{K_2}(t)/\varphi_{K_2^a}(x) \le r^{1-1/K_2}\varphi_{K_2}(x)/\varphi_{K_2^a}(x),$$

$$(4.18)\qquad H(r,t,K_1,1) = t\varphi_{K_1}(r)/\varphi_{K_1^a}(x) \le t^{1-1/K_1}\varphi_{K_1}(x)/\varphi_{K_1^a}(x),$$

$$(4.19)\qquad H(r,t,\infty,K_2) = \varphi_{K_2}(t),$$

$$(4.20)\qquad H(r,t,K_1,\infty) = \varphi_{K_1}(r).$$

The inequalities in (4.17) and (4.18) hold since [AVV1, Lemma 3.24]

$$r^{1/K}\varphi_K(t) \le \varphi_K(rt).$$

For part (1), take $a = 1$. Then (4.11) implies that H has no extreme points in D by Lemma 2.11, $K_j \le K$, $j = 1,2$, and

$$H(r,t,K_1,K_2) \le \sup_{(r,t,K_1,K_2)\in\partial D} H(r,t,K_1,K_2) \le 1$$

by (4.13)–(4.20). Hence, (4.4) follows.

Since by (4.4),

$$\varphi_K(\varphi_{1/K_1}(r)\varphi_{1/K_2}(t)) \ge \varphi_{K_1}(\varphi_{1/K_1}(r))\varphi_{K_2}(\varphi_{1/K_2}(t)) = rt,$$

(4.5) follows from (3.10).

The condition for equalities in (4.4) and (4.5) is clear.

For part (2), we investigate three cases.

Case (i). If $K_j \le K^a$, $j = 1,2$, then $K_1^{1-a} \le K_2^a$ and $K_2^{1-a} \le K_1^a$, yielding $K^{2a-1} \ge 1$, and hence, $a \in [1/2, 1]$. Moreover,

$$\varphi_{K_1}(r)\varphi_{K_2}(t) \le \varphi_{K^a}(r)\varphi_{K^a}(t)$$

and the upper bound in (4.6) follows from Theorems 1.3 and 1.6 and Lemma 2.20(2).

Case (ii). If $K_j \ge K^a$, $j = 1,2$, then $a \in (0, 1/2]$ similarly to Case (i), and

$$\varphi_{K_1}(r)\varphi_{K_2}(t) \ge \varphi_{K^a}(r)\varphi_{K^a}(t) \ge \varphi_{K^a}(rt)$$

by (1.2). The other result is trivial.

Case (iii). Without loss of generality, we may assume that $K_1 > K^a$ and $K_2 < K^a$ with $a \in (0,1)$. Thus, by (4.13) and (4.14), neither $H(r,t,K_1,K_2) \le 1$ nor $H(r,t,K_1,K_2) \ge 1$ holds for all points in D, and hence the result follows. ■

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Sur la conorme essentielle

par

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Résumé. Pour un opérateur T borné sur un espace de Hilbert dans lui-même, nous montrons que $\gamma(\pi(T)) = \sup\{\gamma(T+K) : K\text{opérateur compact}\}$, où γ est la conorme (the reduced minimum modulus) et $\pi(T)$ est la classe de T dans l'algèbre de Calkin. Nous montrons aussi que ce supremum est atteint.

D'autre part, nous montrons que les opérateurs semi-Fredholm caractérisent les points de continuité de l'application $T \mapsto \gamma(\pi(T))$.

Dans ce travail X denotera un espace de Banach et $B(X)$ l'algèbre des opérateurs bornés de X dans lui-même. Si $T \in B(X)$, on notera respectivement $N(T)$, $R(T)$ et $\sigma(T)$ le noyau, l'image et le spectre de T.

Notons par $\gamma(T)$ la *conorme* de T, définie par

$$\gamma(T) = \inf\{\|Tx\| : d(x, N(T)) = 1\} \quad (\gamma(T) = \infty \text{ si } T = 0).$$

Alors (cf. [4], [9])

(0.1) $$\gamma(T) > 0 \quad \text{si et seulement si} \quad R(T) \text{ fermé;}$$

(0.2) $$\gamma(T) = \gamma(T^*).$$

Il est facile de voir que si $V \in B(X)$ est une isométrie alors

(0.3) $$\gamma(T) = \gamma(VT).$$

D'autre part, la conorme joue un rôle important dans la théorie des perturbations des opérateurs semi-Fredholm. Rappelons qu'un opérateur T est dit *semi-Fredholm* (resp. *Fredholm*) si $R(T)$ est fermé et $\min\{\dim N(T), \operatorname{codim} R(T)\} < \infty$ (resp. si $R(T)$ est fermé et $\max\{\dim N(T), \operatorname{codim} R(T)\} < \infty$). Dans ce cas, on définit l'*indice* de T par

$$\operatorname{ind}(T) = \dim N(T) - \operatorname{codim} R(T).$$

Rappelons le résultat suivant ([4, Théorème V.1.6]) :

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