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Almost exactness in normed spaces II

by

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Abstract. In the normed space of bounded operators between a pair of normed spaces, the set of operators which are "bounded below" forms the interior of the set of one-one operators. This note is concerned with the extension of this observation to certain spaces of pairs of operators.

If $T \in BL(X,Y)$ and $S \in BL(Y,Z)$ are bounded operators between normed spaces we shall call the pair $(S,T) \in BL(Y,Z) \times BL(X,Y)$ *weakly exact* if

$$S^{-1}(0) \subseteq \operatorname{cl} T(X), \tag{0.1}$$

whether or not the "chain condition"

$$ST = 0$$

is satisfied, and *almost exact* ([2]; [3], Definition 10.3.1) if there are $k > 0$ and $h > 0$ for which

$$y \in \operatorname{cl}(\mathrm{Disc}_Y(0; h\|Sy\|) + T\,\mathrm{Disc}_X(0; k\|y\|)) \quad \text{for each } y \in Y; \tag{0.2}$$

this means that if $y \in Y$ is arbitrary there is (x_n) in X for which

$$\limsup_n \|y - Tx_n\| \le h\|Sy\| \quad \text{and} \quad \sup_n \|x_n\| \le k\|y\|.$$

For example, if $T = 0$ then (0.1) says that S is one-one, and (0.2) that S is bounded below; if $S = 0$ then (0.1) says that T is dense, and (0.2) that T is almost open [1]. In general it is sufficient for (0.1) if either S is one-one or T dense, and sufficient for (0.2) if either S is bounded below or T almost open. Restricted to the closed subspace of "chains"

$$BL(X,Y,Z) = \{(S,T) \in BL(Y,Z) \times BL(X,Y) : ST = 0\},$$

the condition (0.2) defines an open set ([2]; [3], Theorem 10.3.8; [6], Prop. 10, Ch. 10), although not ([2], (1.4.8), page 261; [3], (10.3.8.8), page 441) relative

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to the full cartesian product space of operators. In this note we show further that the condition (0.2) defines the interior of the condition (0.1), relative to the space of chains; relative to the full product space of pairs of operators both conditions lead to the same interior, pairs (S,T) for which either S is bounded below or T almost open. We begin with the two "one variable" cases:

THEOREM 1. *There is equality*

$$(1.1)\quad \{S\in BL(Y,Z) : S \textit{ bounded below}\} = \operatorname{interior}\{S\in BL(Y,Z) : S \textit{ one-one}\}$$

and equality

$$(1.2)\quad \{T\in BL(X,Y) : T \textit{ almost open}\} = \operatorname{interior}\{T\in BL(X,Y) : T \textit{ dense}\}.$$

Proof. If $S\in BL(Y,Z)$ is bounded below, with $\|y\|\le h\|Sy\|$ for each $y\in Y$, then

$$h\|S'-S\|<1 \Rightarrow \|y\|\le h'\|S'y\| \quad \text{with } (1-h\|S'-S\|)h'=h.$$

This proves that the bounded below operators form an open subset of the one-one operators; conversely, if $S\in BL(Y,Z)$ is not bounded below then there are (y_n) in Y and bounded linear functionals (g_n) in the dual space $Y^\dagger$ for which

$$(1.3)\quad \|y_n\|=\|g_n\|=g_n(y_n)=1 \quad\text{and}\quad \|Sy_n\|\to 0;$$

now the rank one projection $P_n=g_n\odot y_n : w\mapsto g_n(w)y_n$ in $BL(Y,Y)$ satisfies

$$(1.4)\quad \|SP_n\|=\|Sy_n\|\to 0 \quad\text{and}\quad 0\ne y_n\in(S-SP_n)^{-1}(0).$$

This means that S is not in the interior of the one-one operators and proves (1.1). Dually, recall ([3], Theorem 5.5.2) that

$$T \text{ almost open } \Leftrightarrow T^\dagger \text{ bounded below},$$

so that also the almost open operators are an open subset of the dense. Conversely, if $T\in BL(X,Y)$ is not almost open then there are (h_n) in $Y^\dagger$ and (z_n) in Y for which

$$(1.5)\quad \|h_n\|=h_n(z_n)=1 \quad\text{and}\quad \|z_n\|\le 2 \quad\text{and}\quad \|h_nT\|\to 0;$$

this time $Q_n=h_n\odot z_n\in BL(Y,Y)$ gives

$$(1.6)\quad \|Q_nT\|\le 2\|h_nT\|\to 0 \quad\text{and}\quad 0\ne h_n\in(T-Q_nT)^{\dagger-1}(0).$$

This excludes T from the interior of the dense operators and proves (1.2). ∎

In the space of chains, we have

THEOREM 2. *If $W=BL(X,Y,Z)$ is the space of chains then*

$$(2.1)\quad \{(S,T)\in W : (S,T) \textit{ almost exact}\} = \operatorname{interior}_W\{(S,T)\in W : (S,T) \textit{ weakly exact}\}.$$

Proof. If $(S,T)\in BL(X,Y,Z)$ is not almost exact then ([2]; [3], Theorem 10.3.7) either $S^\wedge : Y/\mathrm{cl}\,TX\to Z$ is not bounded below or $T^\vee : X\to S^{-1}(0)$ is not almost open, so that by the proof of Theorem 1 there is either $P_n=g_n\odot y_n : Y\to Y$ for which

$$(2.2)\quad \operatorname{dist}(y_n,TX)=\|g_n\|=1=g_n(y_n) \quad\text{and}\quad \|Sy_n\|\to 0$$

or $Q_n=h_n\odot z_n : Y\to Y$ for which

$$(2.3)\quad \|h_n\|=1=h_n(z_n) \quad\text{and}\quad \|z_n\|\le 2 \quad\text{and}\quad \|h_nT\|\to 0=Sz_n.$$

In the first case

$$(S(I-P_n),T)\in BL(X,Y,Z) \quad\text{and}\quad (S-SP_n)^{-1}(0)\not\subseteq \mathrm{cl}(TX),$$

while in the second

$$(S,(I-Q_n)T)\in BL(X,Y,Z) \quad\text{and}\quad S^{-1}(0)\not\subseteq \mathrm{cl}(T-Q_nT)(X). \ ∎$$

In the full product space, we have

THEOREM 3. *If $W=BL(Y,Z)\times BL(X,Y)$ is the full cartesian product then*

$$(3.1)\quad \{(S,T)\in W : S \textit{ bounded below or } T \textit{ almost open}\} = \operatorname{interior}\{(S,T)\in W : (S,T) \textit{ weakly exact}\}.$$

Proof. If either S is bounded below or T is almost open then it is clear that the pair (S,T) is in the interior. If neither S is bounded below nor T almost open then there are sequences (P_n) and (Q_n) of projections as in (1.4) and (1.6); if in addition we could arrange that $h_n(y_n)\ne 0$, so that $Q_nP_n\ne 0$, then we would have excluded (S,T) from the interior. To do this take (h_n) and (z_n) as in (1.5), and (y_n^0) in Y with $\|y_n^0\|=1$ and $\|Sy_n^0\|\to 0$, and then (y_n^1) in Y with

$$(3.2)\quad \|y_n^1\|\le 1/2 \quad\text{and}\quad \|y_n^1\|\to 0\ne h_n(y_n^0+y_n^1);$$

now take

$$y_n=\frac{y_n^0+y_n^1}{\|y_n^0+y_n^1\|} \quad\text{and}\quad \|g_n\|=1=g_n(y_n). \ ∎$$

In the special case

$$(3.3) \qquad \{0\} \neq S^{-1}(0) \subseteq \operatorname{cl} T(X) \neq Y,$$

in which S is not one-one and T not dense, we have an alternative argument: with (3.3) there are y and z in Y and $g \in Y^\dagger$ for which

$$(3.4) \qquad z \notin \operatorname{cl} T(X) \quad \text{and} \quad y \in S^{-1}(0) \quad \text{and} \quad \|Sz\| = \|y\| = 1 = g(y) \quad \text{and} \quad g(z) = 0;$$

now if $\varepsilon > 0$ is arbitrary and $R = g \odot z : w \mapsto g(w)z$ we claim

$$(3.5) \qquad \|\varepsilon SR\| = \varepsilon\|g\| \quad \text{and} \quad y + \varepsilon z \in (S - \varepsilon SR)^{-1}(0) \quad \text{and} \quad y + \varepsilon z \notin \operatorname{cl} T(X).$$

The first part is clear; for the second note that

$$\varepsilon SR(y + \varepsilon z) = \varepsilon g(y + \varepsilon z) Sz = \varepsilon Sz = S(y + \varepsilon z);$$

finally, if there were $x \in X^{\mathbb{N}}$ for which $y + \varepsilon z = \lim_n T(x_n)$ then

$$\varepsilon z = \lim_n Tx_n - y \in \operatorname{cl} T(X) + S^{-1}(0) = \operatorname{cl} T(X),$$

contradicting the assumption about z.

If in particular $Z = Y = X$ and $T = S^n$ for some $n \in \mathbb{N}$ then it is clear that the projections (P_n) and (Q_n) from the proof of Theorem 3 work for the pair (S, S) as well as for the pair (S, T); thus the interior of the sets of "self-exact" operators (see [4]) $S \in BL(X, X)$ for which the pair (S, S^n) satisfies either (0.1) or (0.2) is, independent of n, the "monothetic" operators, those which are either bounded below or almost open. The interior is unchanged if S is assumed to be "perfect" in the sense of Saphar (see [5]) with or without closed range.

Theorem 2 also tells us that the "Taylor spectrum" of a pair of commuting operators is closed. The result of Wrobel [7], which complements Theorem 2, is also more readable than its extension ([3], Theorem 10.3.9) to incomplete spaces.

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