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Accretive approximation in C^*-algebras

by

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Abstract. The problem of approximation by accretive elements in a unital C^*-algebra suggested by P. R. Halmos is substantially solved. The key idea is the observation that accretive approximation can be regarded as a combination of positive and self-adjoint approximation. The approximation results are proved both in the C^*-norm and in another, topologically equivalent norm.

1. Introduction. For every unital C^*-algebra $\mathcal{A}$ let $Acc_{\mathcal{A}}$ be the set of all accretive elements of $\mathcal{A}$, i.e. the set of all elements with positive real part. For an element a of $\mathcal{A}$ let $Acc_{\mathcal{A}}(a)$ denote the set of all accretive approximants of a. Here an *approximant* means an element x of $Acc_{\mathcal{A}}$ such that $\|a-x\| \le \|a-y\|$ for every element y of $Acc_{\mathcal{A}}$. Furthermore, let the norm $|\!|\!|\cdot|\!|\!|$ be defined by $|\!|\!|a|\!|\!| = \left\|\frac{1}{2}(a^*a+aa^*)\right\|^{1/2}$ (cf. [Bo 2, Be 1]). The accretive approximants in this norm will be called *accretive near-approximants*; the set of all accretive near-approximants will be denoted by $\tilde{A}cc_{\mathcal{A}}(a)$.

The main purpose of this paper is to describe the sets $Acc_{\mathcal{A}}(a)$ and $\tilde{A}cc_{\mathcal{A}}(a)$. The key idea is the observation that accretive approximation is a combination of positive and self-adjoint approximation (Theorem 2.1(c)). As a consequence the real dimensions of the convex sets $Acc_{\mathcal{B}(\mathcal{H})}(A)$ and $\tilde{A}cc_{\mathcal{B}(\mathcal{H})}(A)$ can be computed for every bounded linear operator A on a complex Hilbert space $\mathcal{H}$, and some extreme points can be constructed.

2. Accretive approximation in C^*-algebras. Let $\mathcal{A}$ be a unital C^*-algebra. Then $\mathcal{S}_{\mathcal{A}}$ denotes the set of all self-adjoint elements of $\mathcal{A}$. For every element $a \in \mathcal{A}$ let $\mathcal{S}_{\mathcal{A}}(a)$ (respectively $\tilde{\mathcal{S}}_{\mathcal{A}}(a)$) be the set of all self-adjoint approximants (respectively self-adjoint near-approximants) of a. Similarly $\mathcal{P}_{\mathcal{A}}$ denotes the set of all positive elements of $\mathcal{A}$, and $\mathcal{P}_{\mathcal{A}}(a)$ (respectively $\tilde{\mathcal{P}}_{\mathcal{A}}(a)$) denotes the set of all positive approximants (respectively near-approximants) of a.

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In the case where $\mathcal{A}$ is the C^*-algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ the index $\mathcal{A}$ will be skipped if that does not lead to any confusion.

In this section the problem of accretive approximation suggested by P. R. Halmos (see [Ha]) is discussed for unital C^*-algebras. P. R. Halmos showed that the accretive part of a is an accretive approximant of a in the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$ (cf. Theorem 2.1(b)).

For an element a of a C^*-algebra let $\operatorname{dist}(a, \mathcal{G})$ be the distance from a to a given subset $\mathcal{G}$ of $\mathcal{A}$ measured in the C^*-norm, whereas $\operatorname{dist}_{\|\!|\cdot|\!\|}(a, \mathcal{G})$ denotes the distance measured in the norm $\|\!|\cdot|\!\|$.

First some basic results about the positive and self-adjoint approximation are mentioned. In [Be 1] it is shown that for a normal element a of $\mathcal{A}$ the positive part of the real part is a positive approximant and the real part of a is a self-adjoint approximant (see Theorem 2.1(c)). In the case of self-adjoint approximation the condition of a being normal can be dropped. For $a = b + ic$ with b, c self-adjoint one has $\operatorname{dist}(b + ic, \mathcal{S}_\mathcal{A}) = \operatorname{dist}(ic, \mathcal{S}_\mathcal{A}) = \|c\|$ and $\mathcal{S}_\mathcal{A}(b + ic) = b + \mathcal{S}_\mathcal{A}(ic)$ since the set $\mathcal{S}_\mathcal{A}$ of all self-adjoint elements of $\mathcal{A}$ forms a real vector space. Hence $\|b + ic\| \geq \|c\|$ and $b \in \mathcal{S}_\mathcal{A}(b + ic)$ for b, c self-adjoint. Similar results hold for approximation in the norm $\|\!|\cdot|\!\|$.

THEOREM 2.1. *Let $\mathcal{A}$ be a unital C^*-algebra and let $a = b + ic \in \mathcal{A}$ with b, c self-adjoint. Then*

(a) $\operatorname{dist}(a, \mathcal{A}cc_\mathcal{A}) = \operatorname{dist}_{\|\!|\cdot|\!\|}(a, \mathcal{A}cc_\mathcal{A}) = \operatorname{dist}(b, \mathcal{P}_\mathcal{A})$.

(b) $b^+ + ic \in \mathcal{A}cc_\mathcal{A}(a) \subset \tilde{\mathcal{A}}cc_\mathcal{A}(a)$, *where b^+ denotes the positive part of b.*

(c) $\mathcal{A}cc_\mathcal{A}(a) = \{p + id : p \in \mathcal{P}_\mathcal{A}(b)$ *and* $d \in \mathcal{S}_\mathcal{A}(c - i(b - p))\}$ *and*
$\tilde{\mathcal{A}}cc_\mathcal{A}(a) = \{p + id : p \in \tilde{\mathcal{P}}_\mathcal{A}(b)$ *and* $d \in \tilde{\mathcal{S}}_\mathcal{A}(c - i(b - p))\}$.

Proof. Let $p + id \in \mathcal{A}cc_\mathcal{A}$ with p positive and d self-adjoint. Then, as in the proof of [Ha], Corollary 5,

$$\begin{aligned}\|a - (p + id)\| &= \|(b - p) + i(c - d)\| \geq \|b - p\| \\ &\geq \|b - b^+\| = \|(b + ic) - (b^+ + ic)\|.\end{aligned}$$

Thus $\operatorname{dist}(a, \mathcal{A}cc_\mathcal{A}) = \|b - b^+\| = \operatorname{dist}(b, \mathcal{P}_\mathcal{A})$. Moreover, the accretive part $b^+ + ic$ of a is an accretive approximant of a. Similarly $\operatorname{dist}_{\|\!|\cdot|\!\|}(a, \mathcal{A}cc_\mathcal{A}) = \|\!|b - b^+|\!\| = \|b - b^+\|$ and $b^+ + ic$ is an accretive near-approximant of a. This shows (a) and (b).

(c) Suppose $p + id$ is an accretive approximant of a with p positive and d self-adjoint. Then

$$\|b - b^+\| = \|a - (p + id)\| = \|(b - p) + i(c - d)\| \geq \|b - p\|,$$

i.e., p is a positive approximant of b and

$$\|b - p\| = \|(b - p) + i(c - d)\|.$$

Thus

$$\begin{aligned}\operatorname{dist}(c - i(b - p)), \mathcal{S}_\mathcal{A}) &= \|\operatorname{Im}(c - i(b - p))\| = \|b - p\| \\ &= \|(b - p) + i(c - d)\| = \|(c - i(b - p)) - d\|,\end{aligned}$$

which means that d is a self-adjoint approximant of $c - i(b - p)$. Here $\operatorname{Im}(x)$ denotes the imaginary part of x.

Hence

$$\mathcal{A}cc_\mathcal{A}(a) \subset \{p + id : p \in \mathcal{P}_\mathcal{A}(b) \text{ and } d \in \mathcal{S}_\mathcal{A}(c - i(b - p))\}.$$

For the reverse inclusion suppose $p \in \mathcal{P}_\mathcal{A}(b)$ and $d \in \mathcal{S}_\mathcal{A}(c - i(b - p))$. Then

$$\begin{aligned}\|a - (p + id)\| &= \|(b - p) + i(c - d)\| = \|(c - i(b - p)) - d\| \\ &= \|\operatorname{Im}(c - i(b - p))\| = \|b - p\| = \operatorname{dist}(b, \mathcal{P}_\mathcal{A}),\end{aligned}$$

i.e., $p + id \in \mathcal{A}cc_\mathcal{A}(a)$ by part (a).

This shows $\mathcal{A}cc_\mathcal{A}(a) = \{p + id : p \in \mathcal{P}_\mathcal{A}(b)$ and $d \in \mathcal{S}_\mathcal{A}(c - i(b - p))\}$ and similarly $\tilde{\mathcal{A}}cc_\mathcal{A}(a) = \{p + id : p \in \tilde{\mathcal{P}}_\mathcal{A}(b)$ and $d \in \tilde{\mathcal{S}}_\mathcal{A}(c - i(b - p))\}$. ■

As a consequence of the above theorem the properties of the sets $\mathcal{A}cc_\mathcal{A}(a)$ and $\tilde{\mathcal{A}}cc_\mathcal{A}(a)$ can be described by using the results for self-adjoint and positive approximation.

COROLLARY 2.2. *Let $\mathcal{A}$ be a unital C^*-algebra and let $a = b + ic \in \mathcal{A}$ with b, c self-adjoint.*

(a) *The following conditions are equivalent:*

(1) $0 \in \mathcal{A}cc_\mathcal{A}(a)$;
(2) $-\|a\| \in \sigma(b)$.

(b) *The following conditions are equivalent:*

(1) $0 \in \tilde{\mathcal{A}}cc_\mathcal{A}(a)$;
(2) $-\|\!|a|\!\| \in \sigma(b)$.

Proof. By Theorem 2.1(a) one has $\operatorname{dist}(a, \mathcal{A}cc_\mathcal{A}) = \|b^-\|$, where b^- denotes the negative part of b. Thus $0 \in \mathcal{A}cc_\mathcal{A}(a)$ if and only if $\|a\| = \|b^-\|$, i.e., if and only if $-\|a\| \in \sigma(b)$. Part (b) can be proved in the same way. ■

COROLLARY 2.3. *Let $\mathcal{A}$ be a unital C^*-algebra and let $a = b + ic \in \mathcal{A}$ with b, c self-adjoint. Then the following conditions are equivalent:*

(1) *a has a unique accretive approximant;*
(2) *a has a unique accretive near-approximant;*
(3) *there exists a non-negative number δ such that* $\operatorname{dist}(\lambda, \mathbb{R}^{\geq 0}) = \delta$ *for every* $\lambda \in \sigma(b)$;
(4) *b has a unique positive approximant;*
(5) *b has a unique positive near-approximant.*

Proof. The equivalence of (3), (4) and (5) follows from Theorem 2.5 of [Be 1].

By Theorem 2.1 the implications (2)⇒(1) and (1)⇒(4) hold.

For the implication (3)⇒(2) suppose that there exists a positive number δ such that $\operatorname{dist}(\lambda, \mathbb{R}^{\geq 0}) = \delta$ for every $\lambda \in \sigma(b)$. Then b has a unique positive near-approximant (see [Be 1], Theorem 2.5).

If $\delta = 0$ then $\sigma(b) \subset \mathbb{R}^{\geq 0}$, i.e. b is the unique positive near-approximant of b. Thus $\operatorname{dist}_{\|\cdot\|}(a, \tilde{\mathcal{A}cc}_{\mathcal{A}}(a)) = \|a - (b + ic)\| = 0$, i.e., $b + ic$ is the unique accretive near-approximant of a.

If $\delta > 0$ then $\sigma(b) = \{-\delta\}$, i.e., $b = -\delta$ and 0 is the unique positive near-approximant of b. Moreover, $c + i\delta$ has the unique self-adjoint near-approximant c by Theorem 2.5 of [Be 1]. Hence ic is the unique accretive near-approximant of a by Theorem 2.1(c). This completes the proof. ■

3. Dimension of the set of approximants. In this section the dimension of the convex sets $\mathcal{A}cc_{\mathcal{A}}(a)$ and $\tilde{\mathcal{A}cc}_{\mathcal{A}}(a)$ is discussed for every element $a \in \mathcal{A}$. Recall that the *dimension* of a convex set X is the (real) dimension of the smallest (real) affine subspace including X (cf. [Val]).

Similar to the case of positive and self-adjoint approximation, the dimension of the set of accretive approximants depends heavily on the C^*-algebra $\mathcal{A}$ (see [Be 3], Example 2.1, for positive approximation). However, for the C^*-algebra $\mathcal{B}(\mathcal{H})$ the dimension can be computed by using techniques similar to [Be 3].

In the following remark some notations and properties of positive and self-adjoint approximation in $\mathcal{B}(\mathcal{H})$ are listed which will be used in this section. They can be found in [Bo 2] and [Bo 3].

Remark 3.1. For every operator $A \in \mathcal{B}(\mathcal{H})$ the space $\max A$ is defined by $\max A = \{x \in \mathcal{H} : \|Ax\| = \|A\| \cdot \|x\|\} = \ker(\|A\|^2 - A^*A)$.

If $A = B + iC$ with B, C self-adjoint, then the subspace $\mathcal{H}_1 = \mathcal{H}_1(A)$ is defined by $\mathcal{H}_1(A) = \overline{\operatorname{ran}}(\|C\|^2 - C^2) = (\max C)^{\perp}$.

If S is a self-adjoint near-approximant of A, then $\mathcal{H}_1$ reduces $B - S$ with $(B - S)|_{\mathcal{H}_1^{\perp}} = 0$ and $\mathcal{H}_1^{\perp} \subset \max(A - S)$, since $\operatorname{ran}(B - S) \subset \mathcal{H}_1$ and $\max(A - B) = \mathcal{H}_1^{\perp}$.

Moreover, the subspace $\mathcal{H}_2 = \mathcal{H}_2(A)$ is given by $\mathcal{H}_2(A) = \overline{\operatorname{ran}}\, D_0 \cap \overline{\operatorname{ran}}\, P_0$ with $\delta = \operatorname{dist}(A, \mathcal{P})$, $D_0 = (\delta^2 - C^2)^{1/2}$ and $P_0 = B + D_0$.

If P is a positive near-approximant of A, then $\mathcal{H}_2$ reduces $P_0 - P$ with $(P_0 - P)|_{\mathcal{H}_2^{\perp}} = 0$ and $\mathcal{H}_2^{\perp} \subset \max(A - P)$, since $\operatorname{ran}(P_0 - P) \subset \mathcal{H}_2$ and $\max(A - P_0) = \ker(\delta^2 - (A - P_0)^*(A - P_0)) = \mathcal{H}$.

LEMMA 3.2. *Let $A = B + iC \in \mathcal{B}(\mathcal{H})$ with B, C self-adjoint and let $\mathcal{H}_2 = \mathcal{H}_2(B)$. Then*

$$\dim_{\mathbb{R}} \mathcal{A}cc(A) \leq \dim_{\mathbb{R}} \tilde{\mathcal{A}cc}(A) \leq 2 \cdot (\dim_{\mathbb{C}} \mathcal{H}_2)^2.$$

Proof. Since $\mathcal{A}cc(A) \subset \tilde{\mathcal{A}cc}(A)$ the first inequality follows immediately.

Suppose now that $T = P + iD$ is an accretive near-approximant of A with P positive and D self-adjoint. Then the subspace $\mathcal{H}_2 = \mathcal{H}_2(B)$ reduces $P_0 - P$ with $(P_0 - P)|_{\mathcal{H}_2^{\perp}} = 0$ and $\mathcal{H}_2^{\perp} \subset \max(B - P)$, since P is a positive near-approximant of B by Theorem 2.1(c). Since B^+ is also a positive near-approximant of B this implies that $\mathcal{H}_2$ reduces $P - B^+$ with $(P - B^+)|_{\mathcal{H}_2^{\perp}} = 0$.

Moreover, the subspace $\mathcal{H}_1 = \mathcal{H}_1(C - i(B - P)) = \max(B - P)^{\perp} \subset \mathcal{H}_2$ reduces $C - D$ with $(C - D)|_{\mathcal{H}_1^{\perp}} = 0$, since D is a self-adjoint near-approximant of $C - i(B - P)$.

This shows that the set $\tilde{\mathcal{A}cc}(A) - (B^+ + iC)$ is included in the set

$$\{P' + iD' : P' \in \mathcal{B}(\mathcal{H}) \text{ positive with } \overline{\operatorname{ran}}\, P' \subset \mathcal{H}_2 \text{ and } D' \in \mathcal{B}(\mathcal{H}) \text{ self-adjoint with } \overline{\operatorname{ran}}\, D' \subset \mathcal{H}_2\},$$

which has real dimension $2 \cdot (\dim_{\mathbb{C}} \mathcal{H}_2)^2$. ■

THEOREM 3.3. *Let $A = B + iC \in \mathcal{B}(\mathcal{H})$ with B, C self-adjoint and let $\mathcal{H}_2 = \mathcal{H}_2(B)$. Then*

$$\dim_{\mathbb{R}} \mathcal{A}cc(A) = \dim_{\mathbb{R}} \tilde{\mathcal{A}cc}(A) = 2 \cdot (\dim_{\mathbb{C}} \mathcal{H}_2)^2.$$

Proof. In view of the previous lemma it suffices to prove that $\dim_{\mathbb{R}} \mathcal{A}cc(A) \geq 2 \cdot (\dim_{\mathbb{C}} \mathcal{H}_2)^2$. Fix $n \in \mathbb{N}$ and define the subspace M_n by $M_n := \operatorname{ran} E(D_n)$, where E denotes the spectral measure for B and $D_n := \{\lambda \in \sigma(B) : \operatorname{dist}(\lambda, \mathbb{R}^{\geq 0}) < \operatorname{d}(\sigma(B), \mathbb{R}^{\geq 0}) - 2/n\}$. Here $\operatorname{d}(\sigma(B), \mathbb{R}^{\geq 0})$ denotes the one-sided Hausdorff distance from $\sigma(B)$ to $\mathbb{R}^{\geq 0}$. Then M_n is a B-reducing subspace of $\mathcal{H}_2 = \mathcal{H}_2(B)$ since $\mathcal{H}_2(B) = \mathcal{H}_0(\mathbb{R}^{\geq 0}, B) = \operatorname{ran} E(\{\lambda \in \sigma(B) : \operatorname{dist}(\lambda, \mathbb{R}^{\geq 0}) < \operatorname{d}(\sigma(B), \mathbb{R}^{\geq 0})\})$.

For every positive operator $P' \in \mathcal{B}(\mathcal{H})$ with $\operatorname{ran} P' \subset M_n$ and $\|P'\| \leq 1/n$ and for every self-adjoint operator $D' \in \mathcal{B}(\mathcal{H})$ with $\operatorname{ran} D' \subset M_n$ and $\|D'\| \leq 1/n$ the operator $T = (B^+ + P') + i(C + D')$ is accretive. Moreover,

$$\begin{aligned}
\|A - T\| &= \|(B + iC) - (B^+ + P') - i(C + D')\| \\
&= \|B - (B^+ + P') - iD'\| \\
&= \max\{\|(B - (B^+ + P') - iD')|_{M_n}\|, \|(B - (B^+ + P') - iD')|_{M_n^{\perp}}\|\} \\
&\leq \max\{\|(B - B^+)|_{M_n}\| + \|P'|_{M_n}\| + \|D'|_{M_n}\|, \|(B - B^+)|_{M_n^{\perp}}\|\} \\
&\leq \max\{\|B^-|_{\operatorname{ran} E(D_n)}\| + 2/n, \|B - B^+\|\} \\
&\leq \operatorname{d}(\sigma(B), \mathbb{R}^{\geq 0}) = \operatorname{dist}(B, \mathcal{P}_{\mathcal{A}})
\end{aligned}$$

by the spectral theorem and by the distance formula 1.4 of [Be 1]. Hence T is an accretive approximant of A. Thus the set $\mathcal{A}cc(A)-(B^{+}+iC)$ includes the set

$$\{P'+iD' : P'\in\mathcal{B}(\mathcal{H}) \text{ positive with } \overline{\operatorname{ran}}\,P'\subset M_n \text{ and } \|P'\|\le 1/n \text{ and } D'\in\mathcal{B}(\mathcal{H}) \text{ self-adjoint with } \overline{\operatorname{ran}}\,D'\subset M_n \text{ and } \|D'\|\le 1/n\},$$

which has real dimension $2\cdot(\dim_{\mathbb{C}} M_n)^2$. Since $\mathcal{H}_2=\mathcal{H}_2(B)=\overline{\bigcup_{n\in\mathbb{N}} M_n}$, one has $\lim_{n\to\infty}\dim_{\mathbb{C}} M_n=\dim_{\mathbb{C}}\mathcal{H}_2$, so that

$$\dim_{\mathbb{R}}\mathcal{A}cc(A)\ge 2\cdot(\dim_{\mathbb{C}}\mathcal{H}_2)^2.$$

This completes the proof. ■

4. Extreme points in the set of approximants. In this section the extreme points of the convex sets $\mathcal{A}cc_{\mathcal{A}}(a)$ and $\tilde{\mathcal{A}}cc_{\mathcal{A}}(a)$ are studied for every element a of the unital C^*-algebra $\mathcal{A}$.

First, sufficient conditions are specified for an approximant to be such an extreme point:

PROPOSITION 4.1. *Let $\mathcal{A}$ be a unital C^*-algebra and let $a\in\mathcal{A}$.*

(a) *If $t=p+id\in\mathcal{A}$ is an accretive approximant of a with p positive and d self-adjoint such that $(a-t)^*(a-t)$ or $(a-t)(a-t)^*$ is a scalar, then $t\in\operatorname{ex}\mathcal{A}cc_{\mathcal{A}}(a)$.*

(b) *If $t=p+id\in\mathcal{A}$ is an accretive near-approximant of a with p positive and d self-adjoint such that $(a-t)^*(a-t)+(a-t)(a-t)^*$ is a scalar, then $t\in\operatorname{ex}\tilde{\mathcal{A}}cc_{\mathcal{A}}(a)$.*

Proof. (a) It is a well-known result that every element $x\in\mathcal{A}$ with $x^*x=e$ or $xx^*=e$ is an extreme point of the closed unit ball (see e.g. [Pe], Prop. 1.4.7). Hence the element $a-t$ is an extreme point of the closed ball with center 0 and radius $\|a-t\|$. Since the set $a-\mathcal{A}cc_{\mathcal{A}}(a)$ is included in this ball, the element t is an extreme point of $\mathcal{A}cc_{\mathcal{A}}(a)$.

Similarly part (b) follows from the proposition below. ■

PROPOSITION 4.2. *Let $\mathcal{A}$ be a C^*-algebra with unit e and let $\mathcal{A}_1:=\{a\in\mathcal{A}:\|a\|\le 1\}$ be the closed unit ball in $\mathcal{A}$ for the norm $\|\cdot\|$. Then every element $x\in\mathcal{A}$ with $\frac{1}{2}(x^*x+xx^*)=e$ is an extreme point of $\mathcal{A}_1$.*

Proof. Using the Theorem of Gelfand–Neumark it suffices to prove the statement for $\mathcal{A}=\mathcal{B}(\mathcal{H})$.

Suppose $T\in\mathcal{B}(\mathcal{H})$ with $\frac{1}{2}(T^*T+TT^*)=\mathrm{Id}$. Then $\|T\|=\|\mathrm{Id}\|=1$, i.e. $T\in\mathcal{B}(\mathcal{H})_1$. If $R\in\mathcal{B}(\mathcal{H})$ with $\|T-R\|\le 1$ and $\|T+R\|\le 1$, then Lemma 2.1(b) of [Be 2] implies that $\max(T^*T+TT^*)\subset\ker R$, which means that $R=0$. Hence T is an extreme point of $\mathcal{B}(\mathcal{H})_1$. ■

As a consequence of Proposition 4.1 and Theorem 2.1 some extreme points of $\mathcal{A}cc_{\mathcal{A}}(a)$ and $\tilde{\mathcal{A}}cc_{\mathcal{A}}(a)$ can easily be constructed:

COROLLARY 4.3. *Let $\mathcal{A}$ be a unital C^*-algebra and let $a=b+ic\in\mathcal{A}$ with b,c self-adjoint. Furthermore, let $p\in\mathcal{P}_{\mathcal{A}}(b)=\tilde{\mathcal{P}}_{\mathcal{A}}(b)$. Then*

$$k_p^{(+)}:=p+i((\|b-p\|^2-(b-p)^2)^{1/2}+c)$$

and

$$k_p^{(-)}:=p-i((\|b-p\|^2-(b-p)^2)^{1/2}+c)$$

are both extreme points of $\mathcal{A}cc_{\mathcal{A}}(a)$ and $\tilde{\mathcal{A}}cc_{\mathcal{A}}(a)$.

COROLLARY 4.4. *Let $\mathcal{A}$ be a unital C^*-algebra and let $a\in\mathcal{A}$. Then either a has a unique accretive approximant and near-approximant or $\mathcal{A}cc_{\mathcal{A}}(a)$ and $\tilde{\mathcal{A}}cc_{\mathcal{A}}(a)$ have uncountably many extreme points.*

Proof. If a does not have a unique accretive approximant or near-approximant then $b:=\operatorname{Re}a$ does not have a unique positive approximant by Corollary 2.3. This implies that b has uncountably many positive approximants since the set of all positive approximants is convex. Hence $\mathcal{A}cc_{\mathcal{A}}(a)$ and $\tilde{\mathcal{A}}cc_{\mathcal{A}}(a)$ have uncountably many extreme points by the previous corollary. ■

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