Accretive approximation in $C^*$-algebras

by

RAINER BERNTZEN (Münster)

Abstract. The problem of approximation by accretive elements in a unital $C^*$-algebra suggested by P. R. Halmos is substantially solved. The key idea is the observation that accretive approximation can be regarded as a combination of positive and self-adjoint approximation. The approximation results are proved both in the $C^*$-norm and in another, topologically equivalent norm.

1. Introduction. For every unital $C^*$-algebra $A$ let $\text{Acc}_A$ be the set of all accretive elements of $A$, i.e. the set of all elements with positive real part. For an element $a$ of $A$ let $\text{Acc}_A(a)$ denote the set of all accretive approximants of $a$. Here an approximant means an element $x$ of $\text{Acc}_A$ such that $\|a-x\| \leq \|a-y\|$ for every element $y$ of $\text{Acc}_A$. Furthermore, let the norm $\|\cdot\|$ be defined by $\|a\| = \|\frac{1}{2}(a^*a+aa^*)\|^{1/2}$ (cf. [Bo 2, Be 1]). The accretive approximants in this norm will be called accretive near-approximants; the set of all accretive near-approximants will be denoted by $\tilde{\text{Acc}}_A(a)$.

The main purpose of this paper is to describe the sets $\text{Acc}_A(a)$ and $\tilde{\text{Acc}}_A(a)$. The key idea is the observation that accretive approximation is a combination of positive and self-adjoint approximation (Theorem 2.1(c)). As a consequence the real dimensions of the convex sets $\text{Acc}_{BG}(H)(A)$ and $\tilde{\text{Acc}}_{BG}(H)(A)$ can be computed for every bounded linear operator $A$ on a complex Hilbert space $H$, and some extreme points can be constructed.

2. Accretive approximation in $C^*$-algebras. Let $A$ be a unital $C^*$-algebra. Then $\mathcal{S}_A$ denotes the set of all self-adjoint elements of $A$. For every element $a \in A$ let $\mathcal{S}_A(a)$ (respectively $\tilde{\mathcal{S}}_A(a)$) be the set of all self-adjoint approximants (respectively self-adjoint near-approximants) of $a$. Similarly $\mathcal{P}_A$ denotes the set of all positive elements of $A$, and $\mathcal{P}_A(a)$ (respectively $\tilde{\mathcal{P}}_A(a)$) denotes the set of all positive approximants (respectively near-approximants) of $a$.

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In the case where $A$ is the $C^*$-algebra $B(H)$ of all bounded linear operators on a complex Hilbert space $H$ the index $A$ will be skipped if that does not lead to any confusion.

In this section the problem of accretive approximation suggested by P. R. Halmos (see [Ha]) is discussed for unital $C^*$-algebras. P. R. Halmos showed that the accretive part of $a$ is an accretive approximant of $a$ in the case $A = B(H)$ (cf. Theorem 2.1(b)).

For an element $a$ of a $C^*$-algebra let $\text{dist}(a, \mathcal{G})$ be the distance from $a$ to a given subset $\mathcal{G}$ of $A$ measured in the $C^*$-norm, whereas $\text{dist}_{\| \cdot \|}(a, \mathcal{G})$ denotes the distance measured in the norm $\| \cdot \|$. First some basic results about the positive and self-adjoint approximation are mentioned. In [Be 1] it is shown that for a normal element $a$ of $A$ the positive part of the real part is a positive approximant and the real part of $a$ is a self-adjoint approximant (see Theorem 2.1(c)). In the case of self-adjoint approximation the condition of $a$ being normal can be dropped. For $a = b + ic$ with $b, c$ self-adjoint one has $\text{dist}(b + ic, S_A) = \|ic\|$ and $S_A(b + ic) = b + S_A(ic)$ since the set $S_A$ of all self-adjoint elements of $A$ forms a real vector space. Hence $\|b + ic\| \geq \|c\|$ and $b \in S_A(b + ic)$ for $b, c$ self-adjoint. Similar results hold for approximation in the norm $\| \cdot \|$. Theorem 2.1. Let $A$ be a unital $C^*$-algebra and let $a = b + ic \in A$ with $b, c$ self-adjoint. Then

(a) $\text{dist}(a, \text{Acc}_A) = \text{dist}_{\| \cdot \|}(a, \text{Acc}_A) = \text{dist}(b, P_A)$.

(b) $b^+ + ic \in \text{Acc}_A(a) \subseteq \text{Acc}_A(a)$, where $b^+$ denotes the positive part of $b$.

(c) $\text{Acc}_A(a) = \{ p + id : p \in P_A(b) \text{ and } d \in S_A(c - i(b - p)) \}$ and $\tilde{\text{Acc}}_A(a) = \{ p + id : p \in P_A(b) \text{ and } d \in S_A(c - i(b - p)) \}$.

Proof. Let $p + id \in \text{Acc}_A$ with $p$ positive and $d$ self-adjoint. Then, as in the proof of [Ha], Corollary 5,

$\|a - (p + id)\| = \|(b - p) + i(c - d)\| \geq \|b - p\| \geq \|b - b^+\| = \|(b + ic) - (b^+ + ic)\|$. Thus $\text{dist}(a, \text{Acc}_A) = \|b - b^+\| = \text{dist}(b, P_A)$. Moreover, the accretive part $b^+ + ic$ of $a$ is an accretive approximant of $a$. Similarly $\text{dist}_{\| \cdot \|}(a, \text{Acc}_A) = \|b - b^+\| = \|b - b^+\|$ and $b^+ + ic$ is an accretive near-approximant of $a$. This shows (a) and (b).

(c) Suppose $p + id$ is an accretive approximant of $a$ with $p$ positive and $d$ self-adjoint. Then

$\|b - b^+\| = \|a - (p + id)\| = \|(b - p) + i(c - d)\| \geq \|b - p\|$, i.e., $p$ is a positive approximant of $b$ and

Thus

$\text{dist}(c - i(b - p), S_A) = \|\text{Im}(c - i(b - p))\| = \|b - p\| = \|(b - p) + i(c - d)\| = \|(c - i(b - p) - d)\|$, which means that $d$ is a self-adjoint approximant of $c - i(b - p)$. Here $\text{Im}(z)$ denotes the imaginary part of $z$.

Hence $\text{Acc}_A(a) \subseteq \{ p + id : p \in P_A(b) \text{ and } d \in S_A(c - i(b - p)) \}$. For the reverse inclusion suppose $p \in P_A(b)$ and $d \in S_A(c - i(b - p))$. Then

$\|a - (p + id)\| = \|(b - p) + i(c - d)\| = \|(c - i(b - p) - d)\| = \|\text{Im}(c - i(b - p))\| = \|b - p\| = \text{dist}(b, P_A)$, i.e., $p + id \in \text{Acc}_A(a)$ by part (a).

This shows $\text{Acc}_A(a) = \{ p + id : p \in P_A(b) \text{ and } d \in S_A(c - i(b - p)) \}$ and similarly $\tilde{\text{Acc}}_A(a) = \{ p + id : p \in P_A(b) \text{ and } d \in S_A(c - i(b - p)) \}$. As a consequence of the above theorem the properties of the sets $\text{Acc}_A(a)$ and $\tilde{\text{Acc}}_A(a)$ can be described by using the results for self-adjoint and positive approximation.

Corollary 2.2. Let $A$ be a unital $C^*$-algebra and let $a = b + ic \in A$ with $b, c$ self-adjoint.

(a) The following conditions are equivalent:

1. $0 \in \text{Acc}_A(a)$;
2. $-\|a\| \in \sigma(b)$.

(b) The following conditions are equivalent:

1. $0 \in \tilde{\text{Acc}}_A(a)$;
2. $-\|a\| \in \sigma(b)$.

Proof. By Theorem 2.1(a) one has $\text{dist}(a, \text{Acc}_A) = \|b^-\|$, where $b^-$ denotes the negative part of $b$. Thus $0 \in \text{Acc}_A(a)$ if and only if $\|a\| = \|b^-\|$, i.e., if and only if $-\|a\| \in \sigma(b)$. Part (b) can be proved in the same way. ■

Corollary 2.3. Let $A$ be a unital $C^*$-algebra and let $a = b + ic \in A$ with $b, c$ self-adjoint. Then the following conditions are equivalent:

1. $a$ has a unique accretive approximant;
2. $a$ has a unique accretive near-approximant;
3. there exists a non-negative number $\delta$ such that $\text{dist}(\lambda, \mathbb{R} \geq 0) = \delta$ for every $\lambda \in \sigma(b)$;
4. $b$ has a unique positive approximant;
5. $b$ has a unique positive near-approximant.
Proof. The equivalence of (3), (4) and (5) follows from Theorem 2.5 of [Be 1].

By Theorem 2.1 the implications (2)⇒(1) and (1)⇒(4) hold.

For the implication (3)⇒(2) suppose that there exists a positive number δ such that dist(λ, R²̃) = δ for every λ ∈ σ(A). Then b has a unique positive near-approximant (see [Be 1], Theorem 2.5).

If δ = 0 then σ(b) ⊆ R²̃, i.e. b is the unique positive near-approximant of b. Thus dist q(λ, A_c(a)) = ||a - (b + ic)|| = 0, i.e., b + ic is the unique accretive near-approximant of a.

If δ > 0 then σ(b) = {−δ}, i.e., b = −δ and 0 is the unique positive near-approximant of b. Moreover, c + iδ has the unique self-adjoint near-approximant c by Theorem 2.5 of [Be 1]. Hence ic is the unique accretive near-approximant of a by Theorem 2.1(c). This completes the proof. ■

3. Dimension of the set of approximants. In this section the dimension of the convex sets A_c(a) and A_c̃(a) is discussed for every element a ∈ A. Recall that the dimension of a convex set X is the (real) dimension of the smallest (real) affine subspace including X (cf. [Val]).

Similar to the case of positive and self-adjoint approximation, the dimension of the set of accretive approximants depends heavily on the C^*-algebra A (see [Be 3], Example 2.1, for positive approximation). However, for the C^*-algebra B(H) the dimension can be computed by using techniques similar to [Be 3].

In the following remark some notations and properties of positive and self-adjoint approximation in B(H) are listed which will be used in this section. They can be found in [Bo 2] and [Bo 3].

Remark 3.1. For every operator A ∈ B(H) the space max A is defined by max A = {x ∈ H : ||Ax|| = ||A|| · ||x||} = ker(||A||^2 − A^∗A).

If A = B + ic with B, C self-adjoint, then the subspace H_1 = H_1(A) is defined by H_1(A) = F(K||C||^2 − C^2) = max(C)^⊥.

If S is a self-adjoint near-approximant of A, then H_1 reduces B − S with (B − S)|H_1^⊥ = 0 and H_1^⊥ ⊊ max(A − S), since ran(B − S) ⊊ H_1 and max(A − B) = H_1^⊥.

Moreover, the subspace H_2 = H_2(A) is given by H_2(A) = ran D_0 ∩ ran P_0 with δ = dist(A, P, D_0) = (δ^2 − C^2)^1/2 and P_0 = B + D_0.

If P is a positive near-approximant of A, then H_2 reduces P_0 − P with (P_0 − P)|H_2^⊥ = 0 and H_2^⊥ ⊊ max(A − P), since ran(P_0 − P) ⊊ H_2 and max(A − P_0) = ker(δ^2 − (A − P_0)^∗(A − P_0)) = H.

Lemma 3.2. Let A = B + iC ∈ B(H) with B, C self-adjoint and let H_2 = H_2(B). Then

\[ \text{dim}_{R} A_c(a) \leq \text{dim}_{R} A_c(A) \leq 2 \cdot (\text{dim}_{C} H_2)^2. \]

Proof. Since A_c(A) ⊊ A_c(A) the first inequality follows immediately.

Suppose now that T = P + iD is an accretive near-approximant of A with P positive and D self-adjoint. Then the subspace H_2 = H_2(T) reduces P_0 − P with (P_0 − P)|H_2^⊥ = 0 and H_2^⊥ ⊊ max(B − P), since P is a positive near-approximant of B by Theorem 2.1(c). Since B^+ is also a positive near-approximant of B this implies that H_2 reduces P − B^+ with (P − B^+)|H_2^⊥ = 0.

Moreover, the subspace H_1 = H_1(C − i(B − P)) = max(B − P)^⊥ ⊊ H_2 reduces C − D with (C − D)|H_2 = 0, since D is a self-adjoint near-approximant of C − i(B − P).

This shows that the set A_c(A) − (B^+ + iC) is included in the set

\[ \{ P' + iD' : P' ∈ B(H) \text{ positive with } \text{ran} P' ⊊ H_2 \text{ and } D' ∈ B(H) \text{ self-adjoint with } \text{ran} D' ⊊ H_2 \}, \]

which has real dimension 2 · (dim_{C} H_2)^2. ■

Theorem 3.3. Let A = B + iC ∈ B(H) with B, C self-adjoint and let H_2 = H_2(B). Then

\[ \text{dim}_{R} A_c(A) = \text{dim}_{R} A_c(A) = 2 \cdot (\text{dim}_{C} H_2)^2. \]

Proof. In view of the previous lemma it suffices to prove that \( \text{dim}_{R} A_c(A) \geq 2 \cdot (\text{dim}_{C} H_2)^2 \). Fix \( n ∈ \mathbb{N} \) and define the subspace \( M_n \) by \( M_n := \text{ran} E(D_n) \), where \( E \) denotes the spectral measure for \( B \) and \( D_n := \{ λ ∈ σ(B) : \text{dist}(λ, R^0) < d(σ(B), R^0) - 2/n \} \). Here \( d(σ(B), R^0) \) denotes the one-sided Hausdorff distance from \( σ(B) \) to \( R^0 \). Then \( M_n \) is a \( B \)-reducing subspace of \( H_2 = H_2(B) \) since \( H_2(B) = H_0(R^0, B) = \text{ran} E(\{ λ ∈ σ(B) : \text{dist}(λ, R^0) < d(σ(B), R^0) \}) \).

For every positive operator \( P' ∈ B(H) \) with \( \text{ran} P' ⊊ M_n \) and \( ||P'|| < 1/n \) and for every self-adjoint operator \( D' ∈ B(H) \) with \( \text{ran} D' ⊊ M_n \) and \( ||D'|| < 1/n \) the operator \( T = (B^+ + P') + i(C + D') \) is accretive. Moreover, \( ||A - T|| = ||(B + iC) - (B^+ + P') - i(C + D')|| \)

\[ = ||B - (B^+ + P') - iD'|| \]

\[ = \max \{|(B - (B^+ + P') - iD')|_{M_n}, ||(B - (B^+ + P') - iD')|_{M_n}^2|| \} \]

\[ \leq \max \{|(B - B^+)|_{M_n} + ||P'||_{M_n} + ||D'||_{M_n}, ||(B - B^+)|_{M_n}^2|| \} \]

\[ \leq \max \{|B - \text{ran} E(D_n)| + 2/n, ||B - B^+|| \} \]

\[ \leq d(σ(B), R^0) = \text{dist}(B, P_A) \]
by the spectral theorem and by the distance formula 1.4 of [Be 1]. Hence $T$ is an accretive approximant of $A$. Thus the set $\text{Acc}(A) - (B^* + iC)$ includes the set

$$\{P' + iD' : P' \in B(H) \text{ positive with } \sup_{n} P' \leq M_n \text{ and } \|P'\| \leq 1/n \text{ and } D' \in B(H) \text{ self-adjoint with } \sup_{n} D' \leq M_n \text{ and } \|D'\| \leq 1/n\},$$

which has real dimension $2 \cdot (\dim_{C} M_n)^2$. Since $H_2 = H_2(B) = \bigcup_{n \in \mathbb{N}} M_n$, one has $\lim_{n \to \infty} \dim_{C} M_n = \dim_{C} H_2$, so that

$$\dim_{C} \text{Acc}(A) \geq 2 \cdot (\dim_{C} H_2)^2.$$

This completes the proof. ■

4. Extreme points in the set of approximants. In this section the extreme points of the convex sets $\text{Acc}_A(a)$ and $\hat{\text{Acc}}_A(a)$ are studied for every element $a$ of the unital $C^*$-algebra $A$.

First, sufficient conditions are specified for an approximant to be such an extreme point:

**Proposition 4.1.** Let $A$ be a unital $C^*$-algebra and let $a \in A$.

(a) If $t = p + id \in A$ is an accretive approximant of $a$ with $p$ positive and $d$ self-adjoint such that $(a - t)^*(a - t)$ or $(a - t)(a - t)^*$ is a scalar, then $t \in \text{ex Acc}_A(a)$.

(b) If $t = p + id \in A$ is an accretive near-approximant of $a$ with $p$ positive and $d$ self-adjoint such that $(a - t)^*(a - t) + (a - t)(a - t)^*$ is a scalar, then $t \in \text{ex Acc}_A(a)$.

**Proof.** (a) It is a well-known result that every element $x \in A$ with $x^* x = e$ or $x x^* = e$ is an extreme point of the closed unit ball (see e.g. [Pe], Prop. 1.4.7). Hence the element $a - t$ is an extreme point of the closed ball with center 0 and radius $\|a - t\|$. Since the set $a - \text{Acc}_A(a)$ is included in this ball, the element $t$ is an extreme point of $\text{Acc}_A(a)$.

Similarly part (b) follows from the proposition below. ■

**Proposition 4.2.** Let $A$ be a $C^*$-algebra with unit $e$ and let $A_1 := \{a \in A : \|a\| \leq 1\}$ be the closed unit ball in $A$ for the norm $\|\cdot\|$. Then every element $a \in A$ with $\frac{1}{2}(a^* a + a a^*) = e$ is an extreme point of $A_1$.

**Proof.** Using the Theorem of Gelfand–Neumark it suffices to prove the statement for $A = B(H)$.

Suppose $T \in B(H)$ with $\frac{1}{2}(T^* T + T T^*) = I$, i.e., $T \in B(H_2)$. Then $\|T\| = \|I\| = 1$. Hence $T$ is an extreme point of $B(H_2)$. ■

As a consequence of Proposition 4.1 and Theorem 2.1 some extreme points of $\text{Acc}_A(a)$ and $\hat{\text{Acc}}_A(a)$ can easily be constructed:

**Corollary 4.3.** Let $A$ be a unital $C^*$-algebra and let $a = b + ic \in A$ with $b, c$ self-adjoint. Furthermore, let $p \in \mathcal{P}_A(b) = \mathcal{P}_A(b)$. Then

$$k_p^{(+)} := p + i((\|b - p\|^2 - (b - p)^2)^{1/2} + c)$$

and

$$k_p^{(-)} := p - i((\|b - p\|^2 - (b - p)^2)^{1/2} + c)$$

are both extreme points of $\text{Acc}_A(a)$ and $\hat{\text{Acc}}_A(a)$.

**Corollary 4.4.** Let $A$ be a unital $C^*$-algebra and let $a \in A$. Then either $a$ has a unique accretive approximant and near-approximant or $\text{Acc}_A(a)$ and $\hat{\text{Acc}}_A(a)$ have uncountably many extreme points.

**Proof.** If $a$ does not have a unique accretive approximant or near-approximant then $b := e(a)$ does not have a unique positive approximant by Corollary 2.3. This implies that $b$ has uncountably many positive approximants since the set of all positive approximants is convex. Hence $\text{Acc}_A(a)$ and $\hat{\text{Acc}}_A(a)$ have uncountably many extreme points by the previous corollary. ■

References

[Ha] P. R. Halmos, Positive approximants of operators, ibid. 21 (1972), 961–966.