Almost exactness in normed spaces II

by

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Abstract. In the normed space of bounded operators between a pair of normed spaces, the set of operators which are “bounded below” forms the interior of the set of one-one operators. This note is concerned with the extension of this observation to certain spaces of pairs of operators.

If $T \in BL(X,Y)$ and $S \in BL(Y,Z)$ are bounded operators between normed spaces we shall call the pair $(S,T) \in BL(Y,Z) \times BL(X,Y)$ weakly exact if

\begin{equation}
S^{-1}(0) \subseteq \text{cl}(T(X)),
\end{equation}

whether or not the “chain condition”

$$ST = 0$$

is satisfied, and almost exact ([2]; [3], Definition 10.3.1) if there are $k > 0$ and $h > 0$ for which

\begin{equation}
y \in \text{cl}([\text{Disc}_Y(0;h\|Sy\|) + T \text{Disc}_X(0;k\|y\|)]) \quad \text{for each } y \in Y;
\end{equation}

this means that if $y \in Y$ is arbitrary there is $(x_n)$ in $X$ for which

$$\lim \sup_n \|y - T x_n\| \leq h\|Sy\| \quad \text{and} \quad \sup_n \|x_n\| \leq k\|y\|.$$

For example, if $T = 0$ then (1.1) says that $S$ is one-one, and (1.2) that $S$ is bounded below; if $S = 0$ then (1.1) says that $T$ is dense, and (1.2) that $T$ is almost open [1]. In general it is sufficient for (1.1) if either $S$ is one-one or $T$ dense, and sufficient for (1.2) if either $S$ is bounded below or $T$ almost open. Restricted to the closed subspace of “chains”

$$BL(X,Y,Z) = \{(S,T) \in BL(Y,Z) \times BL(X,Y) : ST = 0\},$$

the condition (0.2) defines an open set ([2]; [3], Theorem 10.3.8; [6], Prop. 10, Ch. 10), although not ([2], (1.4.8), page 261; [3], (10.3.8.8), page 441) relative
to the full cartesian product space of operators. In this note we show further that the condition (0.2) defines the interior of the condition (0.1), relative to the space of chains; relative to the full product space of pairs of operators both conditions lead to the same interior, pairs \((S, T)\) for which either \(S\) is bounded below or \(T\) almost open. We begin with the two "one variable" cases:

**Theorem 1.** There is equality

\[
\{ S \in BL(Y, Z) : S \text{ bounded below} \} = \text{interior}\{ S \in BL(Y, Z) : S \text{ one-one} \}
\]

and equality

\[
\{ T \in BL(X, Y) : T \text{ almost open} \} = \text{interior}\{ T \in BL(X, Y) : T \text{ dense} \}.
\]

**Proof.** If \(S \in BL(Y, Z)\) is bounded below, with \(\|y\| \leq h\|Sy\|\) for each \(y \in Y\), then

\[
h\|S' - S\| < 1 \Rightarrow \|y\| \leq h'\|S'y\| \quad \text{with} \quad (1 - h\|S' - S\|)h' = h.
\]

This proves that the bounded below operators form an open subset of the one-one operators; conversely, if \(S \in BL(Y, Z)\) is not bounded below then there are \((y_n)\) in \(Y\) and bounded linear functionals \((g_n)\) in the dual space \(Y^\dagger\) for which

\[
\|y_n\| = \|g_n\| = g_n(y_n) = 1 \quad \text{and} \quad \|Sy_n\| \to 0;
\]

now the rank one projection \(P_n = g_n \otimes y_n : w \mapsto g_n(w)y_n\) in \(BL(Y, Y)\) satisfies

\[
\|SP_n\| = \|Sy_n\| \to 0 \quad \text{and} \quad 0 \neq y_n \in (S - SP_n)^{-1}(0).
\]

This means that \(S\) is not in the interior of the one-one operators and proves (1.1). Dually, recall [3, Theorem 5.5.2] that

\[
T \text{ almost open} \Leftrightarrow T^\dagger \text{ bounded below},
\]

so that also the almost open operators are an open subset of the dense. Conversely, if \(T \in BL(X, Y)\) is not almost open then there are \((h_n)\) in \(Y^\dagger\) and \((z_n)\) in \(Y\) for which

\[
\|h_n\| = h_n(z_n) = 1 \quad \text{and} \quad \|z_n\| \leq 2 \quad \text{and} \quad \|h_nT\| \to 0;
\]

this time \(Q_n = h_n \otimes z_n \in BL(Y, Y)\) gives

\[
\|Q_nT\| \leq 2\|h_nT\| \to 0 \quad \text{and} \quad 0 \neq h_n \in (T - Q_nT)^{-1}(0).
\]

This excludes \(T\) from the interior of the dense operators and proves (1.2).

In the space of chains, we have

**Theorem 2.** If \(W = BL(X, Y, Z)\) is the space of chains then

\[
(2.1) \quad \{ (S, T) \in W : (S, T) \text{ almost exact} \} = \text{interior}_W \{ (S, T) \in W : (S, T) \text{ weakly exact} \}.
\]

**Proof.** If \((S, T) \not\in BL(X, Y, Z)\) is not almost exact then ([2]; [3, Theorem 10.3.7]) either \(S^\dagger : Y/\text{cl}(TX) \to Z\) is not bounded below or \(T^\dagger : X \to S^{-1}(0)\) is not almost open, so that by the proof of Theorem 1 there is either \(P_n = g_n \otimes y_n : Y \to Y\) for which

\[
(2.2) \quad \text{dist}(y_n, TX) = \|g_n\| = 1 = g_n(y_n) \quad \text{and} \quad \|Sy_n\| \to 0
\]

or \(Q_n = h_n \otimes z_n : Y \to Y\) for which

\[
(2.3) \quad \|h_n\| = h_n(z_n) \quad \text{and} \quad \|z_n\| \leq 2 \quad \text{and} \quad \|h_nT\| \to 0 = S_zn.
\]

In the first case

\[
(S(I - P_n), T) \in BL(X, Y, Z) \quad \text{and} \quad (S - SP_n)^{-1}(0) \not\subseteq \text{cl}(TX),
\]

while in the second

\[
(S, (I - Q_n)T) \in BL(X, Y, Z) \quad \text{and} \quad S^{-1}(0) \not\subseteq \text{cl}(T - Q_nT)(X).
\]

In the full product space, we have

**Theorem 3.** If \(W = BL(X, Y, Z) \times BL(X, Y)\) is the full cartesian product then

\[
(3.1) \quad \{ (S, T) \in W : S \text{ bounded below or } T \text{ almost open} \} = \text{interior}\{ (S, T) \in W : (S, T) \text{ weakly exact} \}.
\]

**Proof.** If either \(S\) is bounded below or \(T\) is almost open then it is clear that the pair \((S, T)\) is in the interior. If neither \(S\) is bounded below nor \(T\) almost open then there are sequences \((P_n)\) and \((Q_n)\) of projections as in (1.4) and (1.6); if in addition we could arrange that \(h_n(y_n) \neq 0\), so that \(Q_nP_n \neq 0\), then we would have excluded \((S, T)\) from the interior. To do this take \((h_n)\) and \((z_n)\) as in (1.5), and \((y_n^0)\) in \(Y\) with \(\|y_n^0\| = 1\) and \(\|Sy_n^0\| \to 0\), and then \((y_n^0)\) in \(Y\) with

\[
(3.2) \quad \|y_n^0\| \leq 1/2 \quad \text{and} \quad \|y_n^0\| \to 0 \neq h_n(y_n^0 + y_n^1);
\]

now take

\[
y_n = \frac{y_n^0 + y_n^1}{\|y_n^0 + y_n^1\|} \quad \text{and} \quad \|g_n\| = 1 = g_n(y_n).
\]

In the special case

\[
(3.3) \quad \|y_n^0\| \leq 1/2 \quad \text{and} \quad \|y_n^0\| \to 0 \neq h_n(y_n^0 + y_n^1);
\]

now take

\[
y_n = \frac{y_n^0 + y_n^1}{\|y_n^0 + y_n^1\|} \quad \text{and} \quad \|g_n\| = 1 = g_n(y_n).
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