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Ergodic theory for the one-dimensional Jacobi operator

by

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Abstract. We determine the number and properties of the invariant measures under the projective flow defined by a family of one-dimensional Jacobi operators. We calculate the derivative of the Floquet coefficient on the absolutely continuous spectrum and deduce the existence of the non-tangential limit of Weyl m-functions in the L^1-topology.

1. Introduction. Let Ω be a compact metric space, T a minimal homeomorphism on Ω and m_0 an ergodic measure for T. Given a real function $V_0 \in C(\Omega)$ we consider the family of Jacobi operators

$$(L_\xi z)(n) = -z(n+1) - z(n-1) + V_0(\xi \cdot n)\, z(n)$$

for $\xi \in \Omega$, where $\xi \cdot n = T^n \xi$. We are mainly interested in the associated spectral problems

$$L_\xi z = Ez \tag{1.1}$$

for $E \in \mathbb{C}$. This family of difference equations, which can be represented in matrix form by

$$\begin{pmatrix} z(n) \\ z(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & V_0(\xi \cdot n) - E \end{pmatrix} \begin{pmatrix} z(n-1) \\ z(n) \end{pmatrix} = M_E(\xi \cdot n) \begin{pmatrix} z(n-1) \\ z(n) \end{pmatrix}, \tag{1.2}$$

defines a flow on the complex bundle $\Omega \times \mathbb{C}^2$, given by $S_E(\xi, \mathbf{z}) = (\xi \cdot 1, M_E(\xi)\mathbf{z})$. Define $U_E(n, \xi) = M_E(\xi \cdot (n-1)) \ldots M_E(\xi)$. Then, if $z_E(n, \xi, \mathbf{z})$ represents the solution of (1.1) corresponding to the initial data $(z_E(-1, \xi, \mathbf{z}), z_E(0, \xi, \mathbf{z}))^t = \mathbf{z}$, one has $S_E^n(\xi, \mathbf{z}) = (\xi \cdot n, (z_E(n-1, \xi, \mathbf{z}), z_E(n, \xi, \mathbf{z}))^t) = (\xi \cdot n, U_E(n, \xi)\, \mathbf{z})$.

The *Lyapunov exponent* of the equation (1.1) corresponding to a value

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of the energy $E \in \mathbb{C}$ is defined as

$$\gamma(E) = \lim_{N\to\infty} \frac{1}{N} \ln(\|U_E(N,\xi)\|);$$

this limit exists and takes the same value almost everywhere on Ω.

The *rotation number* of the family of equations (1.1) corresponding to a real value of E is given by

$$\alpha(E) = \lim_{N\to\infty} \frac{\pi}{2N} \vartheta_E(N,\xi),$$

where $\vartheta_E(N,\xi)$ denotes the number of eigenvalues $\lambda \le E$ for the boundary value problem

$$L_\xi z = Ez \quad \text{in } l^2[-N,N], \quad z(-N-1) = z(N) = 0.$$

This limit is the same for almost every $\xi \in \Omega$ and defines a continuous function on the real line which increases exactly on the spectrum.

If $E \in \mathbb{R}$ the negative Lyapunov exponent $-\gamma(E)$ and the rotation number $\alpha(E)$ agree respectively with the real and imaginary parts of the so-called *Floquet coefficient*, $w(E)$, which is the extension to the real axis of an analytic function with the same name defined on $\mathbb{C} - \mathbb{R}$ in terms of Weyl m-functions. The existence of the latter is guaranteed by the fact that we are dealing with operators which are in the limit-point case. Here we consider the definition of Weyl m-functions based on the concept of *exponential dichotomy* (see Johnson [4]), which provides them with a dynamical meaning. It is proved in [4] that the resolvent set of L_ξ agrees with the set of those values of E for which the family of equations (1.2) admits exponential dichotomy.

Pastur [12] obtains a common spectral decomposition for L_ξ for almost every $\xi \in \Omega$, $\sigma = \sigma_{\text{a.c.}} \cup \sigma_{\text{s.c.}} \cup \sigma_{\text{p.p.}}$. Let $\mathbf{A}$ be the set of those (real) values of the energy with null Lyapunov exponent. Kotani's theory, given in Simon [14] for the Jacobi operator, assures the existence almost everywhere on $\mathbf{A}$ of the non-tangential limit with non-null imaginary part of Weyl m-functions. This finally permits one to identify $\sigma_{\text{a.c.}}$ with the essential closure of $\mathbf{A}$.

Let $P^1(\mathbb{C})$ be the one-dimensional complex projective space. By linearity on the fiber S_E induces a homeomorphism on the complex projective bundle $K_{\mathbb{C}} = \Omega \times P^1(\mathbb{C})$ via the map $\Pi : \Omega \times \mathbb{C}^2 \to K_{\mathbb{C}}$, $(\xi, \mathbf{z}) \mapsto (\xi, Z = z_2/z_1)$. The same symbol S_E will represent the restriction of this homeomorphism to any invariant subset. Writing $Z(n) = z(n)/z(n-1)$ we obtain from (1.1) the family of equations

$$-Z(n+1) - \frac{1}{Z(n)} + V_0(\xi\cdot n) = E \tag{1.3}$$

for $\xi \in \Omega$. Let $Z_E(n,\xi,Z)$ be the solution of (1.3) with initial data $Z_E(0,\xi,Z) = Z$. The homeomorphism S_E on $K_{\mathbb{C}}$ is thus given by $S_E(\xi, Z) = (\xi\cdot 1, Z_E(1,\xi,E))$, and moreover $S_E^n(\xi,Z) = (\xi\cdot n, Z_E(n,\xi,E))$.

Let $M \subset K_{\mathbb{C}}$ be an S_E-invariant subset and $\pi : M \to \Omega$ the projection onto the base. We say that M is a *measurable k-sheet* if there exists an invariant subset $\Omega_0 \subset \Omega$ with $m_0(\Omega_0) = 1$ such that $\operatorname{card} \pi^{-1}(\xi) = k$ for every $\xi \in \Omega_0$ and the multivalued function defined on Ω_0 taking ξ to $\pi^{-1}(\xi) = \{(\xi, Z_1(\xi)), (\xi, Z_2(\xi)), \ldots, (\xi, Z_k(\xi))\}$ is measurable. If such a set exists there is an S_E-invariant measure μ on $K_{\mathbb{C}}$ given by

$$\int_{K_{\mathbb{C}}} g(\xi, Z)\, d\mu = \frac{1}{k} \sum_{j=1}^{k} \int_{\Omega} g(\xi, Z_j(\xi))\, dm_0$$

for every $g \in C(K_{\mathbb{C}})$. One can easily check that μ projects onto m_0 and is concentrated on M. We say that M is an *ergodic k-sheet* if μ is an ergodic measure; for $k = 1$ we simply refer to M as a measurable or ergodic *sheet*.

On the other hand, if $E \in \mathbb{R}$ then $\Omega \times \mathbb{R}^2$ is an S_E-invariant subset of $\Omega \times \mathbb{C}^2$ and $K_{\mathbb{R}} = \Omega \times P^1(\mathbb{R})$ (where $P^1(\mathbb{R})$ is identified with $\mathbb{R}/(\pi\mathbb{Z})$) is an S_E-invariant subset of $K_{\mathbb{C}}$. Representing the projective coordinate of a real solution as $\varphi(n) = \cot^{-1}(x(n)/x(n-1))$ the family of equations (1.1) becomes

$$\varphi(n+1) = \cot^{-1}(-\tan\varphi(n) + V_0(\xi\cdot n) - E). \tag{1.4}$$

We denote by $\varphi_E(n,\xi,\varphi)$ the solution of (1.4) with initial data $\varphi_E(0,\xi,\varphi) = \varphi$; then the restriction of S_E to $K_{\mathbb{R}}$ is given by $S_E(\xi,\varphi) = (\xi\cdot 1, \varphi_E(1,\xi,\varphi))$. Analogously $S_E^n(\xi,\varphi) = (\xi\cdot n, \varphi_E(n,\xi,\varphi))$. The relation $X = \cot\varphi$ gives the change between the two systems of coordinates we have introduced on $K_{\mathbb{R}}$. Let l be the Lebesgue measure on $\mathbb{R}$, l_1 its normalized restriction to $P^1(\mathbb{R})$ and $m_1 = m_0 \otimes l_1$ the complete product measure on $K_{\mathbb{R}}$. Throughout this paper all measures are supposed to be positive and normalized.

A function $q : K_{\mathbb{R}} \to \mathbb{R}$ is said to be *quadratic on the fiber* (or *fiber-quadratic*) if it has the form

$$q(\xi,\varphi) = A(\xi)\cos^2\varphi + B(\xi)\sin^2\varphi + 2\,C(\xi)\sin\varphi\cos\varphi$$

for almost every $(\xi,\varphi) \in K_{\mathbb{R}}$. Let μ be an S_E-invariant measure on $K_{\mathbb{R}}$ absolutely continuous with respect to m_1; we say that μ is a *linear invariant measure* if $d\mu = (1/q)\, dm_1$ and the function q is quadratic on the fiber.

In the first part of this paper we study the ergodic structure of the projective flow and the dynamical consequences of this structure. The interpolation of S_E by a linear continuous flow Φ_E, as given in [4], allows us to transfer the information from the continuous case (see also Delyon–Souillard [2]). We determine the number and properties of the S_E-invariant measures: they turn out to be the restrictions to $K_{\mathbb{R}}$ of invariant measures for the continuous flow. Thus the ergodic classification described in Novo–Obaya [7] is also applicable to the Jacobi case. The number and type of S_E-invariant measures depend directly on the qualitative behavior of the solutions of (1.1). We refer to the *absolutely continuous case* when there exists an invariant measure on $K_{\mathbb{R}}$ absolutely continuous with respect to m_1, and to the *singular case* when every invariant measure for the projective flow is singular with respect to m_1. Both cases present completely different dynamics.

The second part of the paper deals with the behavior of the Floquet coefficient on the absolutely continuous spectrum. Let $\mathbf{A}_2$ be the set of those $E \in \mathbb{R}$ for which $(K_{\mathbb{R}}, S_E)$ admits a linear invariant measure with square integrable density function. Oseledec' ergodic theorem shows that $\mathbf{A}_2 \subset \mathbf{A}$ (see [11]), and we deduce from Kotani's theory that $l(\mathbf{A}-\mathbf{A}_2)=0$. We obtain the derivative of the Floquet coefficient on $\mathbf{A}_2$ and answer a question on the variation of the rotation number posed in Deift–Simon [1]. From the previous ergodic conclusions we calculate the non-tangential limit of Weyl m-functions in the L^1-topology. In particular, the case where the projective bundle decomposes into a collection of identical ergodic sheets is of interest from the spectral and qualitative points of view. These results are an extension of those obtained in Núñez–Obaya [8] for the second order Schrödinger equation.

2. The ergodic measures. In this section we recall the well-known suspension construction and its most basic properties. We also give the explicit relation between the solutions of (1.3) and the trajectories of the suspension and prove that the dynamics is of the same type for both cases. This fact will be fundamental in the next sections since it allows us to transfer the study to a continuous flow.

Let X be a locally compact Hausdorff space and $S: X \to X$ a homeomorphism. We define on $X \times \mathbb{R}$ an equivalence relation by

$$(x,s) \sim (x',s') \;\Leftrightarrow\; s-s'=n \in \mathbb{Z} \text{ and } x'=S^n x.$$

The quotient set $\widehat{X}$ is also a locally compact Hausdorff space; in fact, it is compact if X is. Each equivalence class, denoted by $x\cdot s$, admits a unique representative (x,s) with $s\in[0,1)$. This is the one we will choose whenever fixing a representative is needed. We consider the real flow $\widehat{S}$ on $\widehat{X}$ given by $\widehat{S}(t, x\cdot s) = x\cdot(s+t)$ for $x\cdot s \in \widehat{X}$ and $t\in\mathbb{R}$. This flow $(\widehat{X},\widehat{S})$ is called the *suspension* of (X,S).

If the trajectory $(S^n x)_{n\in\mathbb{Z}}$ of an element $x\in X$ is dense in X, then $(x\cdot(s+t))_{t\in\mathbb{R}}$ is dense in $\widehat{X}$ for every $s\in\mathbb{R}$. In particular, if the homeomorphism S is minimal so is the suspension.

For each $s\in\mathbb{R}$, consider the map $i_s: X\to\widehat{X}$, $x\mapsto x\cdot s$. We can identify X with $i_0(X)\subset\widehat{X}$; thus the restriction of $\widehat{S}$ to $\mathbb{Z}$ and X agrees with the successive iteration of the initial homeomorphism: $\widehat{S}(n,x) = x\cdot n = S^n x$.

Given a measure μ on X we define, for $g\in C(\widehat{X})$,

$$\int_{\widehat{X}} g(x\cdot s)\,d\widehat{\mu} = \int_0^1 \int_X g\circ i_s(x)\,d\mu\,ds. \tag{2.1}$$

It is immediate to check that $\widehat{\mu}$ is a measure and that it is invariant whenever μ is; in fact, if μ is ergodic so is $\widehat{\mu}$. Notice that if $\widehat{X}_0\subset\widehat{X}$ is an $\widehat{S}$-invariant subset such that $\widehat{\mu}(\widehat{X}_0)=1$ then $\mu(\{x \mid x\cdot s\in\widehat{X}_0\})=1$ for every $s\in[0,1]$.

This process allows us to construct $(\widehat{\Omega},\widehat{T})$ and $(\widehat{\Omega\times\mathbb{C}^2},\widehat{S}_E)$, the suspensions of (Ω,T) and $(\Omega\times\mathbb{C}^2,S_E)$ respectively. The space $\widehat{\Omega\times\mathbb{C}^2}$ turns out to be a trivial bundle with base $\widehat{\Omega}$ and its trivialization permits transforming $\widehat{S}_E$ to a new real flow of *skew-product* type. Let $F_E:\Omega\times[0,1]\to \mathrm{GL}(2,\mathbb{C})$ be the homotopy between $\xi\mapsto \mathrm{Id}$ and $\xi\mapsto \left(\begin{smallmatrix}0 & 1\\ -1 & V_0(\xi)-E\end{smallmatrix}\right)$ given by

$$F_E(\xi,s) = \begin{pmatrix}\cos\sigma(s) & \sin\sigma(s)\\ -\sin\sigma(s) & \cos\sigma(s)+\tau(s)(V_0(\xi)-E)\end{pmatrix},$$

where σ and τ are suitable C^∞-functions:

$$\begin{aligned}&\sigma:[0,1]\to[0,\pi/2], \quad \sigma|_{[0,\delta]}=0, \qquad \sigma|_{[1/2-\delta,1]}=\pi/2,\\ &\tau:[0,1]\to[0,1], \qquad \tau|_{[0,1/2+\delta]}=0, \quad \tau|_{[1-\delta,1]}=1,\end{aligned}$$

for some $\delta\in(0,1/4)$. This permits periodic C^∞-extensions of σ' and τ' to the whole real line. The map

$$\zeta_E:\widehat{\Omega\times\mathbb{C}^2}\to\widehat{\Omega}\times\mathbb{C}^2, \qquad (\xi,\mathbf{z})\cdot s\mapsto(\xi\cdot s, F_E(\xi,s)\,\mathbf{z}),$$

$(s\in[0,1))$ is a homeomorphism. We denote by Φ_E the continuous flow obtained from $\widehat{S}_E$ via ζ_E. If $t\in\mathbb{R}$ and $s\in[0,1)$ then

$$\Phi_E(t,\xi\cdot s,\mathbf{z}) = (\xi\cdot(s+t), \widehat{U}_E(t,\xi\cdot s)\,\mathbf{z}),$$

where

$$\widehat{U}_E(t,\xi\cdot s) = F_E(\xi\cdot n, l)\,U_E(n,\xi)\,F_E(\xi,s)^{-1} \tag{2.2}$$

with $n=[s+t]$ and $l=s+t-n$. (As usual, we denote by $[\cdot]$ the integer part of a real number.) The restriction to $\Omega\times\mathbb{C}^2\subset\widehat{\Omega}\times\mathbb{C}^2$ and $\mathbb{Z}$ gives

$$\begin{aligned}\Phi_E(n,\xi,\mathbf{z}) &= (\xi\cdot n, F_E(\xi\cdot n,0)\,U_E(n,\xi)\,F_E(\xi,0)^{-1}\mathbf{z})\\ &= (\xi\cdot n, U_E(n,\xi)\,\mathbf{z}) = S_E^n(\xi,\mathbf{z}),\end{aligned}$$

that is, the homeomorphism S_E is recovered.

PROPOSITION 2.1. *The flow Φ_E is given on $\widehat{\Omega}\times\mathbb{C}^2$ by the family of two-dimensional linear systems*

$$\mathbf{z}' = D_E(\xi\cdot(s+t))\,\mathbf{z}, \tag{2.3}$$

where

$$D_E(\xi\cdot s)=\begin{pmatrix}0 & \sigma'(s)\\ -\sigma'(s)+\tau'(s)\,(V_0(\xi\cdot[s])-E) & 0\end{pmatrix}$$

for $\xi\in\Omega$ *and* $s\in\mathbb{R}$.

Proof. Let $s\in(0,1)$ and take ε such that $s+\varepsilon\in(0,1)$; then

$$\widehat{U}_E(\varepsilon,\xi\cdot s)=F_E(\xi,s+\varepsilon)\,U_E(0,\xi)\,F_E(\xi,s)^{-1}=F_E(\xi,s+\varepsilon)\,F_E(\xi,s)^{-1}.$$

Hence

$$D_E(\xi\cdot s)=\frac{d}{dt}\,\widehat{U}_E(t,\xi\cdot s)\bigg|_{t=0}=\left(\frac{d}{ds}\,F_E(\xi,s)\right)F_E(\xi,s)^{-1}.$$

It is immediate to check that this is the matrix that appears in the statement. For $s=0$ the proof is analogous. ■

Let us denote by $\widehat{\mathbf{z}}_E(t,\xi\cdot s,\mathbf{z})$ the solution of (2.3) with initial data $\widehat{\mathbf{z}}_E(0,\xi\cdot s,\mathbf{z})=\mathbf{z}$; then $\Phi_E(t,\xi\cdot s,\mathbf{z})=(\xi\cdot(s+t),\widehat{\mathbf{z}}_E(t,\xi\cdot s,\mathbf{z}))$. By construction of the suspension we also have $\widehat{\mathbf{z}}_E(n,\xi,\mathbf{z})=(z_E(n-1,\xi,\mathbf{z}),z_E(n,\xi,\mathbf{z}))^t$. Hence one deduces that the Lyapunov exponent of the family of systems (2.3) coincides with the one of the original discrete equation. The analogous result for the rotation number is proved in [4].

Relation (2.2) allows us to express the value of $\widehat{\mathbf{z}}_E(n+t,\xi,\mathbf{z})$ in terms of $z_E(n-1,\xi,\mathbf{z})$ and $z_E(n,\xi,\mathbf{z})$: for $t\in[0,1)$,

$$\begin{aligned}\widehat{z}_{E,1}(n+t,\xi,\mathbf{z})&=\cos\sigma(t)\,z_E(n-1,\xi,\mathbf{z})+\sin\sigma(t)\,z_E(n,\xi,\mathbf{z}),\\ \widehat{z}_{E,2}(n+t,\xi,\mathbf{z})&=-\sin\sigma(t)\,z_E(n-1,\xi,\mathbf{z})\\ &\qquad+[\cos\sigma(t)+\tau(t)(V_0(\xi\cdot n)-E)]\,z_E(n,\xi,\mathbf{z}).\end{aligned}\tag{2.4}$$

As in the discrete case, by linearity on the fiber we can define a new flow on $\widehat{K}_{\mathbb{C}}=\widehat{\Omega}\times P^1(\mathbb{C})$ via the projection $\Pi:\widehat{\Omega}\times\mathbb{C}^2\to\widehat{K}_{\mathbb{C}}$, $(\xi\cdot s,\mathbf{z})\mapsto(\xi\cdot s,Z=z_2/z_1)$. The trajectories of this flow satisfy the Riccati equations

$$Z'=-\sigma'(s+t)(1+Z^2)+\tau'(s+t)(V_0(\xi\cdot[s+t])-E).\tag{2.5}$$

Let $\widehat{Z}_E(t,\xi\cdot s,Z)$ be the solution of (2.5) with initial data $\widehat{Z}_E(0,\xi\cdot s,Z)=Z$; the flow on $\widehat{K}_{\mathbb{C}}$ is given by $\Phi_E(t,\xi\cdot s,Z)=(\xi\cdot(s+t),\widehat{Z}_E(t,\xi\cdot s,Z))$ and by means of (2.4) we can establish that, for $t\in[0,1)$,

$$\begin{aligned}&\widehat{Z}_E(n+t,\xi,Z)\\ &\quad=\frac{-\sin\sigma(t)+[\cos\sigma(t)+\tau(t)(V_0(\xi\cdot n)-E)]\,Z_E(n,\xi,Z)}{\cos\sigma(t)+\sin\sigma(t)\,Z_E(n,\xi,Z)}.\end{aligned}\tag{2.6}$$

Analogously, for $E\in\mathbb{R}$, Φ_E defines a flow on $\widehat{K}_{\mathbb{R}}=\widehat{\Omega}\times P^1(\mathbb{R})$. If $\varphi=\cot^{-1}(x_2/x_1)$ this flow corresponds to the equation

$$\varphi'=\sigma'(s+t)-\tau'(s+t)(V_0(\xi\cdot[s+t])-E)\sin^2\varphi=f_E(\xi\cdot(s+t),\varphi).\tag{2.7}$$

If $\widehat{\varphi}_E(t,\xi\cdot s,\varphi)$ is the solution with the initial data $\widehat{\varphi}_E(0,\xi\cdot s,\varphi)=\varphi$, then $\Phi_E(t,\xi\cdot s,\varphi)=(\xi\cdot(s+t),\widehat{\varphi}_E(t,\xi\cdot s,\varphi))$. Moreover, for $t\in[0,1)$,

$$\begin{aligned}&\widehat{\varphi}_E(n+t,\xi\cdot s,\varphi)\\ &\quad=\cot^{-1}[\cot(\varphi_E(n,\xi,\varphi)+\sigma(t))+\tau(t)(V_0(\xi\cdot n)-E)].\end{aligned}\tag{2.8}$$

The following result characterizes the density function of a measure on $K_{\mathbb{R}}$ absolutely continuous with respect to m_1 and S_E-invariant. The proof is a simple verification and we omit the details.

PROPOSITION 2.2. *Let $p\in L^1(K_{\mathbb{R}},m_1)$ be a positive function. The following assertions are equivalent*:

(i) *the measure μ given on $K_{\mathbb{R}}$ by $d\mu=p\,dm_1$ is S_E-invariant*;

(ii) *p satisfies the functional equation*

$$p(S_E(\xi,\varphi))=p(\xi,\varphi)[\cos^2\varphi+(-\sin\varphi+(V_0(\xi)-E)\cos\varphi)^2].\tag{2.9}$$

If μ is a linear invariant measure with $d\mu=(1/q)\,dm_1$ then q is a fiber-quadratic solution of

$$q(S_E(\xi,\varphi))=q(\xi,\varphi)[\cos^2\varphi+(-\sin\varphi+(V_0(\xi)-E)\cos\varphi)^2]^{-1}.\tag{2.10}$$

Let now $\widehat{m}_0$ be the ergodic measure induced on $\widehat{\Omega}$ by m_0 via (2.1) and $\widehat{m}_1=\widehat{m}_0\otimes l_1$ the complete product measure on $\widehat{K}_{\mathbb{R}}$. The existence of measures on the real projective bundle which are invariant under the flow corresponding to a family of two-dimensional systems and absolutely continuous with respect to the product measure is studied in Obaya–Paramio [10]. It is clear that the results appearing there can be applied to the suspension. The density functions of such measures correspond to the solutions of the functional equation

$$\widehat{p}(\Phi_E(t,\xi\cdot s,\varphi))=\widehat{p}(\xi\cdot s,\varphi)\exp\left\{-\int_0^t\frac{\partial f_E}{\partial\varphi}(\Phi_E(u,\xi\cdot s,\varphi))\,du\right\}.\tag{2.11}$$

Let $\widehat{\mu}$ be a linear invariant measure for $(\widehat{K}_{\mathbb{R}},\Phi_E)$ with $d\widehat{\mu}=(1/\widehat{q})\,dm_1$; then $\widehat{q}$ satisfies the equation

$$\widehat{q}(\Phi_E(t,\xi\cdot s,\varphi))=\widehat{q}(\xi\cdot s,\varphi)\exp\left\{\int_0^t\frac{\partial f_E}{\partial\varphi}(\Phi_E(u,\xi\cdot s,\varphi))\,du\right\}.\tag{2.12}$$

The next results show the connection between the absolutely continuous invariant measures for $(K_{\mathbb{R}},S_E)$ and $(\widehat{K}_{\mathbb{R}},\Phi_E)$; in particular, their density functions are within the same class of integrability.

THEOREM 2.3. *Let $\widehat{p}:\widehat{K}_{\mathbb{R}}\to\mathbb{R}$ and $p:K_{\mathbb{R}}\to\mathbb{R}$ be two measurable functions such that for every $(\xi,\varphi)\in K_{\mathbb{R}}$ and every $s\in[0,1)$,*

$$\widehat{p}(\Phi_E(s,\xi,\varphi))=p(\xi,\varphi)\exp\Big\{-\int_0^s\frac{\partial f_E}{\partial\varphi}(\Phi_E(u,\xi,\varphi))\,du\Big\}. \tag{2.13}$$

Then $\widehat{p}$ is a solution of the functional equation (2.11) if and only if p is a solution of (2.9). Moreover, if $r\ge 1$ then $\widehat{p}\in L^r(\widehat{K}_{\mathbb{R}},\widehat{m}_1)$ if and only if $p\in L^r(K_{\mathbb{R}},m_1)$.

Proof. From (2.8) we deduce that, for $u\in[0,1)$,

$$\begin{aligned}-\frac{\partial f_E}{\partial\varphi}&(\Phi_E(u,\xi,\varphi))\\ &=\tau'(u)(V_0(\xi)-E)\sin 2\widehat{\varphi}_E(u,\xi,\varphi)\\ &=\tau'(u)(V_0(\xi)-E)\frac{2\,[\cot(\varphi+\sigma(u))+\tau(u)(V_0(\xi)-E)]}{1+[\cot(\varphi+\sigma(u))+\tau(u)(V_0(\xi)-E)]^2}\\ &=\tau'(u)(V_0(\xi)-E)\frac{2\,[-\tan\varphi+\tau(u)(V_0(\xi)-E)]}{1+[-\tan\varphi+\tau(u)(V_0(\xi)-E)]^2},\end{aligned}$$

since $\tau'(u)\ne 0$ implies $\sigma(u)=\pi/2$, and $\cot(\varphi+\pi/2)=-\tan\varphi$. Hence

$$\begin{aligned}\exp\Big\{-\int_0^s\frac{\partial f_E}{\partial\varphi}(\Phi_E(u,\xi,\varphi))\,du\Big\}&\\ &=\exp\{\ln(1+[-\tan\varphi+\tau(u)(V_0(\xi)-E)]^2)|_{u=0}^s\}\\ &=\cos^2\varphi+(-\sin\varphi+\tau(s)(V_0(\xi)-E)\cos\varphi)^2.\end{aligned}\tag{2.14}$$

In particular, for $s=1$ we obtain

$$\begin{aligned}\exp\Big\{-\int_0^1\frac{\partial f_E}{\partial\varphi}(\Phi_E(u,\xi,\varphi))\,du\Big\}&\\ &=\cos^2\varphi+(-\sin\varphi+(V_0(\xi)-E)\cos\varphi)^2.\end{aligned}\tag{2.15}$$

Now suppose that $\widehat{p}$ is a solution of (2.11). Since $p(\xi,\varphi)=\widehat{p}(\xi,\varphi)$ one deduces from the above equality that p is a solution of (2.9). Conversely, let p satisfy (2.9). In order to prove that $\widehat{p}$ solves (2.11) in the whole $\widehat{K}_{\mathbb{R}}$ it suffices to show that it solves it for a point of each trajectory and for every $t\in\mathbb{R}$. Take (ξ,φ) and $t=n+s$ with $n\in\mathbb{Z}$ and $s\in[0,1)$. Then

$$\begin{aligned}\widehat{p}(\Phi_E(t,\xi,\varphi))&=\widehat{p}(\Phi_E(s,\Phi_E(n,\xi,\varphi)))\\ &=p(S_E^n(\xi,\varphi))\exp\Big\{-\int_0^s\frac{\partial f_E}{\partial\varphi}(\Phi_E(n+u,\xi,\varphi))\,du\Big\}.\end{aligned}$$

From (2.15) it is easily deduced that for every $n\in\mathbb{Z}$,

$$p(S_E^n(\xi,\varphi))=p(\xi,\varphi)\exp\Big\{-\int_0^n\frac{\partial f_E}{\partial\varphi}(\Phi_E(u,\xi,\varphi))\,du\Big\}.$$

The fact that $\widehat{p}$ is a solution of (2.11) is a consequence of the last two equalities.

Now we prove the last assertion of the theorem. By (2.13),

$$\begin{aligned}\int_{K_{\mathbb{R}}}&p^r(\xi,\varphi)\,dm_1\\ &=\int_{P^1(\mathbb{R})}\int_0^1\int_{\Omega}p^r(\xi,\varphi)\,dm_0\,ds\,dl_1\\ &=\int_{\widehat{\Omega}}\int_{P^1(\mathbb{R})}\widehat{p}^r(\xi\cdot s,\widehat{\varphi}_E(s,\xi,\varphi))\Big[\exp\Big\{\int_0^s\frac{\partial f_E}{\partial\varphi}(\Phi_E(u,\xi,\varphi))\,du\Big\}\Big]^r dl_1\,d\widehat{m}_0\\ &=\int_{\widehat{K}_{\mathbb{R}}}\widehat{p}^r(\xi\cdot s,\varphi)\Big[\exp\Big\{\int_0^s\frac{\partial f_E}{\partial\varphi}(\Phi_E(u,\xi,\widehat{\varphi}_E(-s,\xi\cdot s,\varphi)))\,du\Big\}\Big]^{r-1}d\widehat{m}_1.\end{aligned}$$

The function $|\partial f_E/\partial\varphi|$ is continuous on $\widehat{K}_{\mathbb{R}}$ and hence upper bounded by a constant M. This gives

$$\frac{1}{N}\int_{\widehat{K}_{\mathbb{R}}}\widehat{p}^r(\xi\cdot s,\varphi)\,d\widehat{m}_1\le\int_{K_{\mathbb{R}}}p^r(\xi,\varphi)\,dm_1\le N\int_{\widehat{K}_{\mathbb{R}}}\widehat{p}^r(\xi\cdot s,\varphi)\,d\widehat{m}_1,$$

where $N=\exp(M(r-1))$. Obviously this completes the proof of the statement. ∎

THEOREM 2.4. *Let $\widehat{q}:\widehat{K}_{\mathbb{R}}\to\mathbb{R}$ and $q:K_{\mathbb{R}}\to\mathbb{R}$ be two measurable functions such that for every $(\xi,\varphi)\in K_{\mathbb{R}}$ and every $s\in[0,1)$,*

$$\widehat{q}(\Phi_E(s,\xi,\varphi))=q(\xi,\varphi)\exp\Big\{\int_0^s\frac{\partial f_E}{\partial\varphi}(\Phi_E(u,\xi,\varphi))\,du\Big\}. \tag{2.16}$$

Then $\widehat{q}$ is a fiber-quadratic solution of the functional equation (2.12) if and only if q is a fiber-quadratic solution of (2.10). Moreover, if $r\ge 0$ then $\widehat{q}\in L^r(\widehat{K}_{\mathbb{R}},\widehat{m}_1)$ if and only if $q\in L^r(K_{\mathbb{R}},m_1)$.

Proof. The facts that $\widehat{q}$ satisfies (2.12) if and only if q satisfies (2.10) and $\widehat{q}\in L^r(\widehat{K}_{\mathbb{R}},\widehat{m}_1)$ if and only if $q\in L^r(K_{\mathbb{R}},m_1)$ can be proved in the same way as in the previous theorem. Moreover, since $q(\xi,\varphi)=\widehat{q}(\xi,\varphi)$, it is obvious that if $\widehat{q}$ is quadratic on the fiber so is q. Now suppose that

$$q(\xi,\varphi)=a(\xi)\cos^2\varphi+b(\xi)\sin^2\varphi+2c(\xi)\sin\varphi\cos\varphi$$

for some measurable functions $a, b, c : \Omega \to \mathbb{R}$. Using (2.14) in (2.16) we obtain

$$\widehat{q}(\xi\cdot s,\varphi)=\widehat{a}(\xi\cdot s)\cos^2\varphi+\widehat{b}(\xi\cdot s)\sin^2\varphi+2\widehat{c}(\xi\cdot s)\sin\varphi\cos\varphi,$$

where

$$\begin{aligned}\widehat{a}(\xi\cdot s)&=a(\xi)\cos^2\sigma(s)+b(\xi)\sin^2\sigma(s)-2c(\xi)\sin\sigma(s)\cos\sigma(s),\\ \widehat{b}(\xi\cdot s)&=a(\xi)\sin^2\sigma(s)+b(\xi)(\cos^2\sigma(s)+\tau^2(s)(V_0(\xi)-E)^2)\\ &\qquad+2c(\xi)(\sin\sigma(s)\cos\sigma(s)+\tau(s)(V_0(\xi)-E)),\\ \widehat{c}(\xi\cdot s)&=a(\xi)\sin\sigma(s)\cos\sigma(s)-b(\xi)(\sin\sigma(s)\cos\sigma(s)\\ &\qquad+\tau(s)(V_0(\xi)-E))+c(\xi)(\cos^2\sigma(s)-\sin^2\sigma(s)),\end{aligned}\tag{2.17}$$

which completes the proof of the theorem. Notice that $\widehat{a}(\xi\cdot s)\widehat{b}(\xi\cdot s)-\widehat{c}^2(\xi\cdot s)=a(\xi)\,b(\xi)-c^2(\xi)$ for every $\xi\in\Omega$ and $s\in[0,1]$. By ergodicity on the base it is easy to deduce the existence of an S_E-invariant subset $\Omega_0\subset\Omega$ with $m_0(\Omega_0)=1$ and a constant η such that $\widehat{a}(\xi\cdot s)\,\widehat{b}(\xi\cdot s)-\widehat{c}^2(\xi\cdot s)=\eta$ for every $\xi\in\Omega_0$ and $s\in\mathbb{R}$. ∎

Now it is clear that absolutely continuous or singular dynamics occurs simultaneously for both cases. In consequence, the ergodic classification given in [7] is also valid for the Jacobi case. That is, if an absolutely continuous invariant measure exists then this measure is either ergodic, being necessarily a linear measure and the only invariant measure that projects onto m_0, or non-ergodic, in which case every ergodic measure which projects onto m_0 is concentrated on an ergodic k-sheet. (Notice that an ergodic k-sheet on $(K_{\mathbb{R}}, S_E)$ extends by means of (2.6) to an ergodic k-sheet on $(\widehat{K}_{\mathbb{R}}, \Phi_E)$.) And in the singular case three different options are possible: $(K_{\mathbb{R}}, S_E)$ admits either two different ergodic measures, each of them concentrated on an ergodic sheet, or a unique ergodic measure, concentrated on an ergodic sheet or a 2-sheet.

3. Differentiability of the Floquet coefficient. Consider the family of equations (1.1) and define $\mathbf{A}_2$ as the set of those values of E for which the real projective flow admits a linear invariant measure with square integrable density function. The same kind of argument as used in [10] for the continuous Schrödinger equation allows us to prove the following result:

THEOREM 3.1. *The set* $\mathbf{A}_2$ *is contained in* $\mathbf{A}$ *and* $l(\mathbf{A}-\mathbf{A}_2)=0$.

Throughout this section we will work with a fixed $E_0\in\mathbf{A}_2$, suppose $E_0=0$ and suppress the subscript to simplify the notation. Let μ be a linear invariant measure for the homeomorphism S on $K_{\mathbb{R}}$. Then $d\mu=(1/q)\,dm_1$ with $1/q\in L^2(K_{\mathbb{R}},m_1)$ and

$$q(\xi,\varphi)=A(\xi)\cos^2\varphi+B(\xi)\sin^2\varphi+2C(\xi)\sin\varphi\cos\varphi.$$

Let $\widehat{A}, \widehat{B}$ and $\widehat{C}$ be the coefficients of the fiber-quadratic function $\widehat{q}$ defined from q according to (2.16). Theorem 2.4 assures that $d\widehat{\mu}=(1/\widehat{q})\,d\widehat{m}_1$ is a linear invariant measure for Φ on $\widehat{K}_{\mathbb{R}}$; moreover, there exists an S-invariant subset $\Omega_0\subset\Omega$ with $m_0(\Omega_0)=1$ such that $\widehat{A}(\xi\cdot s)\,\widehat{B}(\xi\cdot s)-\widehat{C}^2(\xi\cdot s)=A(\xi)B(\xi)-C^2(\xi)=1$ for every $\xi\in\Omega_0$ and $s\in\mathbb{R}$.

This section is concerned with the Floquet coefficient. In particular, we will study its directional differentiability at V_0. To this end, we consider the families of equations

$$-z(n+1)-z(n-1)+V_0(\xi\cdot n)z(n)=EV(\xi\cdot n)z(n),\tag{3.1}$$

for $V\in C(\Omega)$, and the corresponding suspensions, given by

$$\mathbf{z}'=\begin{pmatrix}0&\sigma'(s+t)\\ -\sigma'(s+t)+\tau'(s+t)(V_0(\xi\cdot[s+t])-EV(\xi\cdot[s+t]))&0\end{pmatrix}\mathbf{z}.\tag{3.2}$$

Let S_E be the homeomorphism that (3.1) induces on $K_{\mathbb{C}}$ (resp. $K_{\mathbb{R}}$) and Φ_E the continuous flow defined by (3.2) on $\widehat{K}_{\mathbb{C}}$ (resp. $\widehat{K}_{\mathbb{R}}$). We denote by $w_V(E)=-\gamma_V(E)+i\alpha_V(E)$ the Floquet coefficient of (3.1) and (3.2) for $E\in\mathbb{C}$.

We now consider the measurable Perron transformation for $\widehat{m}_0$:

$$\widehat{P}:\widehat{\Omega}\to\mathrm{GL}(2,\mathbb{C}),\quad \xi\cdot s\mapsto\begin{pmatrix}\dfrac{1}{\sqrt{\widehat{A}(\xi\cdot s)}}&0\\ \dfrac{\widehat{C}(\xi\cdot s)}{\sqrt{\widehat{A}(\xi\cdot s)}}&\sqrt{\widehat{A}(\xi\cdot s)}\end{pmatrix}.$$

The linear change of variables

$$\widehat{H}:\widehat{\Omega}\times\mathbb{C}^2\to\widehat{\Omega}\times\mathbb{C}^2,\quad(\xi\cdot s,\mathbf{z})\mapsto(\xi\cdot s,\mathbf{w}),\tag{3.3}$$

with $\mathbf{w}=\widehat{P}(\xi\cdot(s+t))\,\mathbf{z}$ takes the two-dimensional systems (3.2) for $\xi\cdot s\in\widehat{\Omega}_0$ to

$$\mathbf{w}'=\begin{pmatrix}0&\dfrac{\sigma'(s+t)}{\widehat{A}(\xi\cdot(s+t))}\\ -\dfrac{\sigma'(s+t)}{\widehat{A}(\xi\cdot(s+t))}-E\,\tau'(s+t)\,V(\xi\cdot[s+t])\widehat{A}(\xi\cdot(s+t))&0\end{pmatrix}\mathbf{w}.\tag{3.4}$$

It is known that $\widehat{H}$ preserves the rotation number and the Lyapunov exponent (see [10] and [8]) and induces a fractional transformation on the projective bundle that takes $(\widehat{K}_{\mathbb{C}},\Phi_E)$ (resp. $(\widehat{K}_{\mathbb{R}},\Phi_E)$) to a new flow $(\widehat{K}_{\mathbb{C}},\Psi_E)$ (resp. $(\widehat{K}_{\mathbb{R}},\Psi_E)$). Moreover, the invariant measures are taken to invariant measures for the new flows. These facts allow us to make the study of w at V_0 by means of the equations and properties of the transformed flows.

PROPOSITION 3.2. *The rotation number at $E=0$ is given by*

$$\alpha(0)=\int_{\Omega}\cot^{-1}(-C(\xi))\,dm_0.$$

Proof. Let $\mathbf{w}_E(t)$ be a solution of the system (3.4) and $\psi_E(t)=\cot^{-1}(w_{E,2}(t)/w_{E,1}(t))$; then

$$(3.5)\quad \psi_E'=\frac{\sigma'(s+t)}{\widehat{A}(\xi\cdot(s+t))}+E\tau'(s+t)\,V(\xi\cdot[s+t])\widehat{A}(\xi\cdot(s+t))\frac{1-\cos 2\psi_E}{2},$$

whence one derives that

$$\alpha(0)=\int_{\widehat{\Omega}}\frac{\sigma'(s)}{\widehat{A}(\xi\cdot s)}\,d\widehat{m}_0=\int_{\Omega}\int_0^1\frac{\sigma'(s)}{\widehat{A}(\xi\cdot s)}\,ds\,dm_0.$$

According to (2.17),

$$\alpha(0)=\int_{\Omega}\int_0^{1/2}\frac{\sigma'(s)}{A(\xi)\cos^2\sigma(s)+B(\xi)\sin^2\sigma(s)-2C(\xi)\sin\sigma(s)\cos\sigma(s)}\,ds\,dm_0$$
$$=\int_{\Omega}\cot^{-1}(-C(\xi))\,dm_0,$$

which proves the result. ■

We denote by $x(n,\xi,\varphi)$ the solution of the equation (3.1) for $E=0$ with initial data $x(0,\xi,\varphi)+ix(-1,\xi,\varphi)=\exp i\varphi$ and by $\widehat{\mathbf{x}}(t,\xi\cdot s,\varphi)$ the solution of the equation (3.2) for $E=0$ with initial data $\widehat{x}_2(0,\xi\cdot s,\varphi)+i\widehat{x}_1(0,\xi\cdot s,\varphi)=\exp i\varphi$. Given a function $V\in C(\Omega)$ we define

$$q_V(\xi,\varphi)=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}V(\xi\cdot n)\,x^2(n,\xi,\varphi).$$

The Birkhoff ergodic theorem shows that this limit exists for almost every $(\xi,\varphi)\in K_{\mathbb{R}}$ and the function q_V is a solution of the functional equation (2.10) (see [10]). Moreover, by linearity on the fiber we can find an invariant subset $\Omega_0\subset\Omega$ with $m_0(\Omega_0)=1$ and measurable functions $a_V,b_V,c_V:\Omega\to\mathbb{R}$ such that

$$q_V(\xi,\varphi)=a_V(\xi)\cos^2\varphi+b_V(\xi)\sin^2\varphi+2c_V(\xi)\sin\varphi\cos\varphi$$

for every $(\xi,\varphi)\in\Omega_0\times P^1(\mathbb{R})$. In particular, we denote by $q_{(1)}$ the limit associated with $V=1$. Now it is easy to check:

PROPOSITION 3.3. *Let $V\in C(\Omega)$ and*

$$\widehat{q}_V(\xi\cdot s,\varphi)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tau'(s+t)\,V(\xi\cdot[s+t])\,\widehat{x}_1^2(t,\xi\cdot s,\varphi)\,dt.$$

Then the functions q_V and $\widehat{q}_V$ satisfy

$$\widehat{q}_V(\Phi(s,\xi,\varphi))=q_V(\xi,\varphi)\exp\Big\{\int_0^s\frac{\partial f}{\partial\varphi}(\Phi(u,\xi,\varphi))\,du\Big\}.$$

Therefore Theorem 2.4 assures that $\widehat{q}_V$ is a fiber-quadratic solution of (2.12) whose coefficients $\widehat{a}_V,\widehat{b}_V$ and $\widehat{c}_V$ are defined by (2.17) in terms of a_V,b_V and c_V. Moreover, there exist an S-invariant subset $\Omega_0\subset\Omega$ with $m_0(\Omega_0)=1$ and a constant η_V such that $\widehat{a}_V(\xi\cdot s)\,\widehat{b}_V(\xi\cdot s)-\widehat{c}_V^2(\xi\cdot s)=a_V(\xi)b_V(\xi)-c_V^2(\xi)=\eta_V$ for every $\xi\in\Omega_0$ and $s\in\mathbb{R}$.

DEFINITION 3.4. Let $V\in C(\Omega)$. We say that V *belongs to the set* $\mathcal{C}_1$ if $q_V=0$ (almost everywhere) or q_V preserves sign on $K_{\mathbb{R}}$, that is, $\eta_V>0$.

From the existence of a linear invariant measure it can be deduced that $1\in\mathcal{C}_1$; in fact, since Ω is a compact set, any function $V>0$ belongs to $\mathcal{C}_1$ and $q_V>0$.

If $\eta_V>0$ we set $\lambda_V=\operatorname{sign}(q_V)\sqrt{\eta_V}$, $A_V=a_V/\lambda_V$, $B_V=b_V/\lambda_V$ and $C_V=c_V/\lambda_V$; then $A_V(\xi)\,B_V(\xi)-C_V^2(\xi)=1$ for almost every $\xi\in\Omega$. Let $p_V=\lambda_V/q_V$; thus μ_V, given by $d\mu_V=p_V\,dm_1$, is a linear invariant measure with square integrable density function that we call *associated with* V. Clearly the coefficients of the linear invariant measure $\widehat{\mu}_V$ induced by μ_V according to Theorem 2.4 are $\widehat{A}_V=\widehat{a}_V/\lambda_V$, $\widehat{B}_V=\widehat{b}_V/\lambda_V$ and $\widehat{C}_V=\widehat{c}_V/\lambda_V$; they are also related to A_V,B_V and C_V by (2.17).

DEFINITION 3.5. Let $V\in\mathcal{C}_1$ with $\eta_V>0$. We define the *m_V-functions* by

$$m_V^{\pm}:\Omega\to P^1(\mathbb{C}),\qquad \xi\mapsto\frac{-C_V(\xi)\pm i}{A_V(\xi)},$$

and the *$\widehat{m}_V$-functions* by

$$\widehat{m}_V^{\pm}:\widehat{\Omega}\to P^1(\mathbb{C}),\qquad \xi\cdot s\mapsto\frac{-\widehat{C}_V(\xi\cdot s)\pm i}{\widehat{A}_V(\xi\cdot s)}.$$

Notice that the equation $q_V=0$ (resp. $\widehat{q}_V=0$) determines the complex ergodic sheets $M_V^{\pm}=\{(\xi,m_V^{\pm}(\xi))\mid\xi\in\Omega_0\}$ (resp. $\widehat{M}_V^{\pm}$). We can also write $A_V(\xi)=\pm1/\Im m_V^{\pm}(\xi)$, $B_V(\xi)=\pm|m_V^{\pm}(\xi)|^2/\Im m_V^{\pm}(\xi)$ and $C_V(\xi)=\mp\Re m_V^{\pm}(\xi)/\Im m_V^{\pm}(\xi)$.

Recall now the ergodic structure of the projective flow described in the previous section. If either a unique ergodic measure exists or $K_{\mathbb{R}}$ decomposes into ergodic k-sheets with $k \geq 2$, then $(K_{\mathbb{R}}, S)$ admits a unique linear invariant measure μ and every function $V \in C(\Omega)$ determines this measure; that is, $\mathcal{C}_1 = C(\Omega)$, $\mu_V = \mu$ and $m_V^{\pm} = m^{\pm}$. On the contrary, if $k = 1$, then $\mathcal{C}_1$ is a non-trivial subset of $C(\Omega)$ and different functions $V \in \mathcal{C}_1$ determine different linear invariant measures and m_V-functions.

THEOREM 3.6. *If $V \in \mathcal{C}_1$ and $\eta_V > 0$ then the Floquet coefficient has the derivative in the direction of V, namely,*

$$\alpha_V'(0) = \pm\frac{1}{2}\int_{\Omega} V(\xi)\,\frac{|m_V^{\pm}(\xi)|^2}{\Im m_V^{\pm}(\xi)}\,dm_0 \quad \textit{and} \quad \gamma_V'(0) = 0.$$

Proof. Let $\varepsilon \in \mathbb{R}$, $\varepsilon \neq 0$. From (3.5) we can prove the existence of an invariant measure $\widehat{\nu}_\varepsilon$ for $(\widehat{K}_{\mathbb{R}}, \Psi_\varepsilon)$ which projects onto $\widehat{m}_0$ such that

$$\frac{\alpha_V(\varepsilon) - \alpha_V(0)}{\varepsilon} = \int_{\widehat{K}_{\mathbb{R}}} \tau'(s)\,V(\xi)\,\widehat{A}(\xi\cdot s)\,\frac{1-\cos 2\psi}{2}\,d\widehat{\nu}_\varepsilon,$$

$$\frac{\gamma_V(\varepsilon)}{\varepsilon} = \int_{\widehat{K}_{\mathbb{R}}} \tau'(s)\,V(\xi)\,\widehat{A}(\xi\cdot s)\,\frac{\sin 2\psi}{2}\,d\widehat{\nu}_\varepsilon.$$

On the other hand, Proposition 3.3 implies that the function $\widehat{q}_V$ preserves sign on $K_{\mathbb{R}}$; equivalently, that the continuous function $\xi\cdot s \mapsto \tau'(s)\,V(\xi)$ on $\widehat{\Omega}$ belongs to the set $\mathcal{C}_1$ according to the classification given in [8] for the continuous case. The techniques used in the proof of Theorem 3.2 of the cited paper allow us to prove the existence of the limit of the incremental quotient for the Floquet coefficient, namely,

$$\alpha_V'(0) = \frac{1}{2}\int_{\widehat{\Omega}} \tau'(s)\,V(\xi)\,\widehat{A}_V(\xi\cdot s)\,d\widehat{m}_0, \qquad \gamma_V'(0) = 0.$$

Finally, from (2.17) we find

$$\alpha_V'(0) = \frac{1}{2}\int_{\Omega}\int_{1/2}^{1} \tau'(s)\,V(\xi)\,B_V(\xi)\,ds\,dm_0 = \pm\frac{1}{2}\int_{\Omega} V(\xi)\,\frac{|m_V^{\pm}(\xi)|^2}{\Im m_V^{\pm}(\xi)}\,dm_0,$$

which completes the proof of the theorem. ∎

4. L^1-convergence of Weyl m-functions. Let $V \in C(\Omega)$ be a strictly positive function. We consider the family of equations

$$\text{(4.1)} \qquad -z(n+1) - z(n-1) + V_0(\xi\cdot n)\,z(n) = E\,V(\xi\cdot n)\,z(n),$$

which are equivalent to

$$\text{(4.2)} \qquad \begin{pmatrix} z(n) \\ z(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & V_0(\xi\cdot n) - E\,V(\xi\cdot n) \end{pmatrix} \begin{pmatrix} z(n-1) \\ z(n) \end{pmatrix},$$

and the corresponding flows $(\Omega\times\mathbb{C}^2, S_E)$. It is easy to prove that if $\Im E > 0$ then the family (4.2) admits exponential dichotomy. This fact provides the Weyl functions in the direction of V (or Weyl m_V-functions), $m_V^{\pm}(\xi, E)$, which satisfy the equation

$$\text{(4.3)} \qquad -m_V^{\pm}(\xi\cdot(n+1), E) - \frac{1}{m_V^{\pm}(\xi\cdot n, E)} + V_0(\xi\cdot n) = EV(\xi\cdot n).$$

Let $z_V^{\pm}$ be the only solutions of (4.1) that belong to $l^2(\mathbb{Z}^{\pm})$ respectively. Notice that $m_V^{\pm}(\xi\cdot n, E) = z_V^{\pm}(n,\xi,E)/z_V^{\pm}(n-1,\xi,E)$. We consider the suspension of the flow defined by (4.1), given by (3.2). Obviously this family of systems also admits exponential dichotomy. In fact, the only solutions belonging to $L^2(\mathbb{R}^{\pm})$ are $\widehat{\mathbf{z}}_V^{\pm}(t,\xi\cdot s, E)$ defined by continuation in t of the solutions corresponding to the discrete equation. Hence we can introduce $\widehat{m}_V^{\pm}(\xi\cdot s, E) = \widehat{z}_{V,2}^{\pm}(0,\xi\cdot s,E)/\widehat{z}_{V,1}^{\pm}(0,\xi\cdot s,E)$, Weyl $\widehat{m}_V$-functions for the suspension. Notice that

$$\text{(4.4)} \qquad \begin{aligned} (\widehat{m}_V^{\pm})'(\xi\cdot s, E) = &-\sigma'(s)(1 + (\widehat{m}_V^{\pm})^2(\xi\cdot s, E)) \\ &+ \tau'(s)(V_0(\xi) - E\,V(\xi)), \end{aligned}$$

with $(\widehat{m}_V^{\pm})'(\xi\cdot s, E) = (d/dt)\,\widehat{m}_V^{\pm}(\xi\cdot(s+t), E)|_{t=0}$ and $s \in [0,1)$; moreover, $\widehat{m}_V^{\pm}(\xi\cdot s, E)$ and $m_V^{\pm}(\xi, E)$ are related by the corresponding equality (2.6).

The maps $\widehat{m}_V^{\pm}(\xi\cdot s, E)$ and $m_V^{\pm}(\xi, E)$ are jointly continuous in both variables; furthermore, for each fixed $\xi \in \Omega$ and $s \in [0,1)$ they are Herglotz functions on the upper half-plane $\Im E > 0$ (see Johnson [3] and Sacker–Sell [13]). Kotani's theory assures the existence of their non-tangential limit with non-null imaginary part for almost every $E \in \mathbf{A}_2$. In this section we will prove this convergence in the L^1-topology. To this end, we take, as before, $E_0 \in \mathbf{A}_2$ and suppose $E_0 = 0$.

The Floquet coefficient in the direction of V is a Herglotz function on $\Im E > 0$, where it is given by ([5])

$$w_V(E) = \pm\int_{\widehat{\Omega}} \sigma'(s)\,\widehat{m}_V^{\pm}(\xi\cdot s, E)\,d\widehat{m}_0;$$

it admits a continuous extension to the real axis. From (2.6) we derive that

$$w_V(E) = \pm\int_{\Omega} \ln m_V^{\pm}(\xi, E)\,dm_0.$$

The positive function V belongs to the set $\mathcal{C}_1$ and hence determines the m_V and $\widehat{m}_V$-functions, according to Definition 3.5.

Consider the sector $S_\delta = \{E \in \mathbb{C} \mid E = |E|\exp i\theta \text{ and } \theta \in [\delta, \pi-\delta]\}$ for each $\delta \in (0, \pi/2]$. Existence of non-tangential limit at $E = 0$ is equivalent to convergence in each set S_δ. The following results appear in [8] for the

Schrödinger equation. The same proofs work for our case, once the results of Section 3 are established.

LEMMA 4.1. *The incremental quotient of w_V has the following limit from the upper half-plane*:

$$\lim_{E\to 0,\,\Im E>0} \frac{w_V(E)-w_V(0)}{E} = \pm\frac{i}{2}\int_{\widehat{\Omega}} \frac{\tau'(s)\,V(\xi)}{\Im \widehat{m}_V^{\pm}(\xi\cdot s)}\,d\widehat{m}_0 = \pm\frac{i}{2}\int_{\Omega} \frac{V(\xi)\,|m_V^{\pm}(\xi)|^2}{\Im m_V^{\pm}(\xi)}\,dm_0.$$

LEMMA 4.2. *The Weyl $\widehat{m}_V$-functions have the following non-tangential limits from the upper half-plane*:

$$\widehat{m}_V^{\pm}(\xi\cdot s) = \lim_{E\to 0,\,n.t.} \widehat{m}_V^{\pm}(\xi\cdot s, E) \quad \textit{in measure.}$$

Convergence in measure can be characterized by means of pointwise convergence of suitable subsequences of the initial family of functions. Since the above set of convergence is invariant we deduce

COROLLARY 4.3. *The Weyl m_V-functions have the following non-tangential limits from the upper half-plane*:

$$m_V^{\pm}(\xi) = \lim_{E\to 0,\,n.t.} m_V^{\pm}(\xi, E) \quad \textit{in measure.}$$

These results allow us to prove

PROPOSITION 4.4. *The following non-tangential limits from the upper half-plane exist*:

(i) $\displaystyle \frac{|m_V^{+}(\xi)|^2}{\Im m_V^{+}(\xi)} = \lim_{E\to 0,\,n.t.} \frac{|m_V^{+}(\xi,E)|^2}{\Im m_V^{+}(\xi,E)}$ *in the L^1-topology*;

(ii) $\displaystyle \frac{1}{\Im m_V^{-}(\xi)} = \lim_{E\to 0,\,n.t.} \frac{1}{\Im m_V^{-}(\xi,E)}$ *in the L^1-topology.*

Proof. Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers with $\Im E_n > 0$ which tends non-tangentially to 0. We can suppose that $(m_V^{\pm}(\xi\cdot s, E_n))_{n\in\mathbb{N}}$ converges to $m_V^{\pm}(\xi\cdot s)$ for almost every $\xi\cdot s\in\widehat{\Omega}$. From the Riccati equations (4.4) we obtain

$$\frac{\gamma_V(E_n)}{\Im E_n} = \pm\int_{\widehat{\Omega}} \frac{\tau'(s)\,V(\xi)}{\Im \widehat{m}_V^{\pm}(\xi\cdot s, E_n)}\,d\widehat{m}_0.$$

From Lemma 4.1 it follows that

$$\lim_{n\to\infty} \pm\int_{\widehat{\Omega}} \frac{\tau'(s)\,V(\xi)}{\Im \widehat{m}_V^{\pm}(\xi\cdot s, E_n)}\,d\widehat{m}_0 = \pm\int_{\widehat{\Omega}} \frac{\tau'(s)\,V(\xi)}{\Im \widehat{m}_V^{\pm}(\xi\cdot s)}\,d\widehat{m}_0;$$

hence we derive that

$$\lim_{E\to 0,\,\text{n.t.}} \pm\frac{\tau'(s)}{\Im \widehat{m}_V^{\pm}(\xi\cdot s, E)} = \pm\frac{\tau'(s)}{\Im \widehat{m}_V^{\pm}(\xi\cdot s)} \quad \text{in the } L^1(\widehat{\Omega},\widehat{m}_0)\text{-topology}$$

as an easy consequence of the Egorov theorem. On the other hand, (2.6) gives

$$\frac{\tau'(s)}{\Im \widehat{m}_V^{\pm}(\xi\cdot s, E)} = \frac{\tau'(s)|m_V^{\pm}(\xi,E)|^2}{\Im m_V^{\pm}(\xi,E) - \Im E\,\tau(s)\,V(\xi)|m_V^{\pm}(\xi,E)|^2},$$

whence one deduces that

$$\frac{\tau'(s)}{\Im \widehat{m}_V^{+}(\xi\cdot s, E)} \ge \frac{\tau'(s)|m_V^{+}(\xi,E)|^2}{\Im m_V^{+}(\xi,E)},$$

$$-\frac{\tau'(s)}{\Im \widehat{m}_V^{-}(\xi\cdot s, E)} \ge -\frac{\tau'(s)|m_V^{-}(\xi,E)|^2}{\Im m_V^{-}(\xi,E) - \Im E\,V(\xi)|m_V^{-}(\xi,E)|^2} = -\frac{\tau'(s)}{\Im m_V^{-}(\xi\cdot 1, E)}.$$

The Vitali theorem shows that $\tau'(s)/\Im m_V^{+}(\xi,E)$ and $\tau'(s)/\Im m_V^{-}(\xi\cdot 1,E)$ respectively converge to $\tau'(s)/\Im m_V^{+}(\xi)$ and $\tau'(s)/\Im m_V^{-}(\xi\cdot 1)$ in the $L^1(\widehat{\Omega},\widehat{m}_0)$-topology, which is equivalent to the statements (i) and (ii). ■

The proof of the analogue of the preceding proposition for the functions $1/\Im m_V^{+}$ and $|m_V^{-}|^2/\Im m_V^{-}$ requires the next result, which can be derived in the same way as the corresponding one in [8].

PROPOSITION 4.5. *The following non-tangential limit from the upper half-plane exists*:

$$\tau'(s)\left|\frac{\Re \widehat{m}_V^{\pm}(\xi\cdot s)}{\Im \widehat{m}_V^{\pm}(\xi\cdot s)}\right| = \lim_{E\to 0,\,n.t.} \tau'(s)\left|\frac{\Re \widehat{m}_V^{\pm}(\xi\cdot s, E)}{\Im \widehat{m}_V^{\pm}(\xi\cdot s, E)}\right|$$

in the $L^1(\widehat{\Omega},\widehat{m}_0)$-topology.

Now we can establish

PROPOSITION 4.6. *The following non-tangential limits from the upper half-plane exist*:

(i) $\displaystyle \frac{1}{\Im m_V^{+}(\xi)} = \lim_{E\to 0,\,n.t.} \frac{1}{\Im m_V^{+}(\xi,E)}$ *in the L^1-topology*;

(ii) $\displaystyle \frac{|m_V^{-}(\xi)|^2}{\Im m_V^{-}(\xi)} = \lim_{E\to 0,\,n.t.} \frac{|m_V^{-}(\xi,E)|^2}{\Im m_V^{-}(\xi,E)}$ *in the L^1-topology.*

Proof. From the Riccati equations (4.4) we obtain

$$(\Re\widehat{m}_V^\pm)'(\xi\cdot s,E) = -\sigma'(s)(1+\Re^2\widehat{m}_V^\pm(\xi\cdot s,E)-\Im^2\widehat{m}_V^\pm(\xi\cdot s,E)) + \tau'(s)(V_0(\xi)-\Re E\,V(\xi)),$$
$$(\Im\widehat{m}_V^\pm)'(\xi\cdot s,E) = -2\sigma'(s)\Re\widehat{m}_V^\pm(\xi\cdot s,E)\Im\widehat{m}_V^\pm(\xi\cdot s,E) - \Im E\tau'(s)V(\xi).$$

A straightforward calculation shows that

$$\frac{d}{dt}\left(\frac{1+|\widehat{m}_V^\pm(\xi\cdot(s+t),E)|^2}{\Im\widehat{m}_V^\pm(\xi\cdot(s+t),E)}\right)\Big|_{t=0} = [\Im E\tau'(s)V(\xi)(1+\Re^2\widehat{m}_V^\pm(\xi\cdot s,E)-\Im^2\widehat{m}_V^\pm(\xi\cdot s,E))]\frac{1}{\Im^2\widehat{m}_V^\pm(\xi\cdot s,E)} + 2\tau'(s)(V_0(\xi)-\Re E\,V(\xi))\frac{\Re\widehat{m}_V^\pm(\xi\cdot s,E)}{\Im\widehat{m}_V^\pm(\xi\cdot s,E)}.$$

From Proposition 4.5 we deduce that

$$\lim_{E\to0,\,\text{n.t.}}\Im E\int_{\widehat\Omega}\tau'(s)\,V(\xi)\,\frac{1+\Re^2\widehat{m}_V^\pm(\xi\cdot s,E)}{\Im^2\widehat{m}_V^\pm(\xi\cdot s,E)}\,d\widehat{m}_0 = -2\int_{\widehat\Omega}\tau'(s)\,V_0(\xi)\,\frac{\Re\widehat{m}_V^\pm(\xi\cdot s)}{\Im\widehat{m}_V^\pm(\xi\cdot s)}\,d\widehat{m}_0$$
$$= -2\int_{\widehat\Omega}\tau'(s)\,V_0(\xi)\left(\tau(s)\,V_0(\xi)\,\frac{|m_V^\pm(\xi)|^2}{\Im m_V^\pm(\xi)}-\frac{\Re m_V^\pm(\xi)}{\Im m_V^\pm(\xi)}\right)d\widehat{m}_0$$
$$= -\int_\Omega\left(V_0^2(\xi)\,\frac{|m_V^\pm(\xi)|^2}{\Im m_V^\pm(\xi)}-2V_0(\xi)\,\frac{\Re m_V^\pm(\xi)}{\Im m_V^\pm(\xi)}\right)dm_0$$
$$= -\int_\Omega\left(\frac{1}{\Im m_V^\pm(\xi\cdot2)}-\frac{1}{\Im m_V^\pm(\xi)}\right)dm_0 = 0,$$

by (2.17) and (1.3). On the other hand,

$$\frac{d}{dt}\left(-\frac{1}{\Im\widehat{m}_V^\pm(\xi\cdot(s+t),E)}\right)\Big|_{t=0} = -\Im E\,\frac{\tau'(s)\,V(\xi)}{\Im^2\widehat{m}_V^\pm(\xi\cdot s,E)} - 2\sigma'(s)\,\frac{\Re\widehat{m}_V^\pm(\xi\cdot s,E)}{\Im\widehat{m}_V^\pm(\xi\cdot s,E)}.$$

Obviously the right hand term belongs to $L^1(\widehat\Omega,\widehat{m}_0)$ and hence

$$\text{(4.5)}\qquad \lim_{E\to0,\,\text{n.t.}}\int_{\widehat\Omega}2\sigma'(s)\,\frac{\Re\widehat{m}_V^\pm(\xi\cdot s,E)}{\Im\widehat{m}_V^\pm(\xi\cdot s,E)}\,d\widehat{m}_0 = -\lim_{E\to0,\,\text{n.t.}}\Im E\int_{\widehat\Omega}\frac{\tau'(s)\,V(\xi)}{\Im^2\widehat{m}_V^\pm(\xi\cdot s,E)}\,d\widehat{m}_0 = 0.$$

By (2.6) for $s\in[0,1/2]$ and the definition of $\widehat{m}_0$ we obtain

$$\int_{\widehat\Omega}2\sigma'(s)\,\frac{\Re\widehat{m}_V^\pm(\xi\cdot s,E)}{\Im\widehat{m}_V^\pm(\xi\cdot s,E)}\,d\widehat{m}_0 = \int_{\widehat\Omega}\sigma'(s)\frac{(|m_V^\pm(\xi,E)|^2-1)\sin2\sigma(s)+2\Re m_V^\pm(\xi,E)\cos2\sigma(s)}{\Im m_V^\pm(\xi,E)}\,d\widehat{m}_0 = \int_\Omega\frac{|m_V^\pm(\xi,E)|^2-1}{\Im m_V^\pm(\xi,E)}\,dm_0.$$

Equality (4.5), Proposition 4.4 and equation (1.3) give

$$\lim_{E\to0,\,\text{n.t.}}\int_\Omega\frac{1}{\Im m_V^+(\xi,E)}\,dm_0 = \int_\Omega\frac{|m_V^+(\xi)|^2}{\Im m_V^+(\xi)}\,dm_0 = \int_\Omega\frac{1}{\Im m_V^+(\xi)}\,dm_0,$$
$$\lim_{E\to0,\,\text{n.t.}}\int_\Omega\frac{|m_V^-(\xi,E)|^2}{\Im m_V^-(\xi,E)}\,dm_0 = \int_\Omega\frac{1}{\Im m_V^-(\xi)}\,dm_0 = \int_\Omega\frac{|m_V^-(\xi)|^2}{\Im m_V^-(\xi)}\,dm_0.$$

Once this is known, (i) and (ii) can be established as a consequence of the Egorov theorem and from the pointwise convergence already known. ∎

We can write $m_V^\pm(\xi,E) = (m_V^\pm(\xi,E)/\sqrt{\pm\Im m_V^\pm(\xi,E)})\sqrt{\pm\Im m_V^\pm(\xi,E)}$. Propositions 4.4, 4.6 and the fact that the map

$$P: L^2(\Omega,m_0)\times L^2(\Omega,m_0)\to L^1(\Omega,m_0),\qquad (g_1,g_2)\mapsto g_1g_2,$$

is continuous lead us to the main result of this section:

THEOREM 4.7. *The Weyl m_V-functions have the following non-tangential limits from the upper half-plane:*

$$m_V^\pm(\xi) = \lim_{E\to0,\,\text{n.t.}} m_V^\pm(\xi,E)\qquad \textit{in the } L^1(\Omega,m_0)\textit{-topology.}$$

5. Some remarkable relations. The results obtained in Sections 3 and 4 permit us now to improve some previously known relations.

Fix $E_0\in\mathbf{A}_2$. As proved in Theorem 3.6, the derivative in the direction of the parameter is

$$\text{(5.1)}\qquad \alpha'(E_0) = \frac12\int_\Omega\frac{|m_{(1)}^+(\xi,E_0)|^2}{\Im m_{(1)}^+(\xi,E_0)}\,dm_0 = \frac12\int_\Omega\frac{1}{\Im m_{(1)}^+(\xi,E_0)}\,dm_0.$$

This means, in particular, that the known inequality

$$\alpha'(E_0) \geq \frac{1}{2} \int_\Omega \lim_{E\to E_0,\,\text{n.t.}} \frac{1}{\Im m^+_{(1)}(\xi, E)}\, dm_0$$

obtained in [1] from Kotani's theory is in fact an equality. The derivative of the rotation number is lower semicontinuous on $\mathbf{A}_2$, as proved for the continuous case in [9].

On the other hand, the Jensen inequality implies that

$$\begin{aligned}\sin\alpha(E_0) &= \sin \int_\Omega \cot^{-1}(-C_{(1)}(\xi, E_0))\, dm_0 \\ &\geq \int_\Omega \frac{1}{\sqrt{A_{(1)}(\xi, E_0)\, B_{(1)}(\xi, E_0)}}\, dm_0;\end{aligned}$$

that is,

$$\begin{aligned}\sin\alpha(E_0) &\geq \int_\Omega \frac{\Im m^+_{(1)}(\xi, E_0)}{|m^+_{(1)}(\xi, E_0)|}\, dm_0 \\ &= \int_\Omega \sqrt{\Im m^+_{(1)}(\xi, E_0)\, \Im m^+_{(1)}(\xi\cdot 1, E_0)}\, dm_0.\end{aligned} \tag{5.2}$$

This substitutes the inequality $\sin\alpha(E_0) \geq \int_\Omega \Im m^+_{(1)}(\xi, E_0)\, dm_0$, posed as an open question in [1], which in general is not possible (see the example below). Moreover, the Schwarz inequality provides, from (5.1) and (5.2), a very simple proof of the relation $2\sin\alpha(E_0)\,\alpha'(E_0) \geq 1$. In fact, if equality holds for a point E_0 then $A_{(1)}$ and $B_{(1)}$ (and hence $C_{(1)}$) are necessarily constant functions. The equations describing the evolution with respect to T of these coefficients, that can be derived from (2.17), permit us to deduce that V_0 is also constant.

The periodic case: an example. Let $\beta_1, \ldots, \beta_n$ be complex numbers. We define the $n\times n$ matrix $\Delta(\beta_1, \ldots, \beta_n) = (\delta_{i,j})$ by

$$\delta_{i,j} = \begin{cases} \beta_i & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and set $D(\beta_1, \ldots, \beta_n) = \det \Delta(\beta_1, \ldots, \beta_n)$. It is immediate that

$$\begin{aligned} D(\beta_1, \ldots, \beta_n) &= \beta_1 D(\beta_2, \ldots, \beta_n) - D(\beta_3, \ldots, \beta_n) \\ &= \beta_n D(\beta_1, \ldots, \beta_{n-1}) - D(\beta_1, \ldots, \beta_{n-2}).\end{aligned}$$

Let us introduce the terminating continued fraction

$$F(\beta_1, \ldots, \beta_n) = \cfrac{1}{\beta_1 - \cfrac{1}{\ddots - \cfrac{1}{\beta_n}}}.$$

It follows from the preceding relation that

$$F(\beta_1, \ldots, \beta_n) = \frac{D(\beta_2, \ldots, \beta_n)}{D(\beta_1, \ldots, \beta_n)}.$$

Let $E \in \mathbb{C}$ with $\Im E > 0$ and $\xi \in \Omega$. To simplify the notation we write $v_n = V_0(\xi\cdot n) - E$. We assume that there exists a natural number k with $v_{n+k} = v_n$ for every $n \in \mathbb{N}$. One can prove by an induction argument that for $n \geq 3$,

$$U_E(n, \xi) = \begin{pmatrix} -D(v_1, \ldots, v_{n-2}) & D(v_0, \ldots, v_{n-2}) \\ -D(v_1, \ldots, v_{n-1}) & D(v_0, \ldots, v_{n-1}) \end{pmatrix},$$

that is, if $\mathbf{e}_1 = (1,0)^t$ and $\mathbf{e}_2 = (0,1)^t$ then $z_E(n, \xi, \mathbf{e}_1) = -D(v_1, \ldots, v_{n-1})$ and $z_E(n, \xi, \mathbf{e}_2) = D(v_0, \ldots, v_{n-1})$. On the other hand, we know that the solution of the equation (1.1),

$$\begin{aligned} z^+(n, \xi, E) &= z_E(n, \xi, \mathbf{e}_1) + m^+_{(1)}(\xi, E)\, z_E(n, \xi, \mathbf{e}_2) \\ &= z_E(n, \xi, \mathbf{e}_2)(m^+_{(1)}(\xi, E) - F(v_0, \ldots, v_{n-1})),\end{aligned}$$

belongs to $l^2(\mathbb{Z}^+)$, which means that

$$\begin{aligned} m^+_{(1)}(\xi, E) &= \lim_{n\to\infty} F(v_0, \ldots, v_n) = F(v_0, \ldots, v_{k-1}, 1/m^+_{(1)}(\xi, E)) \\ &= \frac{D(v_1, \ldots, v_{k-1}, 1/m^+_{(1)}(\xi, E))}{D(v_0, \ldots, v_{k-1}, 1/m^+_{(1)}(\xi, E))} \\ &= \frac{-z_E(k, \xi, \mathbf{e}_1) + z_E(k-1, \xi, \mathbf{e}_1)\, m^+_{(1)}(\xi, E)}{z_E(k, \xi, \mathbf{e}_2) - z_E(k-1, \xi, \mathbf{e}_2)\, m^+_{(1)}(\xi, E)}.\end{aligned}$$

The spectrum in this case is known to consist of the union of k bands (closed intervals) of the real line. If E_0 is a point of its interior then $|\operatorname{tr} U_{E_0}(k, \xi)| < 2$ and

$$m^+_{(1)}(\xi, E_0) = \frac{z_{E_0}(k, \xi, \mathbf{e}_2) - z_{E_0}(k-1, \xi, \mathbf{e}_1)}{2z_{E_0}(k-1, \xi, \mathbf{e}_2)} + i\frac{\sqrt{4 - \operatorname{tr}^2 U_{E_0}(k, \xi)}}{2|z_{E_0}(k-1, \xi, \mathbf{e}_2)|}.$$

Equation (1.3) easily yields $m^+_{(1)}(\xi\cdot 1, E_0), \ldots, m^+_{(1)}(\xi\cdot(k-1), E_0)$. Notice also that, according to the results of Section 3,

$$\alpha(E_0) = \frac{1}{k}\sum_{n=0}^{k-1} \cot^{-1} \frac{\Re m^+_{(1)}(\xi\cdot n, E_0)}{\Im m^+_{(1)}(\xi\cdot n, E_0)}$$

and

$$\alpha'(E_0) = \frac{1}{2k}\sum_{n=0}^{k-1}\frac{1}{\Im m^+_{(1)}(\xi\cdot n, E_0)}.$$

This expression for the derivative of the rotation number coincides with the one appearing in Last [6].

The computation in Mathematica 2.2 of the $m_{(1)}$-functions corresponding to the periodic operator

$$-z(n+1) - z(n-1) + \lambda\cos(n\pi/2+\xi)z(n) = Ez(n)$$

for $\lambda = 1$ and $\xi = 0.1$ provides a graphic sample of the inequality (5.2):

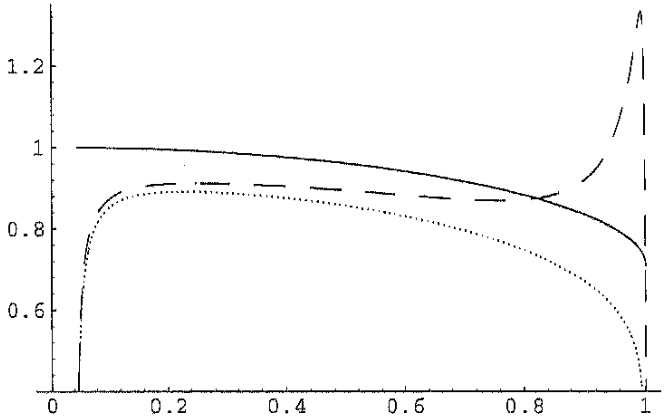

Fig. 1

The solid line represents the function $\sin\alpha(E)$, the dotted line corresponds to

$$\int_\Omega \sqrt{\Im m^+_{(1)}(\xi,E)\,\Im m^+_{(1)}(\xi\cdot 1,E)}\,dm_0,$$

and finally $\int_\Omega \Im m^+_{(1)}(\xi,E)\,dm_0$ is represented by the dashed line, for E within a band of the spectrum. Notice that the third function exhibits a fast growth near one of the edges of the band, before decreasing to zero. This behavior becomes more marked when the period k is increased.

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