[22] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Academic Press, New York, 1986.
[23] F. Weisz, *Cesàro summability of two-dimensional Walsh–Fourier series*, Trans. Amer. Math. Soc., to appear.
[24] —, *Inequalities relative to two-parameter Vilenkin–Fourier coefficients*, Studia Math. 99 (1991), 221–233.
[25] —, *Martingale Hardy Spaces and their Applications in Fourier-Analysis*, Lecture Notes in Math. 1568, Springer, Berlin, 1994.
[26] —, *Martingale Hardy spaces for* $0 < p \leq 1$, Probab. Theory Related Fields 84 (1990), 361–376.
[27] —, *Strong summability of two-dimensional Walsh–Fourier series*, Acta Sci. Math. (Szeged) 60 (1995), 779–803.
[28] J. M. Wilson, *A simple proof of the atomic decomposition for* $H^p(\mathbf{R}^n)$, $0 < p \leq 1$, Studia Math. 74 (1982), 25–33.
[29] —, *On the atomic decomposition for Hardy spaces*, Pacific J. Math. 116 (1985), 201–207.
[30] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press, London, 1959.

DEPARTMENT OF NUMERICAL ANALYSIS
EÖTVÖS L. UNIVERSITY
MÚZEUM KRT. 6-8
H-1088 BUDAPEST, HUNGARY
E-mail: WEISZ@LUDENS.ELTE.HU

Received May 5, 1995 (3461)

STUDIA MATHEMATICA 117 (2) (1996)

Hilbert space representations of the graded analogue of the Lie algebra of the group of plane motions

by

SERGEI D. SILVESTROV (Umeå)

Abstract. The irreducible Hilbert space representations of a $*$-algebra, the graded analogue of the Lie algebra of the group of plane motions, are classified up to unitary equivalence.

1. Introduction. In this article we will study representations, by self-adjoint operators in a Hilbert space, of a certain generalized Lie algebra, the graded analogue of the Lie algebra of the group of plane motions.

For the past twenty years, generalized (coloured) Lie algebras have been an object of constant interest in both mathematics and physics (see for example [2–5, 8–10] and references there). When such an algebra is endowed with an involution $*$, we get a $*$-algebra, and it is an important and interesting problem to describe $*$-representations of this $*$-algebra.

It is well known that representations of three-dimensional Lie algebras play an important role in the representation theory of general Lie algebras and groups. Similarly, one would expect the same to be true for three-dimensional coloured Lie algebras with respect to general coloured Lie algebras.

The representations of non-isomorphic algebras have different structure. It is a simple and attractive idea to start by classifying, up to isomorphism, all coloured Lie algebras and then to describe representations of one representative from each isomorphism class. Unfortunately, the classification, up to isomorphism, of all coloured Lie algebras turns out to be a hopelessly difficult task, in the same way as it is already for Lie algebras. Thus, the idea does not work in the general case.

1991 *Mathematics Subject Classification*: Primary 47D40; Secondary 47D25, 16W55, 17A45.

This research was partially supported by J. C. Kempes Minnes Fond.
The results of this paper were presented by the author at the International Congress of Mathematicians, Zürich, 1994.

However, if we restrict ourselves to three-dimensional coloured Lie algebras the classification can be accomplished and the program just described begins to look more realizable.

In [11, 12] three-dimensional coloured Lie algebras are classified up to isomorphism in terms of their structure constants, that is, in terms of commutation relations between generators. In [7] Hilbert space $*$-representations of the real forms of the graded analogue of the Lie algebra $\mathrm{sl}(2;\mathbb{C})$, one of the non-trivial algebras from the classification, are described. The graded analogue of the Lie algebra of the group of plane motions is another non-trivial algebra in the classification.

2. Formulation of the problem. The main aim of this paper is to classify, up to unitary equivalence, all irreducible triples of both bounded and unbounded Hilbert space self-adjoint operators which satisfy the relations

$$a_1a_2 + a_2a_1 = a_3, \quad a_1a_3 + a_3a_1 = a_2, \quad a_2a_3 + a_3a_2 = 0. \tag{1}$$

A complex associative algebra L with generators a_1, a_2, a_3 and relations (1) is called the *graded analogue of the Lie algebra of the group of plane motions*. When anticommutators in (1) are changed into commutators, we indeed get the relations between generators in the Lie algebra of the group of plane motions or, to be more precise, in its complexification. The above-mentioned triples can be viewed as representations of generators in an associative $*$-algebra L with three generators satisfying (1) and an involution $*$ defined by $a_i^* = a_i$, $i = 1, 2, 3$. At the same time, L can be considered as the universal enveloping algebra of a three-dimensional $\mathbb{Z}_2^3$-graded generalized Lie algebra [10].

Recall that a $\mathbb{Z}_2^n$-*graded* (*coloured*) *generalized Lie algebra* is a $\mathbb{Z}_2^n$-graded linear space

$$X = \bigoplus_{\gamma\in\mathbb{Z}_2^n} X_\gamma$$

with a bilinear multiplication (bracket) $\langle\cdot;\cdot\rangle : X \times X \to X$ satisfying:

Grading Axiom. $\langle X_\alpha; X_\beta\rangle \subseteq X_{\alpha+\beta}$.

Graded Skew-Symmetry. $\langle a; b\rangle = -(-1)^{\Sigma_{i=1}^n \alpha_i\beta_i}\langle b; a\rangle$.

Generalized Jacobi Identity.

$$(-1)^{\Sigma_{i=1}^n \alpha_i\gamma_i}\langle a; \langle b; c\rangle\rangle + (-1)^{\Sigma_{i=1}^n \gamma_i\beta_i}\langle c; \langle a; b\rangle\rangle + (-1)^{\Sigma_{i=1}^n \beta_i\alpha_i}\langle b; \langle c; a\rangle\rangle = 0,$$

for all $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, $\gamma = (\gamma_1, \ldots, \gamma_n)$ in $\mathbb{Z}_2^n$, and $a \in X_\alpha$, $b \in X_\beta$, $c \in X_\gamma$ (see for example [10, 9]).

Here $\sum$ means addition in $\mathbb{Z}_2$. The elements of $\bigcup_{\gamma\in\mathbb{Z}_2^n} X_\gamma$ are called *homogeneous*.

Any $\mathbb{Z}_2^n$-graded (coloured) generalized Lie algebra X can be embedded in its universal enveloping algebra $U(X)$ in such a way that, for homogeneous $a \in X_\alpha$ and $b \in X_\beta$, the bracket $\langle\cdot;\cdot\rangle$ becomes a commutator, $[a; b] = ab - ba$, when $\sum_{i=1}^n \alpha_i\beta_i$ is even, or an anticommutator, $\{a; b\} = ab + ba$, when $\sum_{i=1}^n \alpha_i\beta_i$ is odd [10].

Now take X to be a $\mathbb{Z}_2^3$-graded linear space

$$X = X_{(1,1,0)} \oplus X_{(1,0,1)} \oplus X_{(0,1,1)}$$

with homogeneous basis $a_1 \in X_{(1,1,0)}$, $a_2 \in X_{(1,0,1)}$, $a_3 \in X_{(0,1,1)}$. The homogeneous components graded by the elements of $\mathbb{Z}_2^3$ different from $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$ are zero and so are omitted. If the $\mathbb{Z}_2^3$-graded bilinear multiplication $\langle\cdot;\cdot\rangle$ turns X into a $\mathbb{Z}_2^3$-graded generalized Lie algebra, then $\langle a_i; a_i\rangle = 0$, $i = 1, 2, 3$, and

$$\langle a_1; a_2\rangle = c_{12}a_3, \quad \langle a_2; a_3\rangle = c_{23}a_1, \quad \langle a_3; a_1\rangle = c_{31}a_2.$$

Also, $\langle a; b\rangle = \langle b; a\rangle$ when a and b are in different homogeneous subspaces, and $\langle a; b\rangle = -\langle b; a\rangle$ when a and b are in the same one. Moreover, the generalized Jacobi identity is valid. Now put $c_{12} = 1$, $c_{31} = 1$ and $c_{23} = 0$. The algebra X so defined has L as its universal enveloping algebra.

Let us change generators to transform the relations (1) to a more convenient form. Specifically, put $v_1 = a_1$, $v_2 = a_2 + a_3$, $v_3 = a_2 - a_3$. Then the new generators v_1, v_2 and v_3 are also self-adjoint and satisfy the relations

$$v_1v_2 + v_2v_1 = v_2, \quad v_1v_3 + v_3v_1 = -v_3, \quad v_2^2 - v_3^2 = 0, \tag{2}$$

which follow immediately from (1).

3. Definition of representations. Since the operators required are not necessarily bounded, we need a precise definition of the relations (1) for unbounded operators. This means that we must choose a class of representations to study. Analogously to the Lie algebras case, the representations of this class will be called *integrable*.

Definition 1. An *integrable representation* of the $*$-algebra L in a Hilbert space H is a triple of self-adjoint operators A_1, A_2, A_3 such that the operators $V_1 = A_1$, $V_2 = A_2 + A_3$, $V_3 = A_2 - A_3$ satisfy the relations (2) on a dense linear subset Φ in H, invariant relative to V_1, V_2, V_3, and consisting of entire vectors (see [9]) for V_1, V_2, V_3.

Henceforth only integrable representations will be considered.

Remark 1. All bounded representations, if such exist, are integrable.

Let $E_V(\delta)$ denote the spectral measure (the resolution of the identity) of a self-adjoint operator V. This is a projection-valued probability measure defined for any Borel set $\delta \in B(\mathbb{R})$.

For more general commutation relations $AB = BF(A)$, corresponding to an arbitrary continuous function $F : \mathbb{R} \to \mathbb{R}$, it can be proved (see [9], p. 202) that the definition of unbounded representations on a dense invariant linear subset of entire vectors for A, $F(A)$ and B can be equivalently reformulated in terms of the commutation relations involving only bounded functions of the operators of representation. The first two of the relations (2) are of this type. The above-mentioned continuous function is $F_1(\lambda) = 1 - \lambda$ for the first relation, and $F_2(\lambda) = -1 - \lambda$ for the second, and Definition 1 yields

$$E_{V_1}(\delta)V_2\phi = V_2E_{V_1}(1-\delta)\phi, \quad E_{V_1}(\delta)V_3\phi = V_3E_{V_1}(-1-\delta)\phi$$

for any $\phi \in \Phi$, which is equivalent to

$$E_{V_1}(\delta)V_{2,r} = V_{2,r}E_{V_1}(1-\delta), \quad E_{V_1}(\delta)V_{3,r} = V_{3,r}E_{V_1}(-1-\delta) \tag{3}$$

for any $\delta \in B(\mathbb{R})$, $r = 1, 2, \ldots$ and $V_{i,r} = E_{V_i}([-r, r])V_i$, $i = 2, 3$.

4. Classification of the irreducible representations. Let us proceed to classify the irreducible representations.

DEFINITION 2. A representation of the algebra L is called *irreducible* if any bounded operator which commutes with all operators of the representation (i.e. with all their spectral projections) is a multiple of the identity operator.

Consider a self-adjoint operator $D = V_2^2 = V_3^2$. As shown in the following lemma, D commutes with V_1, V_2, V_3 in the sense of resolution of the identity.

LEMMA 1. *The spectral measure of the operator D commutes with the spectral measures of V_1, V_2, V_3.*

Proof. First, observe that V_1 commutes with V_2^2 and so with D. Next, from the functional calculus for one self-adjoint operator one has

$$f(D)g(V_i) = g(V_i)f(D), \quad i = 2, 3, \tag{4}$$

for any bounded measurable functions f and g. Finally, to complete the proof, take f and g to be the indicator functions of some Borel sets. ∎

Let $\mathcal{F}$ be an iterated function system (see [1]) on $\mathbb{R}^1$ generated by two functions $F_1(\lambda) = 1 - \lambda$, $F_2(\lambda) = -1 - \lambda$.

Remark 2. An iterated function system is a generalization of discrete time dynamical systems to the situation where more than one function is iterated (see for example [1]).

The relations (2) yield $\ker(V_2) = \ker(V_3)$ and the invariance of the subspace $\ker(V_2)$ under V_1. Therefore, if $V = (V_1, V_2, V_3)$ is an irreducible representation, then either $\dim(H) = 1$ or $\ker(V_2) = \ker(V_3) = 0$.

The following two lemmas are concerned with the non-trivial second possibility.

The next lemma is crucial.

LEMMA 2. *In an irreducible representation V_1, V_2, V_3 of the algebra L, the spectrum $\sigma(V_1)$ of the operator V_1 is discrete, simple and based on an orbit of the iterated function system $\mathcal{F}$.*

Proof. If $\delta \in B(\mathbb{R})$ is invariant with respect to $\mathcal{F}$ then, in view of (3), the projection $E_{V_1}(\delta)$ commutes with V_1, V_2 and V_3. Therefore it is either 0 or 1, which means that the spectral measure of V_1 is ergodic with respect to $\mathcal{F}$. The iterated function system $\mathcal{F}$ has a measurable section, a Borel subset of $\mathbb{R}^1$ which meets each orbit just once. For instance, $S = [-1/2, 1/2]$ would serve as such. This property implies that any ergodic measure must be supported on an orbit of $\mathcal{F}$. Now only the simplicity of $\sigma(V_1)$ remains unproved.

Note that since V_1 commutes with V_2^2 and V_3^2, any eigenspace of V_1 is invariant with respect to V_2^2 and V_3^2. Let H_λ be the eigenspace of V_1 associated with an eigenvalue λ. Suppose that $\dim(H_\lambda) \neq 1$. Lemma 1 and the irreducibility of (V_1, V_2, V_3) imply that there exists a proper subspace $H_\lambda^0 \subset H_\lambda$ invariant with respect to V_2^2 and V_3^2. If $\lambda = 1/2$, then the subspace spanned by H_λ^0 and by the subspaces obtained by application to H_λ^0 of the sequence of operators $\{V_3, V_2V_3, V_3V_2V_3, V_2V_3V_2V_3, \ldots\}$ is invariant with respect to V_1, V_2 and V_3. The relations (2) imply that the latter subspace is orthogonal to $H_\lambda \ominus H_\lambda^0$. Hence it is different from H. This contradicts irreducibility. If $\lambda = -1/2$, then the subspace obtained by application to H_λ^0 of the sequence of operators $\{V_2, V_3V_2, V_2V_3V_2, V_3V_2V_3V_2, \ldots\}$ is invariant with respect to V_1, V_2 and V_3, and it is orthogonal to $H_\lambda \ominus H_\lambda^0$. Hence it is different from H. This again contradicts irreducibility. Finally, if $\lambda =]-1/2, 1/2[$, then the subspace obtained by application to H_λ^0 of the sequences of operators $\{V_2, V_3V_2, V_2V_3V_2, V_3V_2V_3V_2, \ldots\}$ and $\{V_3, V_2V_3, V_3V_2V_3, V_2V_3V_2V_3, \ldots\}$ is invariant with respect to V_1, V_2 and V_3, and it is orthogonal to $H_\lambda \ominus H_\lambda^0$. Hence it is different from H, and we get a contradiction with irreducibility which completes the proof. ∎

In the sequel, H_μ and e_μ will denote, respectively, the eigenspace of V_1 and an eigenvector of V_1 associated with an eigenvalue μ.

The following lemma is a corollary to Lemma 2 and the relations (2).

LEMMA 3. *If $e \in H_\lambda$ then $V_2e \in H_{1-\lambda}$ and $V_3e \in H_{-1-\lambda}$.*

Now we are ready to classify the irreducible representations.

THEOREM 1. *To each point of the set*

$$M = ([-1/2, 1/2] \times]0, \infty[) \cup (\mathbb{R} \times \{0\})$$

there corresponds an irreducible triple A_1, A_2, A_3 of self-adjoint operators satisfying (1). *The dimension of any irreducible triple is either* 1 *or* ∞.

(i) $\dim(H) = 1$: *Each point* $(\lambda, 0) \in \mathbb{R} \times \{0\}$ *parametrizes one triple*:

$$A_1 e_\lambda = \lambda e_\lambda, \quad A_2 = 0, \quad A_3 = 0, \tag{5}$$

where H is spanned by the vector e_λ.

(ii) $\dim(H) = \infty$: *Each point* (λ, c) *of* $M_{]-1/2,1/2[} =]-1/2, 1/2[\times]0, \infty[$ *parametrizes one triple*:

$$\begin{aligned} A_1 e^{(k)} &= (-1)^k (\lambda - k) e^{(k)}, \\ A_2 e^{(k)} &= \sqrt{c}(e^{(k+(-1)^k)} + e^{(k-(-1)^k)})/2, \\ A_3 e^{(k)} &= \sqrt{c}(e^{(k+(-1)^k)} - e^{(k-(-1)^k)})/2, \end{aligned} \tag{6}$$

where $\{e^{(k)} = e_{(-1)^k(\lambda-k)} \mid k \in \mathbb{Z}\}$ *is an orthonormal basis consisting of eigenvectors of the operator* A_1.

(iii) $\dim(H) = \infty$: *Each point* (λ, c) *of* $M_{-1/2} \cup M_{1/2} = \{-1/2, 1/2\} \times]0, \infty[$ *parametrizes two unitarily inequivalent triples which are given by the formulas* (7) *and* (8).

The triples corresponding to points of $M_{-1/2} = \{(-1/2, c) \mid c \in]0, \infty[\}$ *are*

$$\begin{aligned} A_1^{\pm} e^{(k)} &= (-1)^k (k - 1/2) e^{(k)}, \\ A_2^{\pm} e^{(k)} &= \begin{cases} \sqrt{c}(e^{(k+(-1)^{k+1})} + e^{(k+(-1)^k)})/2 & \text{if } k \ge 2, \\ \sqrt{c}(e^{(2)} \pm e^{(1)})/2 & \text{if } k = 1, \end{cases} \\ A_3^{\pm} e^{(k)} &= \begin{cases} \sqrt{c}(e^{(k+(-1)^{k+1})} - e^{(k+(-1)^k)})/2 & \text{if } k \ge 2, \\ \sqrt{c}(e^{(2)} \mp e^{(1)})/2 & \text{if } k = 1, \end{cases} \end{aligned} \tag{7}$$

where $\{e^{(k)} = e_{(-1)^k(k-1/2)} \mid k \in \mathbb{N}\}$ *is an orthonormal basis consisting of eigenvectors of the operator* A_1.

The triples corresponding to points of $M_{1/2} = \{(1/2, c) \mid c \in]0, \infty[\}$ *are*

$$\begin{aligned} A_1^{\pm} e^{(k)} &= (-1)^{k+1} (k - 1/2) e^{(k)}, \\ A_2^{\pm} e^{(k)} &= \begin{cases} \sqrt{c}(e^{(k+(-1)^k)} + e^{(k+(-1)^{k+1})})/2 & \text{if } k \ge 2, \\ \sqrt{c}(e^{(2)} \pm e^{(1)})/2 & \text{if } k = 1, \end{cases} \\ A_3^{\pm} e^{(k)} &= \begin{cases} \sqrt{c}(e^{(k+(-1)^k)} - e^{(k+(-1)^{k+1})})/2 & \text{if } k \ge 2, \\ -\sqrt{c}(e^{(2)} \mp e^{(1)})/2 & \text{if } k = 1, \end{cases} \end{aligned} \tag{8}$$

where $\{e^{(k)} = e_{(-1)^{k+1}(k-1/2)} \mid k \in \mathbb{N}\}$ *is an orthonormal basis consisting of eigenvectors of the operator* A_1.

(iv) *The irreducible triples of operators defined by* (5)–(8) *are pairwise unitarily inequivalent. Any non-trivial integrable irreducible triple of self-adjoint operators satisfying* (1) *is unitarily equivalent to one of these.*

Proof. Earlier in this section we found that if (V_1, V_2, V_3) is an irreducible triple, then either $\dim(H) = 1$ or $\ker(V_2) = \ker(V_3) = 0$. The former case is described in the item (i) of the theorem. Let us consider the latter case.

First, from Lemmas 2 and 3, the spectrum $\sigma(V_1)$ of the operator V_1 is discrete and based on some orbit of the iterated function system $\mathcal{F}$. Moreover, $\sigma(V_1)$ is simple, which means that the eigensubspaces H_λ are one-dimensional.

Let e_λ be an orthonormal basis in H consisting of eigenvectors of the operator $V_1 = A_1$. Our aim is to find the eigenvalues of V_1 and to describe the action of V_2 and V_3 in the basis e_λ.

As was mentioned in the proof of Lemma 2, the iterated function system $\mathcal{F}$ has a measurable section $S = [-1/2, 1/2]$. The points $1/2$ and $-1/2$ are fixed by F_1 and F_2 respectively. Let λ be that point where the orbit $\mathrm{Orb}_\lambda(F)$ meets S. Then

$$\mathrm{Orb}_\lambda(F) = \begin{cases} \{(-1)^k(\lambda - k) \mid k \in \mathbb{Z}\} & \text{if } \lambda \in]-1/2, 1/2[, \\ \{(-1)^k(k - 1/2) \mid k \in \mathbb{N}\} & \text{if } \lambda = -1/2, \\ \{(-1)^{k+1}(k - 1/2) \mid k \in \mathbb{N}\} & \text{if } \lambda = 1/2. \end{cases}$$

By Lemma 2, the spectrum $\sigma(V_1)$ coincides with one of these orbits.

Next, since V_2 and V_3 are self-adjoint, from Lemma 1, for some complex numbers $b_2(\lambda)$, $b_3(\lambda)$ we have $V_2 e_\lambda = b_2(\lambda) e_{1-\lambda}$ and $V_3 e_\lambda = b_3(\lambda) e_{-1-\lambda}$, and so $V_2^2 e_\lambda = |b_2(\lambda)|^2 e_\lambda$, $V_3^2 e_\lambda = |b_3(\lambda)|^2 e_\lambda$. A representation unitarily equivalent to $\{V_i\}$ can be chosen so that

$$\begin{aligned} b_j(\lambda) &\ge 0 && \text{if } F_{j-1}(\lambda) \ne \lambda, \\ b_j(\lambda) &\in \mathbb{R}^1 && \text{if } F_{j-1}(\lambda) = \lambda, \ j = 2, 3. \end{aligned} \tag{9}$$

According to Lemma 1, D commutes with the irreducible triple V_1, V_2, V_3 and hence $D = cI$ for some $c \in \mathbb{R}$. Thus, we see from Lemma 3 that

$$b_j(\lambda)^2 e_\lambda = V_j^2 e_\lambda = D e_\lambda = c e_\lambda, \quad j = 2, 3,$$

and hence that $b_2(\lambda)^2 = c$ and $b_3(\lambda)^2 = c$.

It now follows that any irreducible triple satisfying (2) is unitarily equivalent to one of those in the statement of the theorem.

Finally, triples which correspond to different points $(\lambda, c) \in M$ are unitarily inequivalent since they have different $\sigma(D) = \{c\}$ or $\sigma(V_1)$. Inequivalence of the two triples corresponding to a point in $M_{-1/2} \cup M_{1/2}$ can be checked directly.

The change back from the operators V_i to A_i completes the proof. ■

Remark 3. One can write down the matrices of the operators defined by formulas (6)–(8). They are three-diagonal.

Remark 4. It might be of interest to remark that Theorem 1 shows that in any irreducible representation of the relations (1), the operator A_1 is either one-dimensional or unbounded.

Remark 5. It is possible to prove a kind of spectral theorem which gives a decomposition, with respect to some projection-valued measure, of an arbitrary representation as a direct integral of irreducible representations. As in the case of the classical spectral theorem for one self-adjoint operator, such a theorem, reformulated in multiplication operator form, provides the complete list of unitary invariants in terms of the spectral families of measures.

One can conclude from Theorem 1 and Remarks 4 and 5 that there are no triples of non-zero bounded self-adjoint operators satisfying the relations (1).

The same must be true for some broad class of coloured Lie algebras, and an interesting question is: for which?

Acknowledgements. I am very grateful to Professor Hans Wallin and Professor Yuriĭ S. Samoĭlenko for their helpful suggestions and remarks.

References

[1] M. F. Barnsley, *Fractals Everywhere*, Academic Press, New York, 1988.

[2] L. Corwin, Y. Ne'eman and S. Sternberg, *Graded Lie algebras in mathematics and physics (Bose–Fermi symmetry)*, Rev. Modern Phys. 47 (3) (1975), 573–603.

[3] M. V. Karasev and V. P. Maslov, *Non-Lie permutation relations*, Russian Math. Surveys 45 (5) (1990), 51–98.

[4] A. K. Kwaśniewski, *Clifford- and Grassmann-like algebras—old and new*, J. Math. Phys. 26 (1985), 2234–2238.

[5] W. Marcinek, *Generalized Lie algebras and related topics*, 1, 2, Acta Univ. Wratislav. Mat. Fiz. Astronom. 55 (1991).

[6] V. L. Ostrovskiĭ and Yu. S. Samoĭlenko, *Unbounded operators satisfying non-Lie commutation relations*, Rep. Math. Phys. 28 (3) (1989), 93–106.

[7] V. L. Ostrovskiĭ and S. D. Silvestrov, *Representations of the real forms of the graded analogue of a Lie algebra*, Ukrain. Mat. Zh. 44 (11) (1992), 1518–1524 (in Russian).

[8] V. Rittenberg and D. Wyler, *Generalized superalgebras*, Nuclear Phys. B 139 (1978), 189–202.

[9] Yu. S. Samoĭlenko, *Spectral Theory of Families of Self-Adjoint Operators*, Kluwer, Dordrecht, 1990.

[10] M. Scheunert, *Generalized Lie algebras*, J. Math. Phys. 20 (1979), 712–720.

[11] S. D. Silvestrov, *On the classification of 3-dimensional graded ε-Lie algebras*, Research Reports Series, No. 2, Department of Mathematics, Umeå University, 1993, 49 pp.

[12] S. D. Silvestrov, *The classification of 3-dimensional graded ε-Lie algebras*, to appear.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UMEÅ
S-90187 UMEÅ, SWEDEN
E-mail: SERGEI@ABLE.MATH.UMU.SE

Received May 25, 1995
Revised version August 16, 1995 (3474)