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Jordan polynomials can be analytically recognized

by

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Abstract. We prove that there exists a real or complex central simple associative algebra $\mathcal{M}$ with minimal one-sided ideals such that, for every non-Jordan associative polynomial $\mathbf{p}$, a Jordan-algebra norm can be given on $\mathcal{M}$ in such a way that the action of $\mathbf{p}$ on $\mathcal{M}$ becomes discontinuous.

1. Introduction. Among the associative polynomials (elements in the free associative algebra on a countably infinite set of indeterminates) the so-called "Jordan polynomials" are of special interest in Jordan theory. Jordan polynomials are those associative polynomials that can be expressed through the indeterminates by means of the sum and the Jordan product. Well-known examples of Jordan polynomials are $\mathbf{x}^2$ and $\mathbf{xyx}$, whereas the associative product $\mathbf{xy}$ and the tetrad $\mathbf{xyzt}+\mathbf{tzyx}$ are examples of non-Jordan polynomials.

Let A be an associative algebra over $\mathbb{K}$ ($=\mathbb{R}$ or $\mathbb{C}$), and $\|\cdot\|$ be a Jordan-algebra norm on A. Then obviously every Jordan polynomial acts $\|\cdot\|$-continuously on A. If either A is semiprime and $\|\cdot\|$ is complete or A is simple and has a unit element, then the associative product of A is $\|\cdot\|$-continuous ([14], [15], [4]), hence every associative polynomial acts $\|\cdot\|$-continuously on A. An example of $\|\cdot\|$-discontinuity of the associative product of A with $\|\cdot\|$ complete (hence A not semiprime) is given in [14]. The first example of $\|\cdot\|$-discontinuity of the associative product of A with A semiprime (hence $\|\cdot\|$ not complete) appears in [2] (see also [17]), but the algebra A in this example is very far from being simple (and even prime): it is an infinite direct sum of finite-dimensional simple ideals. Very recently an example of $\|\cdot\|$-discontinuity of the associative product of A with A simple (hence neither $\|\cdot\|$ can be complete nor A can have a unit) has been provided in [4],

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A being nothing but the algebra $M_\infty(\mathbb{K})$ of all countably infinite matrices over $\mathbb{K}$ with a finite number of non-zero entries.

The continuity of the action of the tetrad on associative algebras endowed with Jordan-algebra norms has also been discussed in the literature because of its close relation to the so-called "norm-extension problem" [17]. The norm-extension problem, together with the above problem on the continuity of the action of the associative product, becomes of capital importance in the search for normed versions of the Zel'manov prime theorem for Jordan algebras [18] (see [3], [5], [6], [7], and [16]). Let A be an associative algebra over $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) with an involution $*$, assume that A is a "$*$-tight envelope" of its hermitian part $H(A,*)$, and let $|\cdot|$ be an algebra norm on the Jordan algebra $H(A,*)$. Clearly $H(A,*)$ is invariant under the tetrad and, if the action of the tetrad on $H(A,*)$ is $|\cdot|$-discontinuous, then there is no (associative-) algebra norm on A whose topology extends the one of $|\cdot|$ on $H(A,*)$. The converse is also true [17]: if the tetrad acts $|\cdot|$-continuously on $H(A,*)$, then there is an algebra norm on A whose topology coincides on $H(A,*)$ with that of $|\cdot|$. If either $H(A,*)$ is semiprime and $|\cdot|$ is complete or $H(A,*)$ is simple and has a unit element, then the action of the tetrad on $H(A,*)$ is $|\cdot|$-continuous ([17], [4]), hence the norm-extension problem has an affirmative answer in this case.

In a common context, the associative-product-continuity problem and the norm-extension problem are related in the following way. Let A be an associative algebra over $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) with an involution $*$, assume that A is a "$*$-tight envelope" of its hermitian part $H(A,*)$, and let $|\cdot|$ be a Jordan-algebra norm on A. If the action of the tetrad on $H(A,*)$ is $|\cdot|$-discontinuous, then the associative product of A is indeed $|\cdot|$-discontinuous. Therefore the best negative answers to the norm-extension problem (implying negative answers to the associative-product-continuity problem) arise when in the above context, for algebraically "good enough" A, one is able to show the $|\cdot|$-discontinuity of the action of the tetrad on $H(A,*)$. Examples of such a situation are provided in [17] with A semiprime but not prime and, more recently, in [4] with A simple (actually $A = M_\infty(\mathbb{K})$).

The question of the continuity of the action of general associative polynomials on associative algebras endowed with Jordan-algebra (semi-) norms has been first considered in [1]. It is proved there that, for every associative polynomial $\mathbf{p}$ over $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) which is "very non-Jordan" (in a sense that will not be specified here), there exists a prime associative algebra A, together with a Jordan-algebra seminorm $|\cdot|$ on A, such that the action of $\mathbf{p}$ on A becomes $|\cdot|$-discontinuous. Actually, the pair $(A, |\cdot|)$ depends only on the degree of $\mathbf{p}$ and the number of indeterminates involved in $\mathbf{p}$, but the algebra A is not simple and the seminorm $|\cdot|$ is far from being a norm: its kernel is a finite-codimensional subspace of A.

By putting together and refining the arguments in [1] and [4], we prove in the present paper that, for every non-Jordan associative polynomial $\mathbf{p}$ over $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$), there exists a Jordan-algebra norm $|\cdot|$ on the simple associative algebra $M_\infty(\mathbb{K})$ such that the action of $\mathbf{p}$ on $M_\infty(\mathbb{K})$ is $|\cdot|$-discontinuous. In fact, the norm $|\cdot|$ depends only on the degree of $\mathbf{p}$ and the number of indeterminates involved in $\mathbf{p}$, and the $|\cdot|$-discontinuity of the action of $\mathbf{p}$ can be centered in $H(M_\infty(\mathbb{K}),*)$ for a suitable $\mathbb{K}$-linear involution $*$ on $M_\infty(\mathbb{K})$ of arbitrarily prefixed type (hermitian or alternate). This improves in several directions the above-cited result in [1], refines all previously known non-complete negative answers to the associative-product-continuity and norm-extension problems, and has the following nice consequence: Jordan polynomials are precisely those associative polynomials which act continuously on every associative algebra endowed with a Jordan-algebra norm.

2. The main result.

Let A be an associative algebra (with product denoted by juxtaposition) over a field $\mathbb{F}$ (which will be always assumed to be of characteristic not 2). Then the vector space of A with the so-called *Jordan product* of A, defined by

$$a.b := \tfrac{1}{2}(ab + ba),$$

becomes a Jordan algebra, usually denoted by A^+. The subalgebras of A^+ are called *Jordan subalgebras* of A. If A has a (linear algebra) involution $*$, then the set $H(A,*)$, of all $*$-invariant elements in A, is a Jordan subalgebra of A. For real or complex A, a *Jordan-algebra norm* on A is nothing but an algebra norm on A^+, i.e. a norm $\|\cdot\|$ on the vector space of A satisfying $\|a.b\| \le \|a\|\,\|b\|$ for all a, b in A.

Fix a field $\mathbb{F}$. Given a non-empty set $\mathbf{X}$, we denote by $\mathcal{A}(\mathbf{X})$ the free associative algebra over $\mathbb{F}$ on $\mathbf{X}$. When $\mathbf{X}$ is a countably infinite set, the elements in $\mathcal{A}(\mathbf{X})$ are called *associative polynomials* over $\mathbb{F}$, and such a polynomial will be written $\mathbf{p}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ when we are interested in pointing out that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are the indeterminates involved in $\mathbf{p}$. Given an associative polynomial $\mathbf{p} = \mathbf{p}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and elements $a_1, \ldots, a_n$ in an associative algebra A, we denote by $\mathbf{p}(a_1, \ldots, a_n)$ the image of $\mathbf{p}$ in A under the unique homomorphism from $\mathcal{A}(\mathbf{X})$ into A sending $\mathbf{x}_i$ to a_i (for $i = 1, \ldots, n$) and the other elements of $\mathbf{X}$ to zero. In this way we can consider the *action* of $\mathbf{p}$ on A, namely the mapping $(a_1, \ldots, a_n) \mapsto \mathbf{p}(a_1, \ldots, a_n)$ from $A \times \overset{n}{\ldots} \times A$ into A. The action of $\mathbf{p}$ on a subset S of A will mean the restriction of the above mapping to $S \times \overset{n}{\ldots} \times S$. We will say that a subset S of A is *invariant* under $\mathbf{p}$ if the action of $\mathbf{p}$ on S is actually valued in S.

Given an algebra B, we denote by $M_\infty(B)$ the algebra of all countably infinite matrices over B with a finite number of non-zero entries. If B has an involution $*$, then $M_\infty(B)$ has a "canonical" involution (also denoted by $*$)

consisting in transposing a given matrix and applying the original involution to each entry. The proof of our main theorem consists of two independent results, interesting in their own right (Propositions 1 and 2 below). The first one is a greatly improved version of Proposition 3 of [4].

PROPOSITION 1. *Let n be a natural number, $(B, \|\cdot\|)$ an associative normed algebra over $\mathbb{K}$ $(=\mathbb{R}$ or $\mathbb{C})$, and J be a closed Jordan subalgebra of B. Then there exists a Jordan-algebra norm $|\!|\!|\cdot|\!|\!|$ on $M_\infty(B)$ making discontinuous the action on $M_\infty(B)$ of every associative polynomial $\mathbf{p}$ of degree $\leq n$ such that J is not invariant under $\mathbf{p}$. Moreover, if B has an involution $*$, and if J is contained in $H(B,*)$, then the norm $|\!|\!|\cdot|\!|\!|$ can be chosen in such a way that the action on $H(M_\infty(B),*)$ of every polynomial $\mathbf{p}$ as above is $|\!|\!|\cdot|\!|\!|$-discontinuous.*

The proof of this proposition follows essentially the lines of that of [4, Proposition 3]. Some extra difficulties are, however, to be overcome, and this is done in the next lemmas.

LEMMA 1. *Let E and F be normed spaces over $\mathbb{K}$ $(=\mathbb{R}$ or $\mathbb{C})$ and f be a mapping from E to F. Assume that f is continuous at zero and that there is a natural number k such that $f(\lambda x) = \lambda^k f(x)$ for all λ in $\mathbb{K}$ and x in E. Then there exists a non-negative real number M such that $\|f(x)\| \leq M\|x\|^k$ for all x in E.*

Proof. Let B_E and B_F denote the closed unit balls of E and F, respectively. Since $f(0) = 0$, the continuity of f at zero implies the existence of a positive number R such that $f(RB_E) \subseteq B_F$. Therefore $\|f(x)\| \leq R^{-k}\|x\|^k$ for all x in E. ∎

LEMMA 2. *Let E and F be topological vector spaces over $\mathbb{K}$ $(=\mathbb{R}$ or $\mathbb{C})$ and $f_1, \ldots, f_n$ be mappings from E to F. Assume that, for all λ in $\mathbb{K}$, x in E, and $k = 1, \ldots, n$, the equality $f_k(\lambda x) = \lambda^k f_k(x)$ holds. Then the mapping $\sum_{k=1}^n f_k$ is continuous at zero if and only if all the mappings f_k $(k = 1, \ldots, n)$ are.*

Proof. Writing $f := \sum_{k=1}^n f_k$ and choosing pairwise different non-zero elements $\lambda_1, \ldots, \lambda_n$ in $\mathbb{K}$, for all x in E and $i = 1, \ldots n$ we have

$$f(\lambda_i x) = \lambda_i f_1(x) + \lambda_i^2 f_2(x) + \ldots + \lambda_i^n f_n(x),$$

which can be rewritten matricially as

$$\begin{pmatrix} f(\lambda_1 x) \\ \vdots \\ f(\lambda_n x) \end{pmatrix} = A \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix},$$

where A denotes the $n \times n$ matrix over $\mathbb{K}$ given by

$$A := \begin{pmatrix} \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^n \\ \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n & \lambda_n^2 & \cdots & \lambda_n^n \end{pmatrix}.$$

Since the Vandermonde type determinant of A is non-zero, we obtain

$$\begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix} = A^{-1} \begin{pmatrix} f(\lambda_1 x) \\ \vdots \\ f(\lambda_n x) \end{pmatrix},$$

hence there are μ_{ik} in $\mathbb{K}$ such that, for all x in E and $k = 1, \ldots, n$, we have

$$f_k(x) = \sum_{i=1}^n \mu_{ik} f(\lambda_i x).$$

Now clearly all f_k are continuous at zero whenever so is f. ∎

We recall that an associative polynomial is said to be *homogeneous of degree k* if all the monomials involved in its essentially unique decomposition as a linear combination of pairwise different words are of degree k. In general, if we group together the monomials of degree k of a given polynomial $\mathbf{p}$, then we obtain a homogeneous polynomial $\mathbf{p}_k$, called the *homogeneous component of degree k* of $\mathbf{p}$, and we have $\mathbf{p} = \sum_{k=1}^n \mathbf{p}_k$, where n denotes the degree of $\mathbf{p}$.

Proof of Proposition 1. Let $\|\cdot\|$ be an algebra norm on $M_\infty(B)$ satisfying

$$(\$) \qquad \max\{\|b_{ii}\| : i \in \mathbb{N}\} \leq \|(b_{ij})\| \leq \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} \|b_{ij}\|$$

for all (b_{ij}) in $M_\infty(B)$ (e.g., we may take $\|(b_{ij})\| := \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} \|b_{ij}\|$). For an element α in $M_\infty(B)$ and a (not necessarily $\|\cdot\|$-closed) subspace S of $M_\infty(B)$, let us write $\|\alpha + S\| := \inf\{\|\alpha + \beta\| : \beta \in S\}$, so that the mapping $\alpha \mapsto \|\alpha + S\|$ is a seminorm on the vector space of $M_\infty(B)$. Regarding the algebra $M_\infty(B)$ as the algebraic tensor product $M_\infty(\mathbb{K}) \otimes_{\mathbb{K}} B$ and identifying for each k in $\mathbb{N}$ the algebra $M_k(\mathbb{K})$ of all $k \times k$ matrices over $\mathbb{K}$ with the subalgebra of $M_\infty(\mathbb{K})$ of those matrices (λ_{ij}) in $M_\infty(\mathbb{K})$ satisfying $\lambda_{ij} = 0$ whenever either $i > k$ or $j > k$, we may consider the Jordan subalgebra $\mathcal{J}_k$ of $M_\infty(B)$ given by $\mathcal{J}_k := M_{k-1}(\mathbb{K}) \otimes B + e_k \otimes J$, where $M_0(\mathbb{K}) := 0$ and e_k denotes the element $(\lambda_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ in $M_\infty(\mathbb{K})$ given by $\lambda_{ij} = 0$ whenever $(i,j) \neq (k,k)$, and $\lambda_{kk} = 1$.

Finally, we may define a norm $|\!|\!|\cdot|\!|\!|$ on the vector space of $M_\infty(B)$ by

$$|\!|\!|\alpha|\!|\!| := \|\alpha\| + \sum_{i=1}^{\infty} 2^{(n+1)^i}\|\alpha + \mathcal{J}_i\|$$

for all α in $M_\infty(B)$ (note that, for each α in $M_\infty(B)$, the series appearing above has only a finite number of non-zero terms). As in the proof of [4, Proposition 3], Lemma 3 of [17] shows that $|\!|\!|\cdot|\!|\!|$ is a Jordan-algebra norm on $M_\infty(B)$. From the property (\$) of the norm $\|\cdot\|$ on $M_\infty(B)$ we easily deduce that, if k and i are in $\mathbb{N}$ and if b is in B, then

$$\|e_k\otimes b + \mathcal{J}_i\| = \begin{cases} \|b\| & \text{if } k>i,\\ \|b+J\| & \text{if } k=i,\\ 0 & \text{if } k<i.\end{cases}$$

Therefore, for all k in $\mathbb{N}$ and b in B, we obtain

$$|\!|\!|e_k\otimes b|\!|\!| = (1+2^{n+1}+\ldots+2^{(n+1)^{k-1}})\|b\| + 2^{(n+1)^k}\|b+J\|.$$

Let $\mathbf{q}=\mathbf{q}(\mathbf{x}_1,\ldots,\mathbf{x}_s)$ be a homogeneous associative polynomial of degree $m\le n$ such that J is not invariant under $\mathbf{q}$. Since J is closed in $(B,\|\cdot\|)$, we can choose $x_1,\ldots,x_s$ in J such that $\|\mathbf{q}(x_1,\ldots,x_s)+J\|\neq 0$, and so, for k in $\mathbb{N}$,

$$\begin{aligned}
\frac{|\!|\!|\mathbf{q}(e_k\otimes x_1,\ldots,e_k\otimes x_s)|\!|\!|}{\max\{|\!|\!|e_k\otimes x_1|\!|\!|,\ldots,|\!|\!|e_k\otimes x_s|\!|\!|\}^m} &= \frac{|\!|\!|e_k\otimes \mathbf{q}(x_1,\ldots,x_s)|\!|\!|}{\max\{|\!|\!|e_k\otimes x_1|\!|\!|^m,\ldots,|\!|\!|e_k\otimes x_s|\!|\!|^m\}}\\
&= \frac{(1+2^{n+1}+\ldots+2^{(n+1)^{k-1}})\|\mathbf{q}(x_1,\ldots,x_s)\| + 2^{(n+1)^k}\|\mathbf{q}(x_1,\ldots,x_s)+J\|}{(1+2^{n+1}+\ldots+2^{(n+1)^{k-1}})^m\max\{\|x_1\|^m,\ldots,\|x_s\|^m\}}\\
&\ge \frac{2^{(n+1)^k}\|\mathbf{q}(x_1,\ldots,x_s)+J\|}{k^m(2^{(n+1)^{k-1}})^m\max\{\|x_1\|^m,\ldots,\|x_s\|^m\}}\\
&= \frac{2^{(n+1)^{k-1}(n+1-m)}\|\mathbf{q}(x_1,\ldots,x_s)+J\|}{k^m\max\{\|x_1\|^m,\ldots,\|x_s\|^m\}}\\
&\ge \frac{2^{(n+1)^{k-1}}\|\mathbf{q}(x_1,\ldots,x_s)+J\|}{k^m\max\{\|x_1\|^m,\ldots,\|x_s\|^m\}}\to\infty \quad \text{as } k\to\infty.
\end{aligned}$$

Now we may apply Lemma 1, with E equal to the ℓ_∞-sum of s copies of $(M_\infty(B),|\!|\!|\cdot|\!|\!|)$ and F equal to $(M_\infty(B),|\!|\!|\cdot|\!|\!|)$, to deduce that the action of $\mathbf{q}$ on $M_\infty(B)$ is not $|\!|\!|\cdot|\!|\!|$-continuous at zero. Finally, every associative polynomial $\mathbf{p}$ of degree $\le n$ such that J is not invariant under $\mathbf{p}$ has a homogeneous component $\mathbf{q}$ (obviously of degree $\le n$) such that J is not invariant under $\mathbf{q}$. It follows from the above and Lemma 2 that the action of such a polynomial $\mathbf{p}$ on $M_\infty(B)$ is not $|\!|\!|\cdot|\!|\!|$-continuous.

Assume in addition that B has an involution $*$ and J is contained in $H(B,*)$. Then, for $\mathbf{q}$ and $x_1,\ldots,x_s$ as above and for all k in $\mathbb{N}$, $e_k\otimes x_1$, $\ldots, e_k\otimes x_s$ lie in $H(M_\infty(B),*)$, and therefore the preceding argument shows that the action of $\mathbf{q}$ on $H(M_\infty(B),*)$ is discontinuous at zero for the topology of the norm $|\!|\!|\cdot|\!|\!|$. Again Lemma 2 allows us to obtain the $|\!|\!|\cdot|\!|\!|$-discontinuity of the action on $H(M_\infty(B),*)$ of every associative polynomial $\mathbf{p}$ of degree $\le n$ such that J is not invariant under $\mathbf{p}$. ∎

Remark 1. We began the proof of Proposition 1 by choosing an algebra norm $\|\cdot\|$ on $M_\infty(B)$ satisfying

$$\max\{\|b_{ii}\| : i\in\mathbb{N}\} \le \|(b_{ij})\| \le \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} \|b_{ij}\|$$

for all (b_{ij}) in $M_\infty(B)$, and remarked that a simple example of such a norm is the one given by $\|(b_{ij})\| := \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}}\|b_{ij}\|$. For later applications it is interesting to point out that many other examples of such norms on $M_\infty(B)$ can be given. Precisely, imbedding B isometrically into the normed algebra $BL(X)$ of all bounded linear operators on a suitable normed space X [13, p. 4], we can convert the vector space Y of all quasi-null sequences in X into a normed space by fixing $1\le p\le\infty$ and defining, for $y=\{x_n\}$ in Y, $\|y\| := (\sum_{n\in\mathbb{N}}\|x_n\|^p)^{1/p}$ if $p<\infty$ and $\|y\| := \max\{\|x_n\| : n\in\mathbb{N}\}$ if $p=\infty$. Then the imbedding $B\hookrightarrow BL(X)$ induces naturally an algebraic imbedding $M_\infty(B)\hookrightarrow BL(Y)$, and it is enough to restrict to $M_\infty(B)$ the norm of $BL(Y)$ to obtain an algebra norm on $M_\infty(B)$ with the property mentioned above.

The second step for the proof of our main result is of a purely algebraic nature. We recall that the *free special Jordan algebra* $\mathcal{J}(\mathbf{X})$ on a set of indeterminates $\mathbf{X}$ is nothing but the Jordan subalgebra of $\mathcal{A}(\mathbf{X})$ generated by $\mathbf{X}$. $\mathcal{A}(\mathbf{X})$ has a *standard* (or *main*) *involution* $*$ defined as the only involution on $\mathcal{A}(\mathbf{X})$ fixing the elements of $\mathbf{X}$. It follows immediately that $\mathcal{J}(\mathbf{X})\subseteq H(\mathcal{A}(X),*)$. This inclusion is strict unless $\operatorname{card}(\mathbf{X})\le 3$ (if $\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}$ are pairwise different elements in $\mathbf{X}$, then $\mathbf{xyzt}+\mathbf{tzyx}$ is in $H(\mathcal{A}(\mathbf{X}),*)$ but not in $\mathcal{J}(\mathbf{X})$). Elements in $\mathcal{J}(\mathbf{X})$, for a countably infinite set $\mathbf{X}$, are called *Jordan polynomials*.

Given a field $\mathbb{F}$ and a natural number n, the involution $*$ on $M_{2n}(\mathbb{F})$ defined by $a^* := s^{-1}a^ts$, where a^t denotes the transpose of a and $s := \operatorname{diag}\{q,\overset{n}{\ldots},q\}$ with $q := \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, will be called the *symmetric involution* on $M_{2n}(\mathbb{F})$. Since the symmetric involution is of hermitian type, it is cogredient to the transposition whenever $\mathbb{F}$ is algebraically closed [8, Theorem 4, p. 156]. In the case $\mathbb{F}=\mathbb{R}$ the symmetric involution is not cogredient to the transposition, as one can see by realizing that the matrix s above has zero signature and applying [8, Theorem 6, p. 158]. We also recall the familiar *symplectic involution* $*$ on $M_{2n}(\mathbb{F})$ defined by $a^* := s^{-1}a^ts$, where now $s := \operatorname{diag}\{q,\overset{n}{\ldots},q\}$ with $q := \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. It is the standard representative of the unique cogredience class of involutions on $M_{2n}(\mathbb{F})$ of alternate type [8, Theorem 7, p. 161].

PROPOSITION 2. *Let $\mathbb{F}$ be a field, n and m be natural numbers, and, for every natural number p, let $*$ denote either the symmetric or the symplectic involution on $M_{2p}(\mathbb{F})$. Then there exists a natural number d, together with a Jordan subalgebra J of $M_{2d}(\mathbb{F})$ contained in $H(M_{2d}(\mathbb{F}),*)$, such that J is not invariant under any non-Jordan associative polynomial of degree $\le n$ involving at most m indeterminates.*

Proof. Consider a set $\mathbf{Y}$ of cardinality m (say $\mathbf{Y} := \{\mathbf{y}_1,\dots,\mathbf{y}_m\}$) and write $\mathcal{A}(\mathbf{Y}) = \bigoplus_{k\in\mathbb{N}} \mathcal{A}_k$, where $\mathcal{A}_k$ is the vector space of all homogeneous elements in $\mathcal{A}(\mathbf{Y})$ of degree k, so that, if $*$ denotes the main involution on $\mathcal{A}(\mathbf{Y})$, then all the $\mathcal{A}_k$ ($k\in\mathbb{N}$) are $*$-invariant finite-dimensional subspaces of $\mathcal{A}(\mathbf{Y})$. Let I be the $*$-ideal of $\mathcal{A}(\mathbf{Y})$ defined by $I := \bigoplus_{k>n}\mathcal{A}_k$, and write $A := \mathcal{A}(\mathbf{Y})/\mathbf{I}$ and $J := \pi(\mathcal{J}(\mathbf{Y}))$, where π denotes the quotient mapping $\mathcal{A}(\mathbf{Y}) \to \mathcal{A}(\mathbf{Y})/I$. Then A is a finite-dimensional associative algebra with an involution (also denoted by $*$) and J is a Jordan subalgebra of A contained in $H(A,*)$. We claim that J is not invariant under any non-Jordan associative polynomial of degree $\le n$ involving at most m indeterminates. Assume on the contrary that J is invariant under a non-Jordan associative polynomial $\mathbf{p} = \mathbf{p}(\mathbf{x}_1,\dots,\mathbf{x}_s)$ of degree g with $1\le s\le m$ and $1\le g\le n$. Then $\mathbf{p}(\pi(\mathbf{y}_1),\dots,\pi(\mathbf{y}_s)) = \pi(\mathbf{p}(\mathbf{y}_1,\dots,\mathbf{y}_s))$ lies in J, so there is $\mathbf{q}$ in $\mathcal{J}(\mathbf{Y})$ such that $\mathbf{p}(\mathbf{y}_1,\dots,\mathbf{y}_s)-\mathbf{q}$ belongs to I, and therefore, if for k in $\mathbb{N}$ we denote by $\mathbf{q}_k$ the k-homogeneous component of $\mathbf{q}$, we have $\mathbf{p}(\mathbf{y}_1,\dots,\mathbf{y}_s) = \mathbf{q}_1+\dots+\mathbf{q}_g$. Since $\mathbf{q}$ is in $\mathcal{J}(\mathbf{Y})$, every homogeneous component of $\mathbf{q}$ lies in $\mathcal{J}(\mathbf{Y})$ [9, pp. 7–8] and therefore so does $\mathbf{p}(\mathbf{y}_1,\dots,\mathbf{y}_s)$. Now a standard universal-algebra argument shows that $\mathbf{p} = \mathbf{p}(\mathbf{x}_1,\dots,\mathbf{x}_s)$ is a Jordan polynomial, contrary to the assumption.

The proof is completed by showing that, for some natural number d and for both the symmetric and symplectic involution (say also $*$) on $M_{2d}(\mathbb{F})$, there is a one-to-one $*$-homomorphism from A into $M_{2d}(\mathbb{F})$. To this end we follow the lines of the proof of the main result in [12]. Denote by A_1 the unital hull of A, let A_1' be the dual of the vector space of A_1, and, for a in A, consider the linear mappings $\Lambda_a : A_1 \to A_1$ and $\Gamma_a : A_1' \to A_1'$ defined by $\Lambda_a(x) := ax$ for all x in A_1 and $\Gamma_a(f) := f\circ\Lambda_{a^*}$ for all f in A_1'. Then $a\mapsto\Lambda_a$ and $a\mapsto\Gamma_a$ are faithful representations of A on the vector spaces A_1 and A_1', respectively, so that we can consider the (automatically faithful) "direct sum" representation (say $T : a\mapsto T_a$), namely the one of A on $V := A_1\oplus A_1'$ given by $T_a(x,f) := (\Lambda_a(x),\Gamma_a(f))$ for all a in A and (x,f) in V. Given $\varepsilon=\pm1$, the mapping $\langle\cdot,\cdot\rangle$ from $V\times V$ into $\mathbb{F}$ defined by

$$\langle(x,f),(y,g)\rangle := f(y)+\varepsilon g(x)$$

is a non-degenerate ε-hermitian form, i.e., a non-degenerate bilinear form on V satisfying $\langle v_1,v_2\rangle = \varepsilon\langle v_2,v_1\rangle$ for all v_1,v_2 in V. Moreover, for all a in A and $(x,f),(y,g)$ in V, we have

$$\langle T_a(x,f),(y,g)\rangle = \langle(x,f),T_{a^*}(y,g)\rangle,$$

so T_{a^*} is the adjoint operator of T_a relative to $\langle\cdot,\cdot\rangle$, and so T is actually a $*$-representation of A on the ε-self-paired vector space $(V,\langle\cdot,\cdot\rangle)$. Denoting by d the dimension of A_1, taking a basis $\{e_1,\dots,e_d\}$ for A_1, considering the corresponding dual basis $\{\phi_1,\dots,\phi_d\}$ in A_1', and choosing $\{\phi_1,e_1,\phi_2,e_2,\dots,\phi_d,e_d\}$ as a basis for V, linear operators on V are naturally identified with elements in $M_{2d}(\mathbb{F})$ in such a way that, for $\varepsilon=1$ (respectively, -1) taking the adjoint operator relative to $\langle\cdot,\cdot\rangle$ becomes the symmetric (respectively, symplectic) involution on $M_{2d}(\mathbb{F})$. ∎

Both the symmetric and the symplectic involution pass from matrix algebras of the form $M_{2n}(\mathbb{F})$ ($n\in\mathbb{N}$) to $M_\infty(\mathbb{F})$ by simply considering the equality $M_\infty(\mathbb{F}) = \bigcup_{n\in\mathbb{N}} M_{2n}(\mathbb{F})$.

THEOREM. *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, n and m be natural numbers, and denote by $*$ either the symmetric or the symplectic involution on $M_\infty(\mathbb{K})$. Then there exists a Jordan-algebra norm on $M_\infty(\mathbb{K})$ making discontinuous the action on $H(M_\infty(\mathbb{K}),*)$ of every non-Jordan associative polynomial of degree $\le n$ involving at most m indeterminates.*

Proof. By Proposition 2, there exist a natural number d and a subalgebra J of the Jordan algebra $H(M_{2d}(\mathbb{K}),*)$ such that J is not invariant under any non-Jordan associative polynomial of degree $\le n$ involving at most m indeterminates. Writing $B := M_{2d}(\mathbb{K})$ and choosing an arbitrary algebra norm $\|\cdot\|$ on B, the subalgebra J becomes closed in the normed associative algebra $(B,\|\cdot\|)$ because B is finite-dimensional. Now it is enough to apply Proposition 1 to obtain the existence of a Jordan-algebra norm on $M_\infty(B)$ making discontinuous the action on $H(M_\infty(B),*)$ of every non-Jordan associative polynomial of degree $\le n$ involving at most m indeterminates. Finally, observe that $(M_\infty(B),*)$ is $*$-isomorphic to $(M_\infty(\mathbb{K}),*)$. ∎

The next two corollaries are direct consequences of the Theorem. The first one is a negative answer to the "norm-extension problem" for simple algebras which improves Theorem 7 of [4], whereas the second one explains how Jordan polynomials can be analytically recognized.

COROLLARY 1. *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and denote by $*$ either the symmetric or the symplectic involution on $M_\infty(\mathbb{K})$. Then there exists a Jordan-algebra norm $|\cdot|$ on $M_\infty(\mathbb{K})$ such that there is no (associative-) algebra norm on $M_\infty(\mathbb{K})$ whose restriction to $H(M_\infty(\mathbb{K}),*)$ is equivalent to the restriction of $|\cdot|$ to $H(M_\infty(\mathbb{K}),*)$.*

COROLLARY 2. *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and $\mathbf{p}$ be an associative polynomial over $\mathbb{K}$. Then the following assertions are equivalent:*

(i) $\mathbf{p}$ *is a Jordan polynomial.*

(ii) *For every associative algebra A over $\mathbb{K}$, and for every Jordan-algebra norm $|\cdot|$ on A, the action of $\mathbf{p}$ on A is $|\cdot|$-continuous.*

(iii) *For every Jordan-algebra norm $|\cdot|$ on $M_\infty(\mathbb{K})$, the action of $\mathbf{p}$ on $M_\infty(\mathbb{K})$ is $|\cdot|$-continuous.*

Remark 2. Choose in the proof of the Theorem the algebra norm $\|\cdot\|$ on $B := M_{2d}(\mathbb{K})$ equal to the operator norm when B is regarded as the algebra of all bounded linear operators on the Hilbert space $X := \mathbb{K}^{2d}$, and, according to Remark 1, choose in the proof of Proposition 1 the corresponding algebra norm $\|\cdot\|$ on $M_\infty(B)$ equal to the operator norm when $M_\infty(B)$ is regarded as the algebra of all bounded linear operators on the space of quasi-null sequences in X endowed with the ℓ_2-norm. Then we can argue as in Section 3 of [4] so that, if we denote by $|\cdot|$ the pathological Jordan-algebra norm on $\mathcal{M} := M_\infty(\mathbb{K}) \cong M_\infty(B)$, then the Jordan–Banach algebra completion $\mathcal{J}$ of the Jordan normed algebra $(\mathcal{M}^+, |\cdot|)$ is prime non-degenerate and has non-zero socle [10]. Moreover, $\mathcal{J}$ can be viewed as a Jordan subalgebra of the associative algebra $\mathcal{K}(H)$ of all compact operators on the infinite-dimensional separable Hilbert space H over $\mathbb{K}$, and the socle of $\mathcal{J}$ is a central simple associative subalgebra of $\mathcal{K}(H)$ containing $\mathcal{M}$. Since also both the symmetric and the symplectic involution $*$ on $\mathcal{M}$ can be uniquely extended to involutions on $\mathcal{K}(H)$ in such a way that $\mathcal{J}$ remains $*$-invariant, it follows in particular that the Theorem remains true if $M_\infty(\mathbb{K})$ is replaced by a suitable central simple associative algebra A over $\mathbb{K}$ such that A^+ is algebraically isomorphic to the socle of a prime non-degenerate Jordan–Banach algebra, and $*$ is a suitable involution on A of arbitrary prefixed type. We note that, in view of Baire's theorem and results in [10] and [11], $M_\infty(\mathbb{K})^+$ cannot be isomorphic to the socle of a prime non-degenerate Jordan–Banach algebra.

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