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On Kato non-singularity

by

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Abstract. An exactness lemma offers a simplified account of the spectral properties of the "holomorphic" analogue of normal solvability.

Call a bounded linear operator between Banach spaces *normally solvable* if it has closed range $TX = \mathrm{cl}(TX) \subseteq Y$; by an old theorem of Banach this implies that an equation $Tx = y$ is solvable if there is the implication, for arbitrary bounded linear functionals $g \in Y^\dagger$,

$$gT = 0 \Rightarrow gy = 0. \tag{0.1}$$

When X and Y are Hilbert spaces then normal solvability implies that T is *regular*, or "relatively Fredholm", in the sense of having a (bounded) *generalized inverse*, $T^\wedge : Y \to X$, for which

$$T = TT^\wedge T \tag{0.2}$$

(so that if $y = Tx$ can be solved then $x = T^\wedge y$ is a solution). Goldberg and others have tried to make a "spectrum" out of this, collecting ([6], Definition VI.7.1) the complex numbers λ for which $T - \lambda I$ is not normally solvable, but nothing works: for example the operator $0 : X \to X$ has empty spectrum, and ([2], § 2.8) there are simple examples which show that this spectrum is not closed, and does not satisfy the spectral mapping theorem (either way, even for the polynomial z^2).

For a Hilbert space $X = Y$ Mbekhta ([13], [14]) has examined a "holomorphic" analogue of normal solvability, which of course coincides with regularity: we may call the operator $T : X \to X$ *holomorphically regular* or "Kato invertible" if there exists a neighbourhood U of 0 in $\mathbb{C}$ and a holomorphic function $T_z^\wedge : U \to X$ for which

$$T - \lambda I = (T - \lambda I)T_\lambda^\wedge(T - \lambda I) \quad \text{for each } \lambda \in U. \tag{0.3}$$

The work of Mbekhta shows that, on a Hilbert space, the spectrum derived from "holomorphic regularity" is non-empty, closed and subject to the

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spectral mapping theorem. We have offered an extension of this to Banach spaces [8], and here wish to consider the corresponding extension to Banach spaces of the holomorphic analogue of normal solvability. The key to the definition is the observation ([13], Théorème 2.6; [8], Theorem 9) that if $T : X \to X$ is bounded and linear then

(0.4) $$T \text{ holomorphically regular} \Leftrightarrow T \text{ regular hyperexact},$$

where T is called *hyperexact* if

(0.5) $$T^{-1}(0) \subseteq T^\infty(X) = \bigcap_n T^n(X),$$

i.e. its null space is included in its "hyperrange". There are ([8], Theorem 7) various equivalent versions of hyperexactness and related concepts; it is easy to see that, with no topology, T is hyperexact if and only if it is *perfect* in the sense of Saphar ([20], Definition 2). Hyperexactness by itself need not give a good spectrum: for example, any operator which is either one-one or onto satisfies this condition. We do get part of the spectral mapping theorem: if $T : X \to X$ and $S : X \to X$ commute, in the sense that $ST = TS$, then

(0.6) $$ST \text{ hyperexact} \Rightarrow S, T \text{ hyperexact},$$

with the reverse implication if either S is a power of T,

(0.7) $$S = T^n \quad \text{for some } n \in \mathbb{N},$$

or the pair (S, T) satisfies the "middle exactness" condition of Taylor ([8], (3.4)):

(0.8) $$(-S \quad T)^{-1}(0) \subseteq \begin{pmatrix} T \\ S \end{pmatrix}(X).$$

This is enough for the spectral mapping theorem for non-constant polynomials. An example of Müller ([18], Example 2.2) shows that the implication (0.6) cannot be reversed in general. To extend (0.6) to holomorphic regularity we are able to use a simple lemma ([8], Theorem 3): if $T : X \to Y$ and $S : Y \to Z$ are bounded and linear between normed spaces, and if there are bounded linear $S' : Z \to Y$ and $T' : Y \to X$ for which

(0.9) $$S'S + TT' = I,$$

then

(0.10) $$ST \text{ regular} \Leftrightarrow S, T \text{ regular}.$$

Our main result in this note is the analogue of (0.10) for normal solvability:

1. THEOREM. *If $T : X \to Y$ and $S : Y \to Z$ are bounded linear operators between Banach spaces then*

(1.1) $$S^{-1}(0) \cap TX = \{0\},\ S^{-1}(0) + TX \textit{ closed} \Rightarrow TX \textit{ closed}$$

and, even if X is not complete,

(1.2) $$SY,\ S^{-1}(0) + TX \textit{ closed} \Rightarrow STX \textit{ closed} \Rightarrow S^{-1}(0) + TX \textit{ closed}.$$

In particular, if

(1.3) $$S^{-1}(0) \subseteq TX$$

then

(1.4) $$SY,\ TX \textit{ closed} \Rightarrow STX \textit{ closed} \Rightarrow TX \textit{ closed}.$$

Proof. The implication (1.1) is an application ([7], Theorem 4.8.2) of the open mapping theorem, the "lemma of Neuberger": renorm the operator range TX ([10], Lemma 1). The first part of (1.2) is a lemma of Kato ([9], Lemma 331): consider the quotient $(S^{-1}(0)+TX)/S^{-1}(0)$. The second part of (1.2) ([10], Lemma 1) reduces to the remark

(1.5) $$S^{-1}(0) + T(X) = S^{-1}(STX).$$

(1.2) is converted to (1.4) by (1.3). ■

The argument of (1.1) extends to the case in which $S^{-1}(0) \cap TX$ is finite-dimensional, but such a condition cannot entirely be eliminated: let $ST = 0$ with TX not closed. The assumption that SY is closed can be neither removed from the left hand side of (1.4) nor added to the right: take S to be one-one and either $T = I$ or $T = 0$. It is also not possible to add the closedness of STX to the right hand side of (1.1): take $T = I$ and S to be one-one. Theorem 1 offers two different reasons why the product of two closed range operators should again have closed range (compare also the lemma of Bouldin [1]). Mbekhta and Laursen [10] are concerned with "central multipliers" T on a semiprime algebra X, for which $T^{-1}(0) \cap TX = \{0\}$; they use (1.1) and (1.2) to deduce that all the powers T^n have closed range. Our interest here is of course with operators T which satisfy the Saphar condition $T^{-1}(0) \subseteq T^\infty(X) \subseteq TX$.

We shall call $T : X \to X$ *Kato non-singular* if it is normally solvable and hyperexact. Combining (0.6) and (1.4), specialised to $X = Y = Z$, shows that if S and T commute then

(1.6) $$ST \text{ Kato non-singular} \Rightarrow S, T \text{ Kato non-singular},$$

with the reverse implication if either (0.7) or (0.8) holds. To see that the spectrum derived from Kato non-singularity is closed, we might look for a "holomorphic" characterization (cf. [14], Théorème 2.7): call a point $x \in X$ a *holomorphic kernel point* for $T : X \to X$ if there exist a neighbourhood U of 0 in $\mathbb{C}$ and a holomorphic function $f : U \to X$ for which

(1.7) $$f(0) = x \quad \text{and} \quad (T - \lambda I)f(\lambda) = 0 \quad \text{for each } \lambda \in U;$$

more generally, call x a *consorted kernel point* for T if there exist sequences (x_n) in X and (S_n) in $\mathrm{comm}^{-1}(T)$, the invertible operators commuting with T, for which

(1.8) $$(T - S_n)x_n = 0 \quad \text{and} \quad \|S_n\| + \|x_n - x\| \to 0.$$

2. THEOREM. *If $T : X \to X$ is normally solvable on the Banach space X then the following conditions are equivalent*:

(2.1) *every point of $T^{-1}(0)$ is consorted*;

(2.2) *T is hyperexact*;

(2.3) *every point of $T^{-1}(0)$ is holomorphic.*

Proof. If T is normally solvable and hyperexact then by Theorem 1 every power T^n is normally solvable, so that the hyperrange $T^\infty(X) = \bigcap_n T^n(X)$ is closed, and therefore a Banach space in its own right. The hyperexactness implies in particular ([8], (7.8)) that the induced operator $T^\wedge : T^\infty(X) \to T^\infty(X)$ is onto:

(2.4) $$y = Tx_0 = T^{n+1}x_n \Rightarrow x_0 - T^n x_n \in T^{-1}(0) \subseteq T^\infty(X) \Rightarrow x_0 \in T^n(X)$$

for each $n \in \mathbb{N}$. By the open mapping theorem $T^\wedge$ is therefore open, with $k > 0$ for which

(2.5) $$y \in T^\infty(X) \Rightarrow y = Tx_0 \quad \text{with } x_0 \in T^\infty(X) \text{ and } \|x_0\| \le k\|y\|,$$

and hence inductively a sequence (x_n) in $T^\infty(X)$:

(2.6) $$x_n \in T^\infty(X) \Rightarrow x_n = Tx_{n+1} \quad \text{with } x_{n+1} \in T^\infty(X) \text{ and } \|x_{n+1}\| \le k\|x_n\|.$$

Thus we also have $\|x_n\| \le k^{n+1}\|y\|$; now define

(2.7) $$U = \{|z| < k^{-1}\} \quad \text{and} \quad f = \sum_{n=0}^{\infty} z^n x_n : U \to X.$$

This gives implication (2.2)⇒(2.3); conversely, if every kernel point is consorted and $Tx = 0$ then

(2.8) $$x = \lim_n x_n \in \mathrm{cl}\bigcup_n (T - S_n)^{-1}(0) \subseteq \mathrm{cl}\, T^\infty(X) \subseteq \bigcap_n \mathrm{cl}\, T^n(X).$$

The argument is now induction on n, using Theorem 1: if TX and T^nX are both closed then so is $T^{n+1}X$. This shows (2.1)⇒(2.2), and trivially (2.3)⇒(2.1). ∎

Theorem 2 generalises the observation of Finch ([4], Theorem 2) that if T is onto then the "single-valued extension property at 0" implies T is one-one; compare also Schmoeger ([22], Proposition 3) and Laursen and Neumann ([11], Remark 1.6).

Dual to the conditions of Theorem 2, we shall call $y \in X$ a *consorted range point* of T if there are x and (x_n) in X and (S_n) in $\mathrm{comm}^{-1}(T)$ for which

(2.9) $$y = (T - S_n)x_n \quad \text{and} \quad \|S_n\| + \|x_n - x\| \to 0,$$

and a *holomorphic range point* if there is $U \in \mathrm{Nbd}(0)$ and holomorphic $f : U \to X$ for which

(2.10) $$y = (T - \lambda I)f(\lambda) \quad \text{for each } \lambda \in U.$$

These holomorphic range points coincide ([15], Proposition 1.3) with the "coeur analytique" of Mbekhta. Under the conditions of Theorem 2 (cf. [16], Theorem 1.1), we have

3. THEOREM. *If $T : X \to X$ is Kato non-singular on the Banach space X and $y \in X$ then the following are equivalent*:

(3.1) *y is a consorted range point of T*;

(3.2) *$y \in T^\infty(X)$*;

(3.3) *y is a holomorphic range point of T.*

Proof. If (2.9) holds then $y = (T - S_n)x_n \to Tx \in TX$ with

(3.4) $$x_n = S_n^{-1}(Tx_n - y) = TS_n^{-1}(x_n - x) \in TX \quad \text{and} \quad x = \lim_n x_n \in \mathrm{cl}(TX) = TX.$$

For each $k \in \mathbb{N}$ this argument gives

(3.5) $$y \in T^kX \Rightarrow x \in \mathrm{cl}(T^kX) = T^kX \Rightarrow y \in T^{k+1}X,$$

so that (3.2) follows by induction. Conversely, if (3.2) holds then the construction of (2.7) gives (2.10), and trivially again (2.10)⇒(2.9). ∎

The equivalence of (3.2) and (3.3) is also given by Laursen and Neumann ([11], Theorem 1).

The "single-valued extension property" ([3], [4], [16]) says that $T - \lambda I$ has no non-zero holomorphic kernel points for any $\lambda \in \mathbb{C}$; dually, $0 \in \mathbb{C}$ is not in the "local spectrum" $\sigma_T(y)$ of $y \in X$ (this writer would prefer "$\sigma_y(T)$") if and only if $y \in X$ is not a holomorphic range point of T.

On closer examination Theorem 2 does not achieve its aim, to show that the Kato spectrum is closed: this is part of a perturbation theorem of Kato ([9], Theorem 3). Non-emptiness of the Kato spectrum, and the fact that it contains the topological boundary of the usual spectrum, follows from the local constancy of the hyperrange $(T - zI)^\infty(X)$ and of the closure of the hyperkernel $(T - zI)^{-\infty}(0)$ on its complement ([21], Satz 1); the proof seems to need gap theory ([5], Satz 3; [17], Théorème 4.1). More generally, recalling the *reduced minimum modulus*

$$\gamma(T) = \inf\{\|Tx\| : \operatorname{dist}(x, T^{-1}(0)) \ge 1\}, \tag{3.6}$$

we have

4. THEOREM. *If S commutes with T and $\|S\| \le \gamma(T)$ then*

$$T \text{ Kato non-singular} \Rightarrow T - S \text{ Kato non-singular} \tag{4.1}$$

and

$$T - S \text{ invertible, } T \text{ Kato non-singular} \Rightarrow T \text{ invertible.} \tag{4.2}$$

Proof. If S commutes with T then for each $n \in \mathbb{N}$,

$$(T^{-1}S)^n(0) \subseteq T^{-n}(0) \quad \text{and} \quad T^n(X) \subseteq (S^{-1}T)^n(X), \tag{4.3}$$

and hence if T is hyperexact in the sense (0.5) then

$$(T^{-1}S)^n(0) \subseteq (S^{-1}T)^m(X) \quad \text{for each } m, n \in \mathbb{N}. \tag{4.4}$$

But now the stability result of Kato ([9], Theorem 3) says that on the set $\{\lambda \in \mathbb{C} : |\lambda|\,\|S\| < \gamma(T)\}$ the range $(T - \lambda S)(X)$ is closed of constant (possibly infinite) codimension and the null space $(T - \lambda S)^{-1}(0)$ of constant (again possibly infinite) dimension, while (4.4) continues to hold with $T - \lambda S$ in place of T. ∎

Kato's theorem also uses gap theory; we have been unable to find an elementary argument like the proof of the analagous result ([8], (9.5)) for hyperregularity. The stability result of Kato gives immediately a "punctured neighbourhood theorem"; we correct the statement of Lee ([12], Theorem 2) and further simplify the argument of Schmoeger ([23], Theorem 1):

5. THEOREM. *If*

$$T^{-1}(0) + T(X) \text{ is closed and } T^{-1}(0) \cap T(X) \text{ is finite-dimensional} \tag{5.1}$$

then for $S \in \operatorname{comm}^{-1}(T)$ of sufficiently small norm

$$T - S \text{ is normally solvable} \tag{5.2}$$

and

$$\dim(T - S)^{-1}(0) = \dim T^{-1}(0) \cap T^{\infty}(X) < \infty \quad \text{independent of } S. \tag{5.3}$$

Proof. Once again $\operatorname{comm}^{-1}(T) = BL^{-1}(X, X) \cap \operatorname{comm}(T)$ is the "invertible commutant" of $T \in BL(X, X)$, and we write $U^{\sim} : T^{\infty}(X) \to T^{\infty}(X)$ for the operator induced by $U \in \operatorname{comm}(T)$; then it is elementary that ([8], Lemma 6)

$$S \in \operatorname{comm}^{-1}(T) \Rightarrow (T - S)^{-1}(0) \subseteq T^{\infty}(X) \tag{5.4}$$

and

$$\dim T^{-1}(0) \cap T(X) < \infty \Rightarrow T^{\infty}(X) \subseteq T(T^{\infty}(X)). \tag{5.5}$$

It follows that for sufficiently small $S \in \operatorname{comm}^{-1}(T)$,

$$\begin{aligned}\dim(T - S)^{-1}(0) &= \dim(T - S)^{\sim -1}(0) \\ &= \operatorname{index}(T - S)^{\sim} = \operatorname{index} T^{\sim} = \dim\ T^{\sim -1}(0):\end{aligned} \tag{5.6}$$

the fourth equality is (5.5), which says that $T^{\sim}$ is onto, the third equality is the continuity of the index on the Banach space $T^{\infty}(X)$ and the second the fact that the onto mappings on $T^{\infty}(X)$ form an open set; the first equality is just (5.4). ∎

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Accretive approximation in C^*-algebras

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Abstract. The problem of approximation by accretive elements in a unital C^*-algebra suggested by P. R. Halmos is substantially solved. The key idea is the observation that accretive approximation can be regarded as a combination of positive and self-adjoint approximation. The approximation results are proved both in the C^*-norm and in another, topologically equivalent norm.

1. Introduction. For every unital C^*-algebra $\mathcal{A}$ let $Acc_{\mathcal{A}}$ be the set of all accretive elements of $\mathcal{A}$, i.e. the set of all elements with positive real part. For an element a of $\mathcal{A}$ let $Acc_{\mathcal{A}}(a)$ denote the set of all accretive approximants of a. Here an *approximant* means an element x of $Acc_{\mathcal{A}}$ such that $\|a-x\| \le \|a-y\|$ for every element y of $Acc_{\mathcal{A}}$. Furthermore, let the norm $|\!|\!|\cdot|\!|\!|$ be defined by $|\!|\!|a|\!|\!| = \left\|\frac{1}{2}(a^*a+aa^*)\right\|^{1/2}$ (cf. [Bo 2, Be 1]). The accretive approximants in this norm will be called *accretive near-approximants*; the set of all accretive near-approximants will be denoted by $\widetilde{Acc}_{\mathcal{A}}(a)$.

The main purpose of this paper is to describe the sets $Acc_{\mathcal{A}}(a)$ and $\widetilde{Acc}_{\mathcal{A}}(a)$. The key idea is the observation that accretive approximation is a combination of positive and self-adjoint approximation (Theorem 2.1(c)). As a consequence the real dimensions of the convex sets $Acc_{\mathcal{B}(\mathcal{H})}(A)$ and $\widetilde{Acc}_{\mathcal{B}(\mathcal{H})}(A)$ can be computed for every bounded linear operator A on a complex Hilbert space $\mathcal{H}$, and some extreme points can be constructed.

2. Accretive approximation in C^*-algebras. Let $\mathcal{A}$ be a unital C^*-algebra. Then $\mathcal{S}_{\mathcal{A}}$ denotes the set of all self-adjoint elements of $\mathcal{A}$. For every element $a \in \mathcal{A}$ let $\mathcal{S}_{\mathcal{A}}(a)$ (respectively $\tilde{\mathcal{S}}_{\mathcal{A}}(a)$) be the set of all self-adjoint approximants (respectively self-adjoint near-approximants) of a. Similarly $\mathcal{P}_{\mathcal{A}}$ denotes the set of all positive elements of $\mathcal{A}$, and $\mathcal{P}_{\mathcal{A}}(a)$ (respectively $\tilde{\mathcal{P}}_{\mathcal{A}}(a)$) denotes the set of all positive approximants (respectively near-approximants) of a.

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