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Strong convergence theorems for two-parameter Walsh–Fourier and trigonometric-Fourier series

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Abstract. The martingale Hardy space $H_p([0,1)^2)$ and the classical Hardy space $H_p(\mathbb{T}^2)$ are introduced. We prove that certain means of the partial sums of the two-parameter Walsh–Fourier and trigonometric-Fourier series are uniformly bounded operators from H_p to L_p $(0 < p \le 1)$. As a consequence we obtain strong convergence theorems for the partial sums. The classical Hardy–Littlewood inequality is extended to two-parameter Walsh–Fourier and trigonometric-Fourier coefficients. The dual inequalities are also verified and a Marcinkiewicz–Zygmund type inequality is obtained for BMO spaces.

1. Introduction. We introduce the martingale Hardy space $H_p([0,1)^2)$ by the L_p norm of the diagonal maximal function of a two-parameter martingale, and the classical Hardy space $H_p(\mathbb{T}^2)$ by the L_p norm of the non-tangential maximal function of a distribution. It is known that neither the Walsh nor the trigonometric system is a basis in L_1. Moreover, there exist functions in H_1 whose partial sums are not bounded in L_1. Smith [19] and Simon [17] proved the following strong convergence result for one-parameter trigonometric-Fourier and Walsh–Fourier series, respectively:

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|s_k f - f\|_1}{k} = 0,$$

where $f \in H_1$ and $s_k f$ denotes the kth partial sum of the Fourier series.

We generalize these results to two-parameter trigonometric-Fourier and Walsh–Fourier series and verify that there exists a constant C_p depending

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only on p and α such that

$$\frac{1}{\log n \log m} \sum_{\substack{2^{-\alpha} \le k/l \le 2^{\alpha} \\ (k,l) \le (n,m)}} \frac{\|s_{k,l} f\|_1}{kl} \le C_p \|f\|_{H_1},$$

where $\alpha \ge 0$ and H_1 is either $H_1(\mathbb{T}^2)$ or $H_1([0,1)^2)$. The two-parameter analogues of the previous convergence results follow easily from this. For Walsh–Fourier series the previous inequality is extended to the spaces $H_p([0,1)^2)$ $(0 < p < 1)$ as well.

The dual to the previous inequality gives a Marcinkiewicz–Zygmund type inequality. Finally, the classical Hardy–Littlewood inequality ([12]) is extended to two-parameter Walsh–Fourier and trigonometric-Fourier coefficients and the dual inequality is also proved. Another version of the Hardy–Littlewood inequality for two-parameter Walsh–Fourier series was shown by the author in [24], [25].

2. The martingale Hardy space $H_p([0,1)^2)$. In this and the next section the unit square $[0,1)^2$ and the two-dimensional Lebesgue measure λ are considered. We also use the notation $|I|$ for the Lebesgue measure of the set I.

By a *dyadic interval* we mean one of the form $[k2^{-n}, (k+1)2^{-n})$ for some $k, n \in \mathbb{N}$, $0 \le k < 2^n$. Given $n \in \mathbb{N}$ and $x \in [0,1)$ let $I_n(x)$ denote the dyadic interval of length 2^{-n} which contains x. If I_1 and I_2 are dyadic intervals and $|I_1| = |I_2|$ then the set $I := I_1 \times I_2$ is a *dyadic square*. Clearly, the dyadic square of area 2^{-2n} containing $(x,y) \in [0,1)^2$ is $I_n(x) \times I_n(y)$. We also write $I_{n,n}(x,y)$ for this set.

The σ-algebra generated by the dyadic squares $\{I_{n,n}(x) : x \in [0,1)^2\}$ will be denoted by $\mathcal{F}_{n,n}$ $(n \in \mathbb{N})$, more precisely,

$$\mathcal{F}_{n,n} = \sigma\{[k2^{-n}, (k+1)2^{-n}) \times [l2^{-n}, (l+1)2^{-n}) : 0 \le k < 2^n,\ 0 \le l < 2^n\}.$$

The expectation operator and the conditional expectation operator relative to $\mathcal{F}_{n,n}$ $(n \in \mathbb{N})$ are denoted by E and $E_{n,n}$, respectively. We briefly write L_p for the real $L_p([0,1)^2, \lambda)$ space; the norm (or quasinorm) of this space is defined by $\|f\|_p := (E|f|^p)^{1/p}$ $(0 < p \le \infty)$. For simplicity, we assume that $Ef = 0$ for $f \in L_1$.

We investigate one-parameter martingales of the form $f = (f_{n,n}, n \in \mathbb{N})$ with respect to $(\mathcal{F}_{n,n}, n \in \mathbb{N})$ and suppose that $f_{0,0} = 0$. The martingale $f = (f_{n,n}, n \in \mathbb{N})$ is said to be *L_p-bounded* $(0 < p \le \infty)$ if $f_{n,n} \in L_p$ $(n \in \mathbb{N})$ and

$$\|f\|_p := \sup_{n \in \mathbb{N}} \|f_{n,n}\|_p < \infty.$$

If $f \in L_1$ then it is easy to show that the sequence $\widetilde{f} = (E_{n,n} f, n \in \mathbb{N})$

is a martingale. Moreover, if $1 \le p < \infty$ and $f \in L_p$ then $\widetilde{f}$ is L_p-bounded and

$$\lim_{n \to \infty} \|E_{n,n} f - f\|_p = 0;$$

consequently, $\|\widetilde{f}\|_p = \|f\|_p$ (see Neveu [15]). The converse of this result also holds for $1 < p < \infty$ (see Neveu [15]): if $f = (f_{n,n}, n \in \mathbb{N})$ is a martingale, then there exists a function $g \in L_p$ for which $f_{n,n} = E_{n,n} g$ if and only if f is L_p-bounded. If $p = 1$ then there exists $g \in L_1$ as above if and only if f is *uniformly integrable* (Neveu [15]), i.e., if

$$\lim_{\alpha \to \infty} \sup_{n \in \mathbb{N}} \int_{\{|f_{n,n}| > \alpha\}} |f_{n,n}| \, d\lambda = 0.$$

Thus the map $f \mapsto \widetilde{f} := (E_{n,n} f, n \in \mathbb{N})$ is isometric from L_p onto the space of L_p-bounded martingales when $1 < p < \infty$. Consequently, the two spaces can be identified. Similarly, the L_1 space can be identified with the space of uniformly integrable martingales. For this reason a function $f \in L_1$ and the corresponding martingale $(E_{n,n} f, n \in \mathbb{N})$ will be denoted by the same symbol f.

The *maximal function* of a martingale $f = (f_{n,n}, n \in \mathbb{N})$ is defined by

$$f^* := \sup_{n \in \mathbb{N}} |f_{n,n}|.$$

It is easy to see that, in case $f \in L_1$, the maximal function can also be given by

$$f^*(x,y) = \sup_{n \in \mathbb{N}} \frac{1}{|I_{n,n}(x,y)|} \Big| \int_{I_{n,n}(x,y)} f \, d\lambda \Big|.$$

For $0 < p \le \infty$ the *martingale Hardy space* $H_p([0,1)^2)$ consists of all martingales for which

$$\|f\|_{H_p([0,1)^2)} := \|f^*\|_p < \infty.$$

It is well known that for all martingales $f = (f_{n,n}, n \in \mathbb{N})$,

$$\|f^*\|_p \le \frac{p}{p-1} \|f\|_p \quad (1 < p \le \infty),$$

hence $H_p([0,1)^2) \sim L_p$ whenever $1 < p \le \infty$ (see Neveu [15]), where $\sim$ denotes the equivalence of the norms and spaces. Note that $H_p([0,1)^2)$ $(0 < p < \infty)$ can also be defined by the norm

$$\|f\| = \|S(f)\|_p,$$

where

$$S(f) := \Big(\sum_{n=1}^{\infty} |f_{n,n} - f_{n-1,n-1}|^2 \Big)^{1/2}.$$

For this and other equivalent definitions see Weisz [26].

Atomic decomposition provides a useful characterization of Hardy spaces. A bounded measurable function a is a *p-atom* if there exists a dyadic square R such that

(i) $\int_R a\,d\lambda = 0$,
(ii) $\|a\|_\infty \le |R|^{-1/p}$,
(iii) $\{a \ne 0\} \subset R$.

The basic result on atomic decomposition is as follows (see Weisz [25]).

THEOREM A. *A martingale $f = (f_{n,n}, n \in \mathbb{N})$ is in H_p $(0 < p \le 1)$ if and only if there exist a sequence $(a^k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} \mu_k E_{n,n} a^k = f_{n,n} \quad \text{for all } n \in \mathbb{N}, \quad \sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{1}$$

Moreover, the following equivalence of norms holds:

$$\|f\|_{H_p} \sim \inf \Big(\sum_{k=0}^{\infty} |\mu_k|^p \Big)^{1/p}, \tag{2}$$

where the infimum is taken over all decompositions of f of the form (1).

It is proved in Garsia [10] (see also Weisz [25]) that the dual of $H_1([0,1)^2)$ is the $\mathrm{BMO}([0,1)^2)$ space, where $\mathrm{BMO}([0,1)^2)$ consists of all functions $f \in L_2$ for which

$$\|f\|_{\mathrm{BMO}([0,1)^2)} = \sup_{n\in\mathbb{N}} \|(E_{n,n}|f - E_{n,n}f|^2)^{1/2}\|_\infty < \infty.$$

It is easy to see that

$$\|f\|_{\mathrm{BMO}([0,1)^2)} = \sup_R \Big(\frac{1}{|R|} \int_R \Big| f - \frac{1}{|R|} \int_R f\,d\lambda \Big|^2 d\lambda \Big)^{1/2}, \tag{3}$$

where the supremum is taken over all dyadic cubes.

3. Inequalities concerning two-parameter Walsh–Fourier series. First we introduce the Walsh system. Every point $x \in [0,1)$ can be written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}}, \quad 0 \le x_k < 2, \ x_k \in \mathbb{N}.$$

In case there are two different forms, we choose the one for which $\lim_{k\to\infty} x_k = 0$.

The functions

$$r_n(x) := \exp(\pi x_n \sqrt{-1}) \quad (n \in \mathbb{N})$$

are called *Rademacher functions*.

The product system generated by these functions is the *one-dimensional Walsh system*:

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k},$$

where $n = \sum_{k=0}^{\infty} n_k 2^k$, $0 \le n_k < 2$ and $n_k \in \mathbb{N}$.

The Kronecker product $(w_{n,m}; n, m \in \mathbb{N})$ of two Walsh systems is said to be the *two-dimensional Walsh system*. Thus

$$w_{n,m}(x,y) := w_n(x) w_m(y).$$

Recall that the *Walsh–Dirichlet kernels*

$$D_n := \sum_{k=0}^{n-1} w_k \quad (n \in \mathbb{N})$$

satisfy

$$D_{2^n}(x) := \begin{cases} 2^n & \text{if } x \in I_n(0), \\ 0 & \text{if } x \in [0,1) \setminus I_n(0), \end{cases} \tag{4}$$

and

$$D_n = w_n \sum_{k=0}^{\infty} n_k w_{2^k} D_{2^k} \tag{5}$$

for $n \in \mathbb{N}$ (see Fine [8] and Schipp, Wade, Simon and Pál [16]).

If $f \in L_1$ then the number

$$\widehat{f}(n,m) := E(f w_{n,m})$$

is said to be the (n,m)th *Walsh–Fourier coefficient* of f $(n, m \in \mathbb{N})$. Let us extend this definition to martingales as well. If $f = (f_{k,k}, k \in \mathbb{N})$ is a martingale then let

$$\widehat{f}(n,m) := \lim_{k\to\infty} E(f_{k,k} w_{n,m}) \quad (n,m \in \mathbb{N}).$$

Since $w_{n,m}$ is $\mathcal{F}_{k,k}$-measurable for $n, m < 2^k$, it can immediately be seen that this limit does exist. Note that if $f \in L_1$ then $E_{k,k} f \to f$ in L_1 norm as $k \to \infty$, hence

$$\widehat{f}(n,m) = \lim_{k\to\infty} E((E_{k,k} f) w_{n,m}) \quad (n, m, k \in \mathbb{N}).$$

Thus the Walsh–Fourier coefficients of $f \in L_1$ are the same as the ones of the martingale $(E_{k,k} f, k \in \mathbb{N})$ obtained from f.

Denote by $s_{n,m} f$ the (n,m)th partial sum of the Walsh–Fourier series of a martingale f,

$$s_{n,m} f := \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \widehat{f}(k,l) w_{k,l}.$$

It is easy to see that $s_{2^n,2^n} = f_{n,n}$ $(n \in \mathbb{N})$. Obviously,

$$s_{n,m}f(x,y) = (f * (D_n \times D_m))(x,y) := \int_0^1\int_0^1 f(t,u)D_n(x \dot{+} t)D_m(y \dot{+} u)\,dt\,du$$

in case $f \in L_1$, where $\dot{+}$ denotes dyadic addition (see e.g. Schipp, Wade, Simon and Pál [16]). The nth partial sum in the first variable of the Walsh–Fourier series of $f \in L_1$ is defined by

$$(6)\qquad s_n^1 f(x,y) := \int_0^1 f(t,y)D_n(x \dot{+} t)\,dt.$$

The operator s_m^2 is defined analogously. Of course, $s_{n,m} = s_n^1 s_m^2$. We shall also use the following notations for $n,m \in \mathbb{N}$:

$$(7)\qquad \begin{aligned} \widehat{s}_n^1 f(x,m) &:= \int_0^1 s_n^1 f(x,u)w_m(u)\,du,\\ \widehat{s}_m^2 f(n,y) &:= \int_0^1 s_m^2 f(t,y)w_n(t)\,dt. \end{aligned}$$

The main result of this section is the following

THEOREM 1. *Assume that $\alpha \ge 0$, $0 < p \le 1$ and f is an arbitrary martingale from $H_p([0,1)^2)$. Then there exists a constant C_p depending only on p and α such that*

$$\sup_{n,m\ge 2}\left(\frac{1}{\log n \log m}\right)^{[p]} \sum_{\substack{2^{-\alpha}\le k/l\le 2^{\alpha}\\ (k,l)\le(n,m)}} \frac{\|s_{k,l}f\|_p^p}{(kl)^{2-p}} \le C_p\|f\|_{H_p([0,1)^2)}^p,$$

where $[p]$ denotes the integer part of p and $(k,l) \le (n,m)$ means that $k \le n$ and $l \le m$.

We remark that for $p > 1$,

$$(8)\qquad \|s_{k,l}f\|_p \le C_p\|f\|_p \quad (k,l \in \mathbb{N}).$$

Note that the symbol C_p may denote different constants in different contexts. In the proof we follow basically the idea of Simon [17] and verify first the next lemma.

LEMMA 1. *If a is a p-atom $(0 < p \le 1)$ with support $R = I \times J$ and $|I| = |J| = 2^{-K}$ $(K \in \mathbb{N})$ then*

$$(9)\qquad \begin{aligned} \|s_{n,m}a\|_p^p \le C_p + C_pK^{[p]}\int_J |\widehat{s}_m^2 a(n,y)|^p\,dy \\ + C_pK^{[p]}\int_I |\widehat{s}_n^1 a(x,m)|^p\,dx + C_pK^{2[p]}|\widehat{a}(n,m)|^p. \end{aligned}$$

Proof. We may suppose that $I = J = [0, 2^{-K})$. By (8) and by the definition of the atom,

$$(10)\qquad \begin{aligned} \int_I\int_J |s_{n,m}a|^p\,d\lambda &\le |I|^{1-p/2}|J|^{1-p/2}\Big(\int_0^1\int_0^1 |s_{n,m}a|^2\,d\lambda\Big)^{p/2}\\ &\le C_p|I|^{1-p/2}|J|^{1-p/2}\Big(\int_I\int_J |a|^2\,d\lambda\Big)^{p/2} \le C_p. \end{aligned}$$

Let $x \in [0,1)\setminus I =: I^c$. In this case $D_{2^k}(x \dot{+} t)1_I(t) = 0$ for $k \ge K$ (see (4)). Recall that $w_n(x \dot{+} t) = w_n(x)w_n(t)$ and $w_{2^k}(x \dot{+} t) = w_{2^k}(x)$ for $t \in I$ and $k \le K-1$. Using (4) and (5) we obtain

$$\begin{aligned} s_{n,m}a(x,y) &= \int_I s_m^2 a(t,y)D_n(x \dot{+} t)\,dt\\ &= \int_I w_n(x \dot{+} t)\sum_{k=0}^{\infty} n_k w_{2^k}(x \dot{+} t)D_{2^k}(x \dot{+} t)s_m^2 a(t,y)\,dt\\ &= w_n(x)\sum_{k=0}^{K-1} n_k w_{2^k}(x)\int_I D_{2^k}(x \dot{+} t)s_m^2 a(t,y)w_n(t)\,dt\\ &= w_n(x)\sum_{k=0}^{k(x)} n_k w_{2^k}(x)2^k\int_I s_m^2 a(t,y)w_n(t)\,dt, \end{aligned}$$

where $k(x)$ denotes the maximum of the indices $k = 0,1,\dots,K-1$ for which $D_{2^k}(x \dot{+} t) = 2^k$ in case $t \in I$. If $x \in [j2^{-K}, (j+1)2^{-K})$ then $2^{-k(x)} \ge (j+1)2^{-K}$. So

$$\int_{I^c} |s_{n,m}a(x,y)|^p\,dx \le \sum_{j=1}^{2^K-1}\int_{j2^{-K}}^{(j+1)2^{-K}} |s_{n,m}a(x,y)|^p\,dx$$

$$\leq C_p \sum_{j=1}^{2^K-1} 2^{-K} 2^{k(x)p} |\widehat{s}_m^2 a(n,y)|^p$$
$$\leq C_p \sum_{j=1}^{2^K-1} 2^{K(p-1)} j^{-p} |\widehat{s}_m^2 a(n,y)|^p$$
$$\leq C_p K^{[p]} |\widehat{s}_m^2 a(n,y)|^p.$$

Integrating over J we get the second term on the right hand side of (9). Integrating $|s_{n,m}a(x,y)|^p$ over $I \times J^c$ we obtain the third term.

We can show with the same method that

$$s_{n,m}a(x,y) = w_n(x)w_m(y) \sum_{k=0}^{k(x)} \sum_{l=0}^{l(y)} n_k m_l w_{2^k}(x) w_{2^l}(y) 2^k 2^l \widehat{a}(n,m)$$

and

$$\int_{I^c} \int_{J^c} |s_{n,m}a(x,y)|^p \, dx \, dy \leq C_p K^{2[p]} |\widehat{a}(n,m)|^p. \blacksquare$$

Proof of Theorem 1. In addition to Theorem A one can prove that

$$\widehat{f}(k,l) = \sum_{j=0}^{\infty} \mu_j \widehat{a^j}(k,l) \tag{11}$$

(cf. Weisz [25], p. 86). From this it follows that

$$\int_{[0,1)^2} |s_{k,l}f|^p \, d\lambda \leq \sum_{j=0}^{\infty} |\mu_j|^p \int_{[0,1)^2} |s_{k,l}a^j|^p \, d\lambda.$$

Because of this and Theorem A we only have to prove that

$$\sup_{n,m \geq 2} \left(\frac{1}{\log n \log m}\right)^{[p]} \sum_{\substack{2^{-\alpha} \leq k/l \leq 2^\alpha \\ (k,l) \leq (n,m)}} \frac{\|s_{k,l}a\|_p^p}{(kl)^{2-p}} \leq C_p \tag{12}$$

for every p-atom a.

Let a be an arbitrary p-atom with support $I \times J$ and $|I| = |J| = 2^{-K}$ ($K \in \mathbb{N}$). We can suppose again that $I = J = [0, 2^{-K})$. It is easy to see that $\widehat{a}(k,l) = 0$ if $k < 2^K$ and $l < 2^K$, so, in this case, $s_{k,l}a = 0$. Therefore we can suppose that $k \geq 2^K$ or $l \geq 2^K$. Choose $r \in \mathbb{N}$ such that $r - 1 < \alpha \leq r$. If $k \geq 2^K$ then, by the hypothesis,

$$l \geq 2^{-\alpha} k \geq 2^{K-r}.$$

We can assume the same for n and m. Hence

$$\left(\frac{1}{\log n \log m}\right)^{[p]} \sum_{\substack{2^{-\alpha} \leq k/l \leq 2^\alpha \\ (k,l) \leq (n,m)}} \frac{\|s_{k,l}a\|_p^p}{(kl)^{2-p}} \leq \left(\frac{1}{\log n \log m}\right)^{[p]} \sum_{k=2^{K-r}}^{n} \sum_{l=2^{K-r}}^{m} \frac{\|s_{k,l}a\|_p^p}{(kl)^{2-p}}. \tag{13}$$

We use Lemma 1 to estimate the right hand side of (13). First we consider the second term on the right hand side of (9). Since $2^{K-r} \leq n$, we have

$$C_p \left(\frac{1}{\log n \log m}\right)^{[p]} \sum_{k=2^{K-r}}^{n} \sum_{l=2^{K-r}}^{m} K^{[p]} \frac{\int_J |\widehat{s}_l^2 a(k,y)|^p \, dy}{(kl)^{2-p}}$$
$$\leq C_p \left(\frac{1}{\log m}\right)^{[p]} \sum_{l=2^{K-r}}^{m} \frac{1}{l^{2-p}} \sum_{k=2^{K-r}}^{n} \frac{1}{k^{2-p}} \int_J |\widehat{s}_l^2 a(k,y)|^p \, dy$$
$$\leq C_p \left(\frac{1}{\log m}\right)^{[p]} \sum_{l=2^{K-r}}^{m} \frac{1}{l^{2-p}}$$
$$\times \left(\sum_{k=2^{K-r}}^{n} \frac{1}{k^2}\right)^{1-p/2} \left(\sum_{k=2^{K-r}}^{n} \left(\int_J |\widehat{s}_l^2 a(k,y)|^p \, dy\right)^{2/p}\right)^{p/2}.$$

By Hölder's and Parseval's inequalities,

$$\sum_{k=2^{K-r}}^{n} \left(\int_J |\widehat{s}_l^2 a(k,y)|^p \, dy\right)^{2/p} \leq 2^{-K(1-p/2)2/p} \int_0^1 \sum_{k=2^{K-r}}^{n} |\widehat{s}_l^2 a(k,y)|^2 \, dy$$
$$\leq 2^{-2K/p+K} \int_0^1 \int_0^1 |s_l^2 a(x,y)|^2 \, dx \, dy.$$

Applying (8) for one dimension and for a fixed x and the definition of the atom we can conclude that

$$\sum_{k=2^{K-r}}^{n} \left(\int_J |\widehat{s}_l^2 a(k,y)|^p \, dy\right)^{2/p}$$
$$\leq 2^{-2K/p+K} \int_0^1 \int_0^1 |a(x,y)|^2 \, dx \, dy \leq 2^{2K/p-K}.$$

The inequality

$$\sum_{k=2^{K-r}}^{n} \frac{1}{k^2} \leq C 2^{r-K}$$

implies that

$$(14)\quad C_p\Big(\frac{1}{\log n\log m}\Big)^{[p]}\sum_{k=2^{K-r}}^{n}\sum_{l=2^{K-r}}^{m}K^{[p]}\frac{\int_J|\widehat{s}_l^2a(k,y)|^p\,dy}{(kl)^{2-p}} \le C_p\Big(\frac{1}{\log m}\Big)^{[p]}\sum_{l=2^{K-r}}^{m}\frac{1}{l^{2-p}}\le C_p.$$

(14) can be proved similarly for the first and third term of the right hand side of (9). For the fourth term we have

$$(15)\quad C_p\Big(\frac{1}{\log n\log m}\Big)^{[p]}\sum_{k=2^{K-r}}^{n}\sum_{l=2^{K-r}}^{m}K^{2[p]}\frac{\int_J|\widehat{a}(k,l)|^p}{(kl)^{2-p}}$$
$$\le C_p\Big(\sum_{k=2^{K-r}}^{n}\sum_{l=2^{K-r}}^{m}\frac{1}{k^2l^2}\Big)^{1-p/2}\Big(\sum_{k=2^{K-r}}^{n}\sum_{l=2^{K-r}}^{m}|\widehat{a}(k,l)|^2\Big)^{p/2}$$
$$\le C_p2^{-2K(1-p/2)}\int_0^1\int_0^1|a(x,y)|^2\,dx\,dy\le C_p.$$

Combining (14) and (15) we have verified (12) and also Theorem 1. ■

In the one-dimensional case the following corollary was shown by Simon [17] for the Walsh system and recently by Gát [11] for the Vilenkin system.

COROLLARY 1. *If $f\in H_1([0,1)^2)$ then*

$$\frac{1}{\log n\log m}\sum_{\substack{2^{-\alpha}\le k/l\le 2^{\alpha}\\(k,l)\le(n,m)}}\frac{\|s_{k,l}f-f\|_1}{kl}\to 0\quad \text{as } n,m\to\infty.$$

Proof. For every $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $\|f-f_{N,N}\|_{H_1([0,1)^2)}\le\varepsilon$. Since $f_{N,N}$ is a Walsh polynomial, $s_{k,l}f_{N,N}=f_{N,N}$ if k and l are large enough. Theorem 1 implies

$$\frac{1}{\log n\log m}\sum_{\substack{2^{-\alpha}\le k/l\le 2^{\alpha}\\(k,l)\le(n,m)}}\frac{\|s_{k,l}f-f\|_1}{kl}$$
$$\le\frac{1}{\log n\log m}\sum_{\substack{2^{-\alpha}\le k/l\le 2^{\alpha}\\(k,l)\le(n,m)}}\frac{\|s_{k,l}f-s_{k,l}f_{N,N}\|_1}{kl}$$
$$+\frac{1}{\log n\log m}\sum_{\substack{2^{-\alpha}\le k/l\le 2^{\alpha}\\(k,l)\le(n,m)}}\frac{\|s_{k,l}f_{N,N}-f_{N,N}\|_1}{kl}$$
$$+\frac{1}{\log n\log m}\sum_{\substack{2^{-\alpha}\le k/l\le 2^{\alpha}\\(k,l)\le(n,m)}}\frac{\|f_{N,N}-f\|_1}{kl}$$
$$\le C_p\|f-f_{N,N}\|_{H_1([0,1)^2)}+o(1)+\varepsilon\le C_p\varepsilon$$

if n and m are large enough. ■

Another version of the following Hardy–Littlewood type inequality is given in Weisz [24] for two-parameter Vilenkin–Fourier coefficients.

THEOREM 2. *For every martingale $f\in H_p([0,1)^2)$ $(0<p\le 2)$ we have*

$$\sum_{2^{-\alpha}\le k/l\le 2^{\alpha}}\frac{|\widehat{f}(k,l)|^p}{(kl)^{2-p}}\le C_p\|f\|^p_{H_p([0,1)^2)},$$

where C_p depends only on p and α.

Proof. For $0<p\le 1$ the inequality can be proved with the same method as Theorem 1. For $p=2$ it is the Parseval inequality while for $1<p<2$ we obtain it by interpolation (cf. Weisz [25]). ■

We now formulate the dual inequalities to Theorems 1 and 2. The first one is a Marcinkiewicz–Zygmund type inequality for the $\mathrm{BMO}([0,1)^2)$ space. The Marcinkiewicz–Zygmund inequality for $L_p(l_r)$ spaces can be found in Zygmund [30] (Volume 2, p. 225), García-Cuerva and Rubio de Francia [9] (p. 496) and also in Weisz [27].

THEOREM 3. *If $g^{k,l}$ $(2^{-\alpha}\le k/l\le 2^{\alpha},\ k,l\in\mathbb{N})$ are uniformly bounded then*

$$\sup_{n,m\ge 2}\Big\|\frac{1}{\log n\log m}\sum_{\substack{2^{-\alpha}\le k/l\le 2^{\alpha}\\(k,l)\le(n,m)}}\frac{s_{k,l}g^{k,l}}{kl}\Big\|_{\mathrm{BMO}([0,1)^2)}\le C\sup_{2^{-\alpha}\le k/l\le 2^{\alpha}}\|g^{k,l}\|_\infty.$$

Proof. The duality of $H_1([0,1)^2)$ and $\mathrm{BMO}([0,1)^2)$ yields that

$$\Big\|\frac{1}{\log n\log m}\sum_{\substack{2^{-\alpha}\le k/l\le 2^{\alpha}\\(k,l)\le(n,m)}}\frac{s_{k,l}g^{k,l}}{kl}\Big\|_{\mathrm{BMO}([0,1)^2)}$$
$$\le C\sup_{\substack{\|f\|_{H_1([0,1)^2)}\le 1\\ f\in L_2}}\Big|\int_{[0,1)^2}\frac{1}{\log n\log m}\sum_{\substack{2^{-\alpha}\le k/l\le 2^{\alpha}\\(k,l)\le(n,m)}}\frac{s_{k,l}g^{k,l}}{kl}f\,d\lambda\Big|.$$

Since

$$\int_{[0,1)^2}s_{k,l}g^{k,l}f\,d\lambda=\int_{[0,1)^2}s_{k,l}f\,g^{k,l}\,d\lambda,$$

Theorem 1 implies

$$\Big\|\frac{1}{\log n\log m}\sum_{\substack{2^{-\alpha}\le k/l\le 2^{\alpha}\\(k,l)\le(n,m)}}\frac{s_{k,l}g^{k,l}}{kl}\Big\|_{\mathrm{BMO}([0,1)^2)}$$
$$\le C\sup_{\substack{\|f\|_{H_1([0,1)^2)}\le 1\\ f\in L_2}}\frac{1}{\log n\log m}\sum_{\substack{2^{-\alpha}\le k/l\le 2^{\alpha}\\(k,l)\le(n,m)}}\frac{1}{kl}\|s_{k,l}f\|_1\|g^{k,l}\|_\infty$$
$$\le C\sup_{2^{-\alpha}\le k/l\le 2^{\alpha}}\|g^{k,l}\|_\infty. \ \blacksquare$$

THEOREM 4. *If $kl|a_{k,l}|$ $(2^{-\alpha}\le k/l\le 2^{\alpha},\ k,l\in\mathbb{N})$ are uniformly bounded real numbers then*

$$\Big\|\sum_{2^{-\alpha}\le k/l\le 2^{\alpha}}a_{k,l}w_{k,l}\Big\|_{\mathrm{BMO}([0,1)^2)}\le C\sup_{2^{-\alpha}\le k/l\le 2^{\alpha}}kl|a_{k,l}|.$$

Proof. Similarly to the previous proof,

$$\Big\|\sum_{2^{-\alpha}\le k/l\le 2^{\alpha}}a_{k,l}w_{k,l}\Big\|_{\mathrm{BMO}([0,1)^2)}$$
$$\le C\sup_{\substack{\|f\|_{H_1([0,1)^2)}\le 1\\ f\in L_2}}\Big|\int_{[0,1)^2}f\sum_{2^{-\alpha}\le k/l\le 2^{\alpha}}a_{k,l}w_{k,l}\,d\lambda\Big|$$
$$\le C\sup_{\substack{\|f\|_{H_1([0,1)^2)}\le 1\\ f\in L_2}}\sum_{2^{-\alpha}\le k/l\le 2^{\alpha}}\frac{|\widehat{f}_{k,l}|}{kl}kl|a_{k,l}|$$
$$\le C\sup_{\substack{\|f\|_{H_1([0,1)^2)}\le 1\\ f\in L_2}}\|f\|_{H_1([0,1)^2)}\sup_{2^{-\alpha}\le k/l\le 2^{\alpha}}kl|a_{k,l}|. \ \blacksquare$$

Note that this theorem was shown by Ladhawala [13] for one dimension.

4. The classical Hardy space $H_p(\mathbb{T}^2)$. Set $\mathbb{T}:=[-\pi,\pi)$. In this and the next section we denote $L_p(\mathbb{T}^2,\lambda)$ also by L_p and we assume again that $\int_{\mathbb{T}^2}f\,d\lambda=0$ for $f\in L_1$.

Let f be a distribution on $C^\infty(\mathbb{T}^2)$ (briefly $f\in\mathcal{D}'(\mathbb{T}^2)=\mathcal{D}'$). The (n,m)th Fourier coefficient is defined by $\widehat{f}(n,m):=f(e^{-\imath nx}e^{-\imath my})$, where $\imath=\sqrt{-1}$ and $n,m\in\mathbb{Z}$. In particular, if f is an integrable function then

$$\widehat{f}(n,m)=\frac{1}{(2\pi)^2}\int_{\mathbb{T}}\int_{\mathbb{T}}f(x)e^{-\imath nx}e^{-\imath my}\,dx\,dy.$$

Denote by $s_{n,m}f$ the (n,m)th partial sum of the Fourier series of a distribution f,

$$s_{n,m}f(x):=\sum_{k=-n}^{n}\sum_{l=-m}^{m}\widehat{f}(k,l)e^{\imath kx}e^{\imath ly}\qquad(n,m\in\mathbb{N}).$$

For $f\in\mathcal{D}'$ and $t>0$ let

$$u(x,y,t):=(f*P_t)(x,y),$$

where $*$ denotes convolution and

$$P_t(x,y):=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}e^{-t\sqrt{k^2+l^2}}e^{\imath kx+\imath ly}\qquad(x,y\in\mathbb{T})$$

is the Poisson kernel. It is easy to show that $u(x,y,t)$ is a harmonic function and

$$u(x,y,t)=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\widehat{f}(k,l)e^{-t\sqrt{k^2+l^2}}e^{\imath kx+\imath ly}\qquad(t>0)$$

with absolute and uniform convergence (see e.g. Stein and Weiss [21] and Edwards [5]).

Let Γ_α be a cone with vertex at the origin, i.e.

$$\Gamma_\alpha:=\{(x,y,t):\sqrt{x^2+y^2}<\alpha t\}.$$

We denote by $\Gamma'_\alpha(x,y)$ $(x,y\in\mathbb{T})$ the translate of Γ_α with vertex (x,y). Set

$$\Gamma_\alpha(x,y)=\bigcup_{k,l\in\mathbb{Z}}\Gamma'_\alpha(x+k2\pi,y+l2\pi)\cap(\mathbb{T}^2\times(0,\infty)).$$

The non-tangential maximal function is defined by

$$u^*_\alpha(x,y):=\sup_{(x',y',t)\in\Gamma_\alpha(x,y)}|u(x',y',t)|\qquad(\alpha>0).$$

Now we define the Hardy space analogous to $H_p([0,1)^2)$. The space $H_p(\mathbb{T}^2)$ $(0<p\le\infty)$ consists of all distributions f for which $u^*_\alpha\in L_p$; we set

$$\|f\|_{H_p(\mathbb{T}^2)}:=\|u^*_1\|_p.$$

For $0<p<\infty$ Fefferman and Stein [7] proved the equivalence

$$\|u^*_\alpha\|_p\sim\|u^*_1\|_p\sim\|u^+\|_p\qquad(\alpha>0),$$

where

$$u^+(x,y):=\sup_{t>0}|u(x,y,t)|.$$

It is known that if $f\in H_p$ $(0<p<\infty)$ then $f(x,y)=\lim_{t\to 0}u(x,y,t)$ in the sense of distributions (see Fefferman and Stein [7]). Recall that $H_p(\mathbb{T}^2)\sim L_p$ for $1<p<\infty$ (see Fefferman and Stein [7]). The space $H_p(\mathbb{T}^2)$ was investigated by Stein and Weiss [21] (p. 283), Sledd and Stegenga [18] and recently by the author [23].

A *generalized interval* on $\mathbb{T}$ is either an interval $I \subset \mathbb{T}$ or $I = [-\pi, x) \cup [y, \pi)$. A *generalized cube* on $\mathbb{T}^2$ is the Cartesian product $I \times J$ of two generalized intervals with $|I| = |J|$. In this case a bounded measurable function a is called a *p-atom* if there exists a generalized cube R such that (ii) and (iii) hold and, moreover,

(i′) $\int_R a(x,y)x^\alpha y^\beta \, dx\, dy = 0$, where $\alpha, \beta \in \mathbb{N}$ and $\alpha + \beta \le [2(1/p-1)]$, the integer part of $2(1/p-1)$.

In this case the atomic decomposition was proved by Latter [14] and Wilson [28].

THEOREM B. *A distribution f is in H_p $(0 < p \le 1)$ if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} \mu_k a_k = f \quad \text{in the sense of distributions} \tag{16}$$

and the second inequality of (1) *holds. Moreover, the equivalence* (2) *is also true.*

5. Inequalities concerning two-parameter trigonometric-Fourier series. In this section we show the analogues of the results of Section 3.

THEOREM 5. *If $\alpha \ge 0$ and $f \in H_1(\mathbb{T}^2)$ then there exists a constant C depending only on α such that*

$$\sup_{n,m\ge 2} \frac{1}{\log n \log m} \sum_{\substack{2^{-\alpha} \le k/l \le 2^\alpha \\ (k,l)\le(n,m)}} \frac{\|s_{k,l}f\|_1}{kl} \le C\|f\|_{H_1(\mathbb{T}^2)}.$$

As in the proof of Theorem 1 we again need a lemma.

LEMMA 2. *If a is a 1-atom with support $R = I \times J$ and $\pi 2^{-K-1} \le |I| = |J| \le \pi 2^{-K}$ $(K \in \mathbb{N})$ then*

$$\|s_{n,m}a\|_1 \le C + CK\Big(\int_{2J} (|\widehat{s}_m^2 a(n,y)| + |\widehat{s}_m^2 a(-n,y)|)\, dy \tag{17}$$
$$+ \int_{2I} (|\widehat{s}_n^1 a(x,m)| + |\widehat{s}_n^1 a(x,-m)|)\, dx$$
$$+ \int_{2J} (|\widehat{a}(n,y)| + |\widehat{a}(-n,y)|)\, dy$$
$$+ \int_{2I} (|\widehat{a}(x,m)| + |\widehat{a}(x,-m)|)\, dx\Big)$$
$$+ CK^2(|\widehat{a}(n,m)| + |\widehat{a}(-n,m)| + |\widehat{a}(n,-m)| + |\widehat{a}(-n,-m)|,$$

where $s_n^1 a$, $s_m^2 a$, $\widehat{s}_n^1 a(x,m)$, $\widehat{s}_m^2 a(n,y)$, $\widehat{a}(x,m)$ and $\widehat{a}(n,y)$ are defined analogously to (6) *and* (7), *and $2I$ is the generalized interval with the same center as I and with length $2|I|$.*

Proof. The integral of $|s_{n,m}a|$ over $2I \times 2J$ is bounded by C as we have seen in (10).

We can suppose that the center of R is zero. Again,

$$s_{n,m}a(x,y) = \frac{1}{\pi}\int_I s_m^2 a(t,y) D_n(x-t)\, dt, \tag{18}$$

where the *Dirichlet kernels* D_n are given by

$$D_n(t) := \frac{1}{2}\sum_{k=-n}^{n} e^{ikt} = \frac{1}{2}\frac{\sin(n+1/2)t}{\sin(t/2)} \quad (n \in \mathbb{N})$$

(see e.g. Torchinsky [22] or Zygmund [30]). First we investigate the integral

$$\int_I s_m^2 a(t,y) D_n^*(x-t)\, dt$$
$$= \int_I s_m^2 a(t,y) \frac{\sin n(x-t)}{x-t}\, dt$$
$$+ \int_I s_m^2 a(t,y) \sin n(x-t)\left[\frac{1}{2\tan(x-t)/2} - \frac{1}{x-t}\right] dt = (A) + (B),$$

where the *modified Dirichlet kernels* D_n^* are defined by

$$D_n^*(t) := \frac{D_{n-1}(t) + D_n(t)}{2} = \frac{\sin nt}{2\tan(t/2)} \quad (n \in \mathbb{N}).$$

Since $\frac{1}{2\tan(t/2)} - \frac{1}{t} \in L_\infty$ and (8) also holds in this case, we conclude that

$$\int_{(2I)^c}\int_{2J} |(B)|\, d\lambda \le C\int_{2J}\int_I |s_m^2 a(t,y)|\, dt\, dy \tag{19}$$
$$\le C2^{-K}\Big(\int_{\mathbb{T}}\int_{\mathbb{T}} |a(t,y)|^2\, dt\, dy\Big)^{1/2} \le C.$$

For $t \in I$ and $x \notin 2I$ we have $|t/x| \le 1/2$, so

$$(A) = \int_I s_m^2 a(t,y) \frac{\sin n(x-t)}{x} \sum_{k=0}^{\infty} \left(\frac{t}{x}\right)^k dt$$
$$= \int_I s_m^2 a(t,y) \frac{\sin n(x-t)}{x}\, dt + \sum_{k=1}^{\infty} \frac{1}{x^{k+1}} \int_I s_m^2 a(t,y) t^k \sin n(x-t)\, dt$$
$$= (C) + (D).$$

Hence

$$(20) \quad \int_{(2I)^c} \int_{2J} |(C)|\, d\lambda \le \int_{(2I)^c} \int_{2J} \frac{1}{|x|} \Big| \int_I s_m^2 a(t,y) e^{-\imath nt}\, dt \Big|\, dx\, dy$$
$$+ \int_{(2I)^c} \int_{2J} \frac{1}{|x|} \Big| \int_I s_m^2 a(t,y) e^{\imath nt}\, dt \Big|\, dx\, dy$$
$$\le CK \int_{2J} |\widehat{s}_m^2 a(n,y)|\, dy + CK \int_{2J} |\widehat{s}_m^2 a(-n,y)|\, dy.$$

If $t \in I$ and $x \notin 2I$ then $|x| \ge |I|$ and $|t| \le |I|/2$. Consequently,

$$(21) \quad \int_{(2I)^c} \int_{2J} |(D)|\, d\lambda$$
$$\le \sum_{k=1}^{\infty} \frac{|I|^{-k}}{k} \frac{|I|^k}{2^k} \int_{2J} \int_I |s_m^2 a(t,y)|\, dt\, dy$$
$$\le C \sum_{k=1}^{\infty} \frac{1}{k 2^k} \Big(\int_{2J} \int_I |s_m^2 a(t,y)|^2\, dt\, dy \Big)^{1/2} |I|^{1/2} |J|^{1/2} \le C.$$

Since $D_n(t) - D_n^*(t) = \frac{1}{2}\cos(nt)$, we have

$$\int_I s_m^2 a(t,y)[D_n(x-t) - D_n^*(x-t)]\, dt$$
$$= \frac{1}{4} \int_I s_m^2 a(t,y)[e^{\imath n(x-t)} + e^{-\imath n(x-t)}]\, dt.$$

Therefore

$$(22) \quad \int_{(2I)^c} \int_{2J} \Big| \int_I s_m^2 a(t,y)[D_n(x-t) - D_n^*(x-t)]\, dt \Big|\, dx\, dy$$
$$\le C \int_{2J} |\widehat{s}_m^2 a(n,y)|\, dy + C \int_{2J} |\widehat{s}_m^2 a(-n,y)|\, dy.$$

Combining (18)–(22) we can establish that the integral of $|s_{n,m}a|$ over $(2I)^c \times 2J$ is bounded by the right hand side of (17). The same holds if we integrate $|s_{n,m}a|$ over $2I \times (2J)^c$.

Obviously,

$$s_{n,m} a(x,y) = \frac{1}{\pi^2} \int_I \int_J a(t,u) D_n^*(x-t) D_m^*(y-u)\, dt\, du$$
$$+ \frac{1}{\pi^2} \int_I \int_J a(t,u)[D_n(x-t) - D_n^*(x-t)] D_m^*(y-u)\, dt\, du$$
$$+ \frac{1}{\pi^2} \int_I \int_J a(t,u) D_n^*(x-t)[D_m(y-u) - D_m^*(y-u)]\, dt\, du$$
$$+ \frac{1}{\pi^2} \int_I \int_J a(t,u)[D_n(x-t) - D_n^*(x-t)][D_m(y-u)$$
$$- D_m^*(y-u)]\, dt\, du.$$

Now the integral of $|s_{n,m}a|$ over $(2I)^c \times (2J)^c$ can be calculated by the same method as above. The details are left to the reader. ■

Proof of Theorem 5. In this case (11) follows from the L_1-convergence of the series in (16). So we have to prove the theorem only for 1-atoms.

Let a be an arbitrary 1-atom with support $R = I \times J$ and $\pi 2^{-K-1} \le |I| = |J| \le \pi 2^{-K}$ ($K \in \mathbb{N}$). We can suppose again that the center of R is zero. The proof of Theorem 5 is more complicated than the one of Theorem 1 because, in this case, $s_{k,l}a \ne 0$ if $k < 2^K$ and $l < 2^K$. Clearly,

$$\sum_{\substack{2^{-\alpha} \le k/l \le 2^{\alpha} \\ (k,l) \le (n,m)}} \frac{\|s_{k,l}a\|_1}{kl} \le \sum_{\substack{2^{-\alpha} \le k/l \le 2^{\alpha} \\ (k,l) \le (2^K \wedge n, 2^K \wedge m)}} \frac{\|s_{k,l}a\|_1}{kl} + \sum_{k=2^{K-r}}^{n} \sum_{l=2^{K-r}}^{m} \frac{\|s_{k,l}a\|_1}{kl}.$$

Using Lemma 2 one can estimate the second term on the right hand side divided by $\log n \log m$ by a constant depending only on α in the same way as in Theorem 1. We will estimate the first term with $\|s_{k,l}a\|_1$ replaced by $K \int_{2J} |\widehat{s}_l^2 a(k,y)|\, dy$, $K \int_{2J} |\widehat{a}(k,y)|\, dy$ and $K^2 |\widehat{a}(k,l)|$. The other cases are all similar.

Since $|D_l| \le l + 1/2$, we have

$$|s_l^2 a(x,y)| = \frac{1}{\pi} \Big| \int_J a(x,u) D_l(y-u)\, dt \Big| \le \frac{2}{\pi} l |I|^{-1}$$

and so

$$\int_{2J} |\widehat{s}_l^2 a(k,y)|\, dy = \int_{2J} \Big| \int_I s_l^2 a(x,y) e^{-\imath kx}\, dx \Big|\, dy \le C l 2^{-K}.$$

Hence

$$\frac{1}{\log n\log m}\sum_{\substack{2^{-\alpha}\le|k|/l\le 2^{\alpha}\\ (|k|,l)\le(2^K\wedge n,2^K\wedge m)}} K\frac{\int_{2J}|\widehat{s}_l^2a(k,y)|\,dy}{|k|l}\le\frac{CK2^{-K}}{\log n}\sum_{\substack{2^{-\alpha}\le k/l\le 2^{\alpha}\\ (|k|,l)\le(2^K\wedge n,2^K\wedge m)}}\frac{1}{k}.$$

As $k2^{-\alpha}\le l\le k2^{\alpha}$, we conclude that

$$\frac{1}{\log n\log m}\sum_{\substack{2^{-\alpha}\le|k|/l\le 2^{\alpha}\\ (|k|,l)\le(2^K\wedge n,2^K\wedge m)}} K\frac{\int_{2J}|\widehat{s}_l^2a(k,y)|\,dy}{|k|l}\le\frac{CK2^{-K}}{\log n}\sum_{k=1}^{2^K\wedge n}\frac{1}{k}k(2^{\alpha}-2^{-\alpha})\le\frac{CK2^{-K}(2^K\wedge n)}{\log n}.$$

If $n\ge 2^K$ then the last term is bounded by C. On the other hand, the function $x/\log x$ is increasing whenever $x\ge e$. Therefore $n/\log n\le C2^K/K$ if $n\le 2^K$, which yields again that the corresponding term is bounded.

One can verify similarly to (23) below that

$$\int_{2J}|\widehat{a}(k,y)|\,dy\le C|k|2^{-K}.$$

So the proof for the expression $\int_{2J}|\widehat{a}(k,y)|\,dy$ is the same as above.

Finally, using (23) for $p=1$ and $L=0$ we obtain

$$\frac{1}{\log n\log m}\sum_{\substack{2^{-\alpha}\le|k|/|l|\le 2^{\alpha}\\ (|k|,|l|)\le(2^K\wedge n,2^K\wedge m)}} K^2\frac{|\widehat{a}(k,l)|}{|k|\cdot|l|}\le\frac{CK^2}{\log n\log m}\sum_{\substack{2^{-\alpha}\le|k|/|l|\le 2^{\alpha}\\ (|k|,|l|)\le(2^K\wedge n,2^K\wedge m)}}\frac{|I|}{|l|}+\frac{|I|}{|k|}$$
$$\le\frac{CK^2}{\log n\log m}\sum_{\substack{2^{-\alpha}\le k/l\le 2^{\alpha}\\ (k,l)\le(2^K\wedge n,2^K\wedge m)}}\frac{|I|}{k}.$$

If $m\ge n$ then this is bounded by

$$\frac{CK^22^{-K}}{(\log n)^2}\sum_{k=1}^{2^K\wedge n}C\le\frac{CK^22^{-K}(2^K\wedge n)}{(\log n)^2}.$$

If $n\ge 2^K$ then this term is bounded by C. The function $x/(\log x)^2$ is increasing whenever $x\ge e^2$. So $n/(\log n)^2\le C2^K/K^2$ in case $n\le 2^K$, which shows the boundedness of the last term if $m\ge n$. If $n\ge m$ then

$$\frac{1}{\log n\log m}\sum_{\substack{2^{-\alpha}\le|k|/|l|\le 2^{\alpha}\\ (|k|,|l|)\le(2^K\wedge n,2^K\wedge m)}} K^2\frac{|\widehat{a}(k,l)|}{|k|\cdot|l|}\le\frac{CK^22^{-K}}{(\log m)^2}\sum_{l=1}^{2^K\wedge m}\sum_{k=l2^{-\alpha}}^{l2^{\alpha}}\frac{1}{k}\le\frac{CK^22^{-K}(2^K\wedge m)}{(\log m)^2}.$$

The boundedness of this expression can be shown as above. ∎

Note that Theorem 5 cannot be proved with the same method for $0<p<1$.

Since the trigonometric polynomials are dense in $H_1(\mathbb{T}^2)$, one can show the following result in the same way as Corollary 1. Corollary 2 was proved by Smith [19] in the one-parameter case.

COROLLARY 2. *If $f\in H_1(\mathbb{T}^2)$ then*

$$\frac{1}{\log n\log m}\sum_{\substack{2^{-\alpha}\le k/l\le 2^{\alpha}\\ (k,l)\le(n,m)}}\frac{\|s_{k,l}f-f\|_1}{kl}\to 0\quad \text{as } n,m\to\infty.$$

Now the analogue of Theorem 2 is proved.

THEOREM 6. *For every distribution $f\in H_p(\mathbb{T}^2)$ $(0<p\le 2)$ we have*

$$\sum_{2^{-\alpha}\le|k|/|l|\le 2^{\alpha}}\frac{|\widehat{f}(k,l)|^p}{|kl|^{2-p}}\le C_p\|f\|^p_{H_p(\mathbb{T}^2)},$$

where C_p depends only on p and α.

Proof. First we show the inequality for $0<p\le 1$. Since the series in (16) converges in the sense of distributions, it is sufficient to show the theorem for p-atoms.

Let a be an arbitrary p-atom with support $R=I\times J$ and $\pi 2^{-K-1}\le|I|=|J|\le\pi 2^{-K}$ $(K\in\mathbb{N})$. We can suppose again that the center of R is zero. Again,

$$\sum_{2^{-\alpha}\le|k|/|l|\le 2^{\alpha}}\frac{|\widehat{a}(k,l)|^p}{|kl|^{2-p}}\le\sum_{\substack{2^{-\alpha}\le|k|/|l|\le 2^{\alpha}\\ (|k|,|l|)\le(2^K,2^K)}}\frac{|\widehat{a}(k,l)|^p}{|kl|^{2-p}}+\sum_{|k|=2^{K-r}}^{\infty}\sum_{|l|=2^{K-r}}^{\infty}\frac{|\widehat{a}(k,l)|^p}{|kl|^{2-p}}.$$

The second term on the right hand side can be majorized by a constant depending only on α in the same way as in Theorem 1. To estimate the first term, observe that

$$|\widehat{a}(k,l)| = \left| \frac{1}{(2\pi)^2} \int_I \int_J a(x,y) e^{-\imath(kx+ly)}\, dx\, dy \right|$$
$$= \left| \frac{1}{(2\pi)^2} \int_I \int_J a(x,y) \left(e^{-\imath(kx+ly)} - \sum_{j=0}^{L} \frac{(-\imath(kx+ly))^j}{j!} \right) dx\, dy \right|$$

by (i′), where $L \le [2(1/p-1)]$. Therefore

$$|\widehat{a}(k,l)| \le C \int_I \int_J |a(x,y)| \frac{|kx+ly|^{L+1}}{(L+1)!}\, dx\, dy,$$

which means that

$$|\widehat{a}(k,l)|^p \le C_p |k|^{p(L+1)} |I|^{p(L+3)-2} + C_p |l|^{p(L+1)} |I|^{p(L+3)-2}. \tag{23}$$

Consequently,

$$\sum_{\substack{2^{-\alpha} \le |k|/|l| \le 2^\alpha \\ (|k|,|l|) \le (2^K, 2^K)}} \frac{|\widehat{a}(k,l)|^p}{|kl|^{2-p}} \le C_p \sum_{\substack{2^{-\alpha} \le k/l \le 2^\alpha \\ (k,l) \le (2^K, 2^K)}} \frac{k^{p(L+1)} |I|^{p(L+3)-2}}{(kl)^{2-p}}$$
$$\le C_p 2^{-K(p(L+3)-2)} \sum_{\substack{2^{-\alpha} \le k/l \le 2^\alpha \\ (k,l) \le (2^K, 2^K)}} \frac{k^{p(L+1)}}{k^{4-2p}}$$
$$\le C_p 2^{-K(p(L+3)-2)} \sum_{k=1}^{2^K} k^{p(L+3)-3} \le C_p$$

whenever $p(L+3)-2>0$, but this is true if we choose $L=[2(1/p-1)]$. The proof of Theorem 6 is finished if $0<p\le 1$. For $1<p<2$ we get the theorem by interpolation (cf. Fefferman, Rivière and Sagher [6]). ■

The $\mathrm{BMO}(\mathbb{T}^2)$ space is defined in this case also by (3), however, the supremum is now taken over all generalized cubes. Fefferman and Stein [7] verified that the dual of $H_1(\mathbb{T}^2)$ is $\mathrm{BMO}(\mathbb{T}^2)$.

The dual inequalities to Theorems 5 and 6 are given without proofs since they can be proved similarly to Theorems 3 and 4.

THEOREM 7. *If $g^{k,l}$ $(2^{-\alpha} \le k/l \le 2^\alpha,\ k,l \in \mathbb{N})$ are uniformly bounded then*

$$\sup_{n,m\ge 2} \Bigg\| \frac{1}{\log n \log m} \sum_{\substack{2^{-\alpha} \le k/l \le 2^\alpha \\ (k,l) \le (n,m)}} \frac{s_{k,l} g^{k,l}}{kl} \Bigg\|_{\mathrm{BMO}(\mathbb{T}^2)} \le C \sup_{2^{-\alpha} \le k/l \le 2^\alpha} \|g^{k,l}\|_\infty.$$

THEOREM 8. *If $|kl| \cdot |a_{k,l}|$ $(2^{-\alpha} \le |k|/|l| \le 2^\alpha,\ k,l \in \mathbb{Z})$ are uniformly bounded real numbers then*

$$\Bigg\| \sum_{2^{-\alpha} \le |k|/|l| \le 2^\alpha} a_{k,l} e^{\imath(kx+ly)} \Bigg\|_{\mathrm{BMO}(\mathbb{T}^2)} \le C \sup_{2^{-\alpha} \le |k|/|l| \le 2^\alpha} |kl| \cdot |a_{k,l}|.$$

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Hilbert space representations of the graded analogue of the Lie algebra of the group of plane motions

by

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Abstract. The irreducible Hilbert space representations of a $*$-algebra, the graded analogue of the Lie algebra of the group of plane motions, are classified up to unitary equivalence.

1. Introduction. In this article we will study representations, by self-adjoint operators in a Hilbert space, of a certain generalized Lie algebra, the graded analogue of the Lie algebra of the group of plane motions.

For the past twenty years, generalized (coloured) Lie algebras have been an object of constant interest in both mathematics and physics (see for example [2–5, 8–10] and references there). When such an algebra is endowed with an involution $*$, we get a $*$-algebra, and it is an important and interesting problem to describe $*$-representations of this $*$-algebra.

It is well known that representations of three-dimensional Lie algebras play an important role in the representation theory of general Lie algebras and groups. Similarly, one would expect the same to be true for three-dimensional coloured Lie algebras with respect to general coloured Lie algebras.

The representations of non-isomorphic algebras have different structure. It is a simple and attractive idea to start by classifying, up to isomorphism, all coloured Lie algebras and then to describe representations of one representative from each isomorphism class. Unfortunately, the classification, up to isomorphism, of all coloured Lie algebras turns out to be a hopelessly difficult task, in the same way as it is already for Lie algebras. Thus, the idea does not work in the general case.

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