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Solution operators for convolution equations on the germs of analytic functions on compact convex sets in $\mathbb{C}^N$

by

S. N. MELIKHOV (Rostov-na-Donu) and
SIEGFRIED MOMM (Düsseldorf)

Abstract. If $G \subset \mathbb{C}^N$ is compact and convex it is known for a long time that the nonzero constant coefficients linear partial differential operators (of finite or infinite order) are surjective on the space of all analytic functions on G. We consider the question whether solutions of the inhomogeneous equation can be given in terms of a continuous linear operator. For instance we characterize those sets G for which this is always the case.

Introduction. For a given compact convex set $G \subset \mathbb{C}^N$, let $A(G)$ denote the space of all germs of analytic functions on G. This space is endowed with its natural inductive limit topology. If $K \subset \mathbb{C}^N$ is another compact convex set, for each analytic functional $\mu \in A(\mathbb{C}^N)' \setminus \{0\}$ carried by K, a continuous linear operator is given by

$$T_\mu : A(G+K) \to A(G), \quad T_\mu(h)(z) = \langle \mu, h(z+\cdot)\rangle, \quad z \in G.$$

If $K = \{0\}$, the convolution operator T_μ is a partial differential operator on $A(G)$ (of finite or infinite order) and can be written as $T_\mu(f) = \sum_{\alpha \in \mathbb{N}_0^N} a_\alpha f^{(\alpha)}$, $f \in A(G)$. The coefficients are determined by the entire function $\widehat{\mu}(z) = \mu(e^{zw}) = \sum_{\alpha\in\mathbb{N}_0^N} a_\alpha z^\alpha$. In this case T_μ is surjective. If $K \neq \{0\}$, a characterization of surjective operators T_μ is known when G has a nonempty interior.

In the present paper, we investigate whether a given surjective operator $T_\mu : A(G+K) \to A(G)$ admits a continuous linear right inverse $R : A(G) \to A(G+K)$, i.e. we investigate whether it is possible to find solutions $R(f) \in A(G+K)$ of the convolution equation $T_\mu(R(f)) = f$ which depend on $f \in A(G)$ in a continuous and linear way.

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To formulate our result, we assume that the origin of $\mathbb{C}^N$ is contained in the relative interior of the convex compact set G. Let $H : \mathbb{C}^N \to [0,\infty[$ be the support function of G and denote by $v_H : \mathbb{C}^N \to [-\infty,\infty[$ the extremal plurisubharmonic function introduced in [21]: v_H is the largest plurisubharmonic function on $\mathbb{C}^N$ with $v_H \le H$ and with $v_H(z) \le \log|z| + O(1)$ as $z \to 0$. If $P_H \subset \mathbb{C}^N$ denotes the set of all $z \in \mathbb{C}^N$ for which $v_H(z) = H(z)$, then there is an upper semicontinuous function $C_H : \{z \in \mathbb{C}^N \mid |z| = 1\} =: S \to [0,\infty[$ such that

$$P_H = \{\lambda a \mid a \in S,\ 1/C_H(a) \le \lambda < \infty\}.$$

THEOREM I. *The following statements are equivalent:*

(i) *Each nonzero partial differential operator $T_\mu : A(G) \to A(G)$ admits a continuous linear right inverse.*

(ii) *There is $\delta > 0$ such that $C_H \ge \delta$ on S.*

(iii) *There is a neighborhood of infinity (i.e. the complement of a compact set) on which $v_H = H$. In particular, the interior of G is not empty.*

(iv) *There is a plurisubharmonic function $u \le H$ on $\mathbb{C}^N$ with $u(0) < 0$ and $u = H$ on a neighborhood of infinity.*

If $N = 1$, we can also characterize whether a single convolution operator $T_\mu : A(G+K) \to A(G)$ admits a continuous linear right inverse. For this sake let $A_\mu \subset S$ be the set of all accumulation points of $(\overline{a}/|a|)_{\{a\in\mathbb{C}\mid\hat{\mu}(a)=0\}}$.

THEOREM II. *Let $N = 1$, K a compact convex set in $\mathbb{C}$ and μ be a nonzero analytic functional on $A(\mathbb{C})$ which is carried by K. If the convolution operator $T_\mu : A(G+K) \to A(G)$ is surjective, the following are equivalent.*

(i) *$T_\mu : A(G+K) \to A(G)$ admits a continuous linear right inverse.*

(ii) *There is some $\delta > 0$ with $C_H \ge \delta$ on A_μ.*

In particular, the equivalent conditions of Theorem I *imply that for every compact convex set $K \subset \mathbb{C}$ all surjective convolution operators $T_\mu : A(G+K) \to A(G)$ admit continuous linear right inverses.*

By [21], when G is not pluripolar and $g_G : \mathbb{C}^N \to [0,\infty[$ is the pluricomplex Green function of $\mathbb{C}^N\backslash G$ with pole at infinity, the function C_H may be replaced by a function D_G which measures the growth of g_G at ∂G: We put $G_x := \{z \in \mathbb{C}^N \mid g_G(z) \le x\}$, we denote the support function of G_x by H_x, for all $x > 0$, and define

$$D_G(a) := \lim_{x\downarrow 0} \frac{H_x(a) - H(a)}{x} \in [0,\infty[, \qquad a \in S.$$

By [21], Theorem I implies:

THEOREM III. *For $N = 1$ let G be a compact convex set in $\mathbb{C}$ with $\#G > 1$. The following are equivalent:*

(i) *Each nonzero differential operator $T_\mu : A(G) \to A(G)$ admits a continuous linear right inverse.*

(ii) *For each (some) biholomorphic mapping $\psi : \{z \in \mathbb{C} \mid |z| > 1\} \to \mathbb{C}\backslash G$ with $\psi(\infty) = \infty$, there is $\delta > 0$ with $|\psi'| \ge \delta$.*

(iii) *There is $\delta > 0$ such that $G + \delta x U \subset G_x$ for all $x > 0$, where U denotes the unit disc in $\mathbb{C}$.*

For the evaluation of these conditions, we refer to [21] and to classical results from function theory. For example the equivalent conditions of Theorem III hold if the boundary of G is of class C^λ for some $\lambda > 1$. They do not hold if ∂G has a corner. The results of the present paper extend results of [16], [17] and [20], where the dual case of *open* convex $G \subset \mathbb{C}^N$ was investigated. There are two special cases for which different but equivalent versions of Theorem II have been obtained earlier or simultaneously, respectively: If $K = G = \{0\}$ it follows from Meise and Taylor [13] that only the nonzero differential operators of finite order admit continuous linear right inverses. If $K = \{0\}$ and $G = [-1,1] \subset \mathbb{R}$, Langenbruch [9] proves that a nonzero operator T_μ admits a continuous linear right inverse if and only if $A_\mu \subset \{-i, i\}$.

For the proof of our results, we extend the technique which has been developed in [20] using ideas of Meise and Taylor [14]. Doing this we extend some results of Langenbruch and Momm [10] to the case of several variables. Crucial for this extension is the application of a slightly improved version of a result of Langenbruch [8] on the existence of a continuous linear right inverse for the $\overline{\partial}$-operator on weighted L^2-spaces.

1. Preliminaries. Throughout this paper, for all $z, w \in \mathbb{C}^N$ and $r > 0$, we will use the following abbreviations: $\langle w, z\rangle := \sum_{j=1}^N w_j\overline{z}_j$, $|z| := \langle z, z\rangle^{1/2}$, $U(z,r) := \{w \in \mathbb{C}^N \mid |w - z| < r\}$, $U(r) := U(0,r)$, $B(z,r) := \{w \in \mathbb{C}^N \mid |w - z| \le r\}$, $B(r) := B(0,r)$, and $S := \partial B(1)$, $\mathbb{R}_+ := [0,\infty[$. If $A \subset \mathbb{C}^N$, we write $\Gamma(A) := \{ta \mid t \ge 0, a \in A\}$. By $\operatorname{int} G$ (resp. $\overline{G}$) we denote the interior (resp. closure) of a set G.

We refer to the book of Meise and Vogt [15] for standard notations and results from functional analysis. The book of Schneider [23] may be consulted for elementary facts about convex sets.

NOTATION. For the sequel, we fix a compact convex set G of $\mathbb{C}^N$ which contains the origin in its relative interior. Let H be its support function, i.e.

$$H(z) = \sup_{w\in G} \Re\langle w, z\rangle \in [0,\infty[, \qquad z \in \mathbb{C}^N.$$

If $K \subset \mathbb{C}^N$ is convex and compact, the support function of K will always be denoted by L. Sometimes it is useful to consider $\overline{H}$ (resp. $\overline{L}$) defined by $\overline{H}(z) := H(\overline{z})$, $z \in \mathbb{C}^N$ (resp. $\overline{L}(z) := L(\overline{z})$).

1.1. *Function spaces*. For each open set $D \subset \mathbb{C}^N$, we denote by $A(D)$ (resp. $A^\infty(D)$) the space of all (bounded) analytic functions on D. Let $K \subset \mathbb{C}^N$ be convex and compact. Let G_x, $x > 0$, be a family of bounded convex domains of $\mathbb{C}^N$ with $G = \bigcap_{x>0} G_x$ and such that $\overline{G}_x \subset G_y$ for all $0 < x < y$. We put $G + K := \{z + w \mid z \in G,\ w \in K\}$ and denote by $A(G+K)$ the space of all germs of analytic functions on $G + K$, i.e.

$$A(G+K) := \bigcup_{x>0} A^\infty(G_x + K),$$

endowed with the usual inductive limit topology defined by the norms

$$|f|_x := \sup_{z\in G_x+K} |f(z)|, \quad f \in A^\infty(G_x + K),\ x > 0.$$

If H_x is the support function of G_x, $x > 0$, we denote by A^0_{H+L} the Fréchet space of all entire functions f on $\mathbb{C}^N$ with

$$\|f\|_x := \sup_{z\in\mathbb{C}^N} |f(z)| \exp(-H_x(\overline{z}) - L(\overline{z})) < \infty \quad \text{for all } x > 0.$$

If $L = 0$ we write $A^0_H = A^0_{H+L}$. If $H = 0$ we write $A^0 = A^0_H$.

1.2. Remark. If $\Omega \subset \mathbb{C}^N$ is a bounded convex domain with $0 \in \Omega$ and with support function ω, let A^2_ω denote the Hilbert space of all entire functions on $\mathbb{C}^N$ with

$$\int_{\mathbb{C}^N} |f(z)|^2 \exp(-2\omega(z))\, d\lambda(z) < \infty.$$

It may be well known and can be found in Taylor [25], Thm. 3, that the linear span of the exponentials $\{\exp\langle\cdot,\overline{w}\rangle \mid w \in \Omega\}$ is dense in A^2_ω. This shows that also for each nonpluripolar set $K \subset \Omega$, the linear span of $\{\exp\langle\cdot,\overline{w}\rangle \mid w \in K\}$ is dense in A^2_ω. Otherwise by the Hahn–Banach Theorem, there would be a functional $\nu \in A^{2\prime}_\omega\backslash\{0\}$ such that the analytic function

$$\widehat{\nu}(w) := \nu(\exp\langle\cdot,\overline{w}\rangle), \quad w \in \Omega,$$

is not identically zero but vanishes on K. This would prove that K is pluripolar.

1.3. *Convolution operators*. Let $K \subset \mathbb{C}^N$ be convex and compact. We fix an analytic functional $\mu \in A(K)'$. Then

$$T_\mu(f)(z) := \mu(f(z+\cdot)), \quad z \in G,\ f \in A(G+K),$$

defines a continuous linear operator $T_\mu : A(G+K) \to A(G)$. By the Laplace transform $\widehat{\mu}(z) := \mu(e^{\langle\cdot,\overline{z}\rangle})$, $z \in \mathbb{C}^N$ (see Hörmander [3], Thm. 4.5.3), we may identify $A(K)'$ and A^0_L. If $K = \{0\}$, the series $P(z) := \widehat{\mu}(z) = \sum_{\alpha\in\mathbb{N}_0^N} a_\alpha z^\alpha$ converges in the topology of $A^0_L = A^0$ and so does $\mu = \sum_{\alpha\in\mathbb{N}_0^N} a_\alpha \delta_0^{(\alpha)}$ in $A(K)'$, where δ_0 is the functional of evaluation at 0. In this case

$$P(D)f := \sum_{\alpha\in\mathbb{N}_0^N} a_\alpha f^{(\alpha)} = T_\mu(f), \quad f \in A(G).$$

$P(D) : A(G) \to A(G)$ is called a *differential operator*. We will say that $T_\mu : A(G+K) \to A(G)$ admits a *solution operator* if there is a continuous linear map $R : A(G) \to A(G+K)$ with $T_\mu \circ R = \mathrm{id}_{A(G)}$.

The following well known results about the Laplace transformation and the duality theory of Fréchet spaces can be found in Hörmander [3], Thm. 4.5.3, and in Meise and Vogt [15], respectively.

1.4. Duality. *Let $K \subset \mathbb{C}^N$ be convex and compact. The Laplace transform, given by $\mathcal{F}(\nu)(z) := \nu(e^{\langle\cdot,\overline{z}\rangle})$, defines by restriction a Fréchet space isomorphism $\mathcal{F} : A(G+K)' \to A^0_{H+L}$. Moreover, for all $0 < x_2 < x_1$ there is $C > 0$ with*

(i) $\|\mathcal{F}(\nu)\|_{x_1} \le |\nu|^*_{x_1}$ *for all* $\nu \in A(G+K)'$ *and*
(ii) $|\mathcal{F}^{-1}(f)|^*_{x_1} \le C\|f\|_{x_2}$ *for all* $f \in A^0_{H+L}$,

where $|\cdot|^*_x$ denotes the dual norm of $|\cdot|_x$. Let $\mu \in A(K)'\backslash\{0\}$ be such that $T_\mu : A(G+K) \to A(G)$ is surjective. Identifying $A(G+K)'$ and $A(G)'$ with A^0_{H+L} and A^0_H, respectively, the transposed map $T^t_\mu : A(G)' \to A(G+K)'$ is the multiplication operator $M_{\widehat{\mu}} : A^0_H \to A^0_{H+L}$, $M_{\widehat{\mu}}(f) = \widehat{\mu}\cdot f$. By duality theory for Fréchet–Schwartz spaces, the following holds: T_μ is surjective if and only if $\widehat{\mu}\cdot A^0_H$ is a closed subspace of A^0_{H+L} (the latter being true by hypothesis). T_μ has a solution operator on $A(G)$ if and only if the quotient map $\pi : A^0_{H+L} \to A^0_{H+L}/(\widehat{\mu}\cdot A^0_H)$ has a continuous linear right inverse.

For the following notion compare Ehrenpreis [1], and see Sigurdsson [24] for further references. In Proposition 1.6 we collect well known results on the surjectivity of convolution operators $T_\mu : A(G+K) \to A(G)$. These results have a long history. For this history, in particular concerning much older results in the case $N = 1$, we refer to the literature cited in the proof of Proposition 1.6.

1.5. Definition. Let $K \subset \mathbb{C}^N$ be convex and compact and let $\mu \in A(K)'$. If $A \subset S$ is closed, μ and $\widehat{\mu}$ will be called *slowly decreasing* (or of *regular growth*) on the cone $\Gamma(A)$ if the following holds: For each $\varepsilon > 0$ there is $R > 0$ such that for all $z \in \Gamma(A)$ with $|z| \ge R$ there is $w \in B(z, \varepsilon|z|)$ with $|\widehat{\mu}(w)| \ge \exp(L(\overline{w}) - \varepsilon|w|)$. If $A = S$ we simply say that μ and $\widehat{\mu}$ are slowly decreasing.

1.6. Proposition. *Let $K \subset \mathbb{C}^N$ be convex and compact and let $\mu \in A(K)'$.*

(a) *If $\widehat{\mu}$ is slowly decreasing (i.e. on $\mathbb{C}^N$), then $T_\mu : A(G+K) \to A(G)$ is surjective and $(\widehat{\mu}\cdot A(\mathbb{C}^N)) \cap A^0_{H+L} = \widehat{\mu}\cdot A^0_H$. If $K=\{0\}$, then each $\mu \in A(K)'\backslash\{0\}$ is slowly decreasing.*

(b) *Let $\Gamma_{\overline{H}}$ denote the support of $(dd^c\overline{H})^N$. If $T_\mu : A(G+K) \to A(G)$ is surjective, then $\widehat{\mu}$ is slowly decreasing on the cone $\Gamma_{\overline{H}}$. If $\operatorname{int} G \neq \emptyset$, then $T_\mu : A(G+K) \to A(G)$ is surjective if and only if $\widehat{\mu}$ is slowly decreasing on the cone $\Gamma_{\overline{H}}$. In this case again, $(\widehat{\mu}\cdot A(\mathbb{C}^N)) \cap A^0_{H+L} = \widehat{\mu}\cdot A^0_H$.*

(c) *Let $N=1$. If $T_\mu : A(G+K) \to A(G)$ is surjective, then $(\widehat{\mu}\cdot A(\mathbb{C})) \cap A^0_{H+L} = \widehat{\mu}\cdot A^0_H$.*

Proof. (a) The assertion for arbitrary $K\subset\mathbb{C}^N$ compact and convex follows from Morzhakov [22] (see also [18]). The assertion for $K=\{0\}$ is true by Martineau [12], Thm. 7 and Lemme 15.

(b) This is essentially contained in Krivosheev [7]. An explicit reference is [18], Prop. 2.3 and Thm. 3.9.

(c) For each $x>0$, we consider the inductive limit space A_{H_x+L} which consists of all entire functions f on $\mathbb{C}$ with

$$\sup_{z\in\mathbb{C}} |f(z)|\exp(-H_x(\overline{z}) - L(\overline{z}) + |z|/n) < \infty \quad \text{for some } n\in\mathbb{N}.$$

By Krasičkov-Ternovski [6], Thm. 4.4, for each $x>0$ the set $\widehat{\mu}\cdot\mathbb{C}[z]$ is dense in $(\widehat{\mu}\cdot A(\mathbb{C}))\cap A_{H_x+L}$. (From [6], Thm. 4.4(1), it follows that the rational functions which are constructed in [6], Thm. 4.4, are in fact polynomials in the present situation.) Since $\mathbb{C}[z]\subset A^0_H$, also $\widehat{\mu}\cdot A^0_H$ is dense in $(\widehat{\mu}\cdot A(\mathbb{C}))\cap A_{H_x+L}$ for all $x>0$. Thus $\widehat{\mu}\cdot A^0_H$ is dense in $(\widehat{\mu}\cdot A(\mathbb{C}))\cap A^0_{H+L} = (\widehat{\mu}\cdot A(\mathbb{C}))\cap \operatorname{proj}_{x\to 0} A_{H_x+L}$. Since T_μ is surjective, by duality theory, the subspace $\widehat{\mu}\cdot A^0_H$ is closed in A^0_{H+L}. Thus the assertion follows.

NOTATION. We consider the Fréchet space

$$L^2_H := \Big\{ f\in L^2_{\mathrm{loc}}(\mathbb{C}^N) \,\Big|\, \|f\|_x := \Big(\int_{\mathbb{C}^N} |f(z)|^2 \exp(-2H_x(\overline{z}))\,d\lambda(z)\Big)^{1/2} < \infty \text{ for all } x>0\Big\}.$$

By $L^2_{H(0,1)}$, we denote the corresponding Fréchet space of all $\overline{\partial}$-closed $(0,1)$-forms with coefficients in L^2_H. If $\Omega\subset\mathbb{C}^N$ is open, we consider the Fréchet space

$$W^2_H(\mathbb{C}^N,\Omega) := \{f\in L^2_{\mathrm{loc}}(\mathbb{C}^N) \mid f\in L^2_H,\ \overline{\partial} f\in L^2_{H(0,1)} \text{ and } f|\Omega\in A(\Omega)\}$$

endowed with the norms $(\|f\|_x^2 + \|\overline{\partial} f\|_x^2)^{1/2}$, $x>0$. By the mean value property of analytic functions, we have $A^0_H = W^2_H(\mathbb{C}^N,\mathbb{C}^N)$. Finally, we define $W^2_H := W^2_H(\mathbb{C}^N,\emptyset)$.

1.7. LEMMA. *The continuous linear map*

$$\overline{\partial} : W^2_H \to L^2_{H(0,1)}, \qquad f\mapsto \overline{\partial} f,$$

is surjective (with kernel A^0_H). Moreover, for every $g\in L^2_{H(0,1)}$, $x>0$, and $q>1$ there is $f\in W^2_H$ with $\overline{\partial} f = g$ and

$$\int_{\mathbb{C}^N} |f|^2 \exp(-2\overline{H}_x - 2\log(1+|z|^2))\,d\lambda \le q \int_{\mathbb{C}^N} |g|^2\exp(-2\overline{H}_x)\,d\lambda.$$

Proof. As in Meise and Taylor [13], Prop. 2.1, we apply Hörmander [3], 4.4.2, together with the Mittag-Leffler Lemma (which can be applied in view of Remark 1.2). The quantitative remark holds by the proof of the Mittag-Leffler Lemma.

We will apply the following slight extension of a result of Langenbruch [8]. To avoid technical definitions, we state it only for the situation which will be considered in this paper.

1.8. PROPOSITION. *Let $\Omega\subset\mathbb{C}^N$ be open and assume that for each $a\in\mathbb{C}^N\backslash\Omega$ there is a plurisubharmonic function u_a on $\mathbb{C}^N$ with $u_a(a)\ge 0$ satisfying the following condition: For each $y>0$ there are $x>0$ and $C>0$ with*

$$u_a(z)\le C + H_y(\overline{z}) - H_x(\overline{a}) \quad \text{for all } z\in\mathbb{C}^N,\ a\in\mathbb{C}^N\backslash\Omega.$$

Then there is a continuous linear projection $P : W^2_H(\mathbb{C}^N,\Omega)\to A^0_H = W^2_H(\mathbb{C}^N,\mathbb{C}^N)$.

Proof. If for *each* $a\in\mathbb{C}^N$ there is a plurisubharmonic function u_a on $\mathbb{C}^N$ with $u_a(a)\ge 0$ and such that for each $y>0$ there are $x>0$ and $C>0$ with

$$u_a(z)\le C + H_y(\overline{z}) - H_x(\overline{a}) \quad \text{for all } z,a\in\mathbb{C}^N,$$

then by Langenbruch [8], Thm. 1.3 and Remark 1.11 (applied with $r(z):=1$, $z\in\mathbb{C}^N$), there would be a continuous linear projection $P : W^2_H = W^2_H(\mathbb{C}^N,\emptyset)\to A^0_H$. If we put formally $u_a :\equiv 0$ for all $a\in\Omega$, then the proof shows that our assertion is true. (Note that in the proof in [8], Thm. 1.3, the absence of the upper bounds for u_a for $a\in\Omega$ does not affect the results of [8], Lemma 1.5, on the projections π_k, $k=N,\ldots,1$, which are defined for all compactly supported $(0,k)$-forms f with coefficients in $L^2_{\mathrm{loc}}(\mathbb{C}^N)$ such that $\overline{\partial} f$ is a $(0,k+1)$-form with coefficients in $L^2_{\mathrm{loc}}(\mathbb{C}^N)$. Since $\overline{\partial} f|\Omega\equiv 0$ for all $f\in A(\Omega)$, a small straightforward modification of the proof in [8] gives the desired estimate for $P:=\pi_0 : W_H(\mathbb{C}^N,\Omega)\to A^0_H$,

$$\pi_0(f) := f - \sum_{m\in\mathbb{N}} r_{m0}(\pi_1(h_m\overline{\partial} f)),$$

where we use the notation of [8].)

2. Solution operators. From Hörmander [2], Lemma 3.2, we recall the following:

2.1. Lemma. *For $\zeta\in\mathbb{C}^N$ and $r>0$, let g,P be analytic in $U(\zeta,4r)$ such that g/P is also analytic in $U(\zeta,4r)$. Then*
$$|g(\zeta)/P(\zeta)|\le\sup_{|\zeta-w|<4r}|g(w)|\sup_{|\zeta-w|<4r}|P(w)|/(\sup_{|\zeta-w|<r}|P(w)|)^2.$$

Notation. If F is an entire function, we put $V(F):=\{z\in\mathbb{C}^N\mid F(z)=0\}$. Its tangent cone at infinity is defined by
$$V_\infty(F):=\{ta\mid t\ge 0,\ a=\lim_{j\to\infty}a_j/|a_j|$$
$$\text{for some sequence } (a_j)_{j\in\mathbb{N}} \text{ in } V(F) \text{ with } \lim_{j\to\infty}|a_j|=\infty\}.$$
We note that $\operatorname{dist}(a,V_\infty(F))=o(|a|)$ as $a\in V(F)$ tends to infinity. This assertion is void (as is $V_\infty(F)$) if $V(F)$ is bounded.

2.2. Lemma. *Let $K\subset\mathbb{C}^N$ be convex and compact and let $\mu\in A(K)'$ be slowly decreasing on $V_\infty(\widehat{\mu})$. Then there is a locally bounded function $r:\mathbb{C}^N\to[1,\infty[$ with $r(z)=o(|z|)$ as $z\to\infty$ and such that for each $\varepsilon>0$ there is $R>0$ with the following property: Whenever $z\in\mathbb{C}^N$ and $|z|\ge R$ satisfies $U(z,r(z))\cap V(\widehat{\mu})\ne\emptyset$, then there is $w\in U(z,(1+\varepsilon)r(z))$ with $|\widehat{\mu}(w)|\ge\exp(L(\overline{w})-\varepsilon|w|)$. Put*
$$r'(z):=\sup\{|z-w|+2r(w)\mid w\in\mathbb{C}^N,\ |z-w|\le r(z)+r(w)\},\qquad z\in\mathbb{C}^N.$$
Then $2r\le r'$ and $r'(z)=o(|z|)$ as $|z|\to\infty$. If $U(z,r(z))\cap U(w,r(w))\ne\emptyset$, then also $U(w,2r(w))\subset U(z,r'(z))$.

Proof. Put $A:=S\cap V_\infty(\widehat{\mu})$. By Definition 1.5, for each $j\in\mathbb{N}$ there is $R_j>0$ such that for each $z\in V_\infty(\widehat{\mu})$ with $|z|\ge R_j$ there is $w\in U(z,|z|/j)$ with $|\widehat{\mu}(w)|\ge\exp(L(\overline{w})-|w|/j)$. We may assume that $R_j\ge j$, $R_{j+1}>R_j$ and that $V(\widehat{\mu})\backslash U(R_j)\subset\Gamma(A+U(1/j))$ for all $j\in\mathbb{N}$.

We put $r(z):=|z|/j$ if $R_j\le|z|<R_{j+1}$ for some $j\in\mathbb{N}$, and $r(z):=1$ if $|z|<R_1$. Direct computation shows that the functions r and r' have the desired properties.

2.3. *Auxiliary spaces.* Let $K\subset\mathbb{C}^N$ be convex and compact and let $\mu\in A(K)'$ be slowly decreasing on $V_\infty(\widehat{\mu})$. For each open set $\Omega\in\mathbb{C}^N$, let $A^2(\Omega)$ be the Hilbert space of all square integrable functions in $A(\Omega)$. Let $I(\Omega)$ be its closed subspace $I(\Omega)=(\widehat{\mu}\cdot A(\Omega))\cap A^2(\Omega)$. We put $E_\Omega:=A^2(\Omega)/I(\Omega)$ and for $x_\Omega\in E(\Omega)$,
$$|x_\Omega|_\Omega:=\inf_{\xi\in x_\Omega}|\xi|_2=\inf_{\xi\in x_\Omega}\Big(\int_\Omega|\xi|^2\,d\lambda\Big)^{1/2}.$$

We choose $r':\mathbb{C}^N\to[1,\infty[$ according to 2.2, and set $\widetilde{r}:=16r'$. For each $z\in\mathbb{C}^N$, we write $\Omega(z):=U(z,\widetilde{r}(z))$. We consider the Fréchet space
$$A^0_{H+L}(\widehat{\mu}):=\Big\{x=(f_{\Omega(z)}+I(\Omega(z)))_{z\in\mathbb{C}^N}\in\prod_{z\in\mathbb{C}^N}E_{\Omega(z)}\ \Big|\ \|x\|_y<\infty\quad\text{for all } y>0,$$
$$f_{\Omega(z)}-f_{\Omega(w)}\in I(\Omega(z)\cap\Omega(w))\text{ whenever }\Omega(z)\cap\Omega(w)\ne\emptyset\Big\},$$
where
$$\|x\|_y:=\sup_{z\in\mathbb{C}^N}|x_{\Omega(z)}|_{\Omega(z)}\exp(-H_y(\overline{z})-L(\overline{z})).$$
We note that $E_{\Omega(z)}=0$ if $\Omega(z)\cap V(\widehat{\mu})=\emptyset$.

2.4. Proposition. *Let $K\subset\mathbb{C}^N$ be convex and compact and let $\mu\in A(K)'$ be slowly decreasing on $V_\infty(\widehat{\mu})$ such that $(\widehat{\mu}\cdot A(\mathbb{C}^N))\cap A^0_{H+L}=\widehat{\mu}\cdot A^0_H$. Then the linear map*
$$\varrho:A^0_{H+L}/(\widehat{\mu}\cdot A^0_H)\to A^0_{H+L}(\widehat{\mu}),\qquad\varrho(f+\widehat{\mu}\cdot A^0_H):=(f|\Omega(z)+I(\Omega(z)))_{z\in\mathbb{C}^N},$$
is an isomorphism of Fréchet spaces. To be more precise, for all $0<y_2<y_1$, there is $C>0$ with

(a) $\|\varrho(f+\widehat{\mu}\cdot A^0_H)\|_{y_1}\le C\|f+\widehat{\mu}\cdot A^0_H\|_{y_2}$ *for all $f\in A^0_{H+L}$,*
(b) $\|\varrho^{-1}(x)\|_{y_1}\le C\|x\|_{y_2}$ *for all $x\in A^0_{H+L}(\widehat{\mu})$.*

Proof. As in [20], we roughly follow the proof of Meise and Taylor [14], Thm. 12. By direct computation, we see that the map ϱ is well defined and continuous in such a way that (a) holds. We are going to prove that ϱ is surjective and that ϱ^{-1} is continuous and satisfies (b).

Let $x=(x_{\Omega(z)})_{z\in\mathbb{C}^N}\in A^0_{H+L}(\widehat{\mu})$. We fix $0<y_3<y_2<y_1$. For each $z\in\mathbb{C}^N$, let $f_z\in A^2(\Omega(z))$ be unique with $f_z+I(\Omega(z))=x_{\Omega(z)}$ and minimal norm, i.e. with
$$|f_z|_2=\inf_{f\in x_{\Omega(z)}}|f|_2=|x_{\Omega(z)}|_{\Omega(z)}.$$
Since $x\in A^0_{H+L}(\widehat{\mu})$, for all $y>0$ we obtain
$$|f_z|_2\le\|x\|_y\exp(H_y(\overline{z})+L(\overline{z})),\qquad z\in\mathbb{C}^N.$$
Since $|f_z|^2$ is subharmonic, for all $z\in\mathbb{C}^N$ we get
$$(1)\qquad|f_z(\zeta)|\le(\operatorname{vol}_{2N}(U(\zeta,\widetilde{r}(z)/2)))^{-1/2}|f_z|_2\quad\text{ if }\zeta\in U(z,\widetilde{r}(z)/2).$$
By the definition of $A^0_{H+L}(\widehat{\mu})$, for all $z,w\in\mathbb{C}^N$ with $\Omega(z)\cap\Omega(w)\ne\emptyset$, there is $h_{z,w}\in A(\Omega(z)\cap\Omega(w))$ with
$$f_z-f_w=\widehat{\mu}h_{z,w}\quad\text{ on }\Omega(z)\cap\Omega(w).$$
Now, for each $z\in\mathbb{C}^N$, we put $\Omega'(z):=U(z,r(z))\subset U(z,\widetilde{r}(z)/20)$. If $\Omega'(z)\cap V(\widehat{\mu})\ne\emptyset$, we denote by f'_z the restriction of f_z to $\Omega'(z)$. If $\Omega'(z)\cap$

$V(\widehat{\mu}) = \emptyset$, we put $f'_z :\equiv 0$ on $\Omega'(z)$. For all $z, w \in \mathbb{C}^N$ with $\Omega'(z) \cap \Omega'(w) \neq \emptyset$ we define analytic functions $h'_{z,w}$ on $\Omega'(z) \cap \Omega'(w)$ by the restriction of $h_{z,w}$ to $\Omega'(z) \cap \Omega'(w)$ if $\Omega'(z)$ and $\Omega'(w)$ do intersect $V(\widehat{\mu})$. Otherwise we put $h'_{z,w} := (f'_z - f'_w)/\widehat{\mu}$. These functions trivially satisfy

$$f'_z - f'_w = \widehat{\mu} h'_{z,w} \quad \text{on } \Omega'(z) \cap \Omega'(w). \tag{2}$$

Let $z, w \in \mathbb{C}^N$ with $\Omega'(z) \cap \Omega'(w) \neq \emptyset$. For all $z \in \mathbb{C}^N$ and $\zeta \in \Omega'(z)$ we set $r_\zeta(z) := (9/80)\widetilde{r}(z) > 0$ and get (since $2r \le r' = \widetilde{r}/16$)

$$U(z, 2r(z)) \subset U(z, \widetilde{r}(z)/16) \subset U(\zeta, r_\zeta(z)),$$
$$U(\zeta, 4r_\zeta(z)) \subset U(z, \widetilde{r}(z)/2).$$

If $\Omega'(z)$ and $\Omega'(w)$ do not intersect $V(\widehat{\mu})$, then $h'_{z,w} \equiv 0$, by (2) and the definition of f'_z and f'_w. In the other case we may assume that $\Omega'(z) \cap V(\widehat{\mu}) \neq \emptyset$. We put $r_\zeta := \min\{r_\zeta(z), r_\zeta(w)\}$. By Lemma 2.2, for all $\zeta \in \Omega'(z) \cap \Omega'(w)$ we obtain

$$U(z, 2r(z)) \subset U(z, r'(z)) \cap U(w, r'(w)) \subset U(\zeta, r_\zeta)$$

and

$$U(\zeta, 4r_\zeta) \subset U(z, \widetilde{r}(z)/2) \cap U(w, \widetilde{r}(w)/2).$$

Thus by Lemmas 2.1 and 2.2, and by (1), for all $0 < y' < y$ there is $C_1 > 0$ not depending on x such that for all $z, w \in \mathbb{C}$ and $\zeta \in \Omega'(z) \cap \Omega'(w)$,

$$|h'_{z,w}(\zeta)| = |(f'_z(\zeta) - f'_w(\zeta))/\widehat{\mu}(\zeta)| \le C_1 \|x\|_{y'} \exp H_y(\overline{\zeta}).$$

We are going to find $a_z \in A(\Omega'(z))$, $z \in \mathbb{C}^N$, having appropriate bounds such that $h'_{z,w} = a_z - a_w$ on $\Omega'(z) \cap \Omega'(w)$ for all $z, w \in \mathbb{C}^N$.

As in [20] we choose a sequence $(z_j)_{j\in\mathbb{N}}$ in $\mathbb{C}^N$ such that $U(z_j, \widetilde{r}(z_j)/40)$, $j \in \mathbb{N}$, is a cover of $\mathbb{C}^N$ and such that $\Omega'(z_j)$, $j \in \mathbb{N}$, is locally finite in the following sense: each $z \in \mathbb{C}^N$ has a neighborhood which meets at most $l(z)$ sets $\Omega'(z_j)$ and $\log l(z) = O(\log(1 + |z|))$ as $|z| \to \infty$. Thus there is $C_2 > 0$ and there are functions $\phi_j \in \mathcal{D}(\Omega'(z_j))$, $j \in \mathbb{N}$, with $\sum_j \phi_j = 1$, $0 \le \phi_j \le 1$ and $|\overline{\partial}\phi_j(z)| \le C_2 l(z)$ for all $z \in \mathbb{C}^N$ and $j \in \mathbb{N}$. For each $z \in \mathbb{C}^N$ we define

$$h_z := \sum_{j=1}^{\infty} (\phi_j h'_{z,z_j}) | \Omega'(z).$$

Since the sum is locally finite, h_z is in $C^\infty(\Omega'(z))$. By (2), for all $z, w \in \mathbb{C}^N$ we have

$$h_z - h_w = \sum_{j\in\mathbb{N}} \phi_j h'_{z,w} = h'_{z,w} \quad \text{on } \Omega'(z) \cap \Omega'(w)$$

and in particular,

$$\overline{\partial} h_z = \overline{\partial} h_w \quad \text{on } \Omega'(z) \cap \Omega'(w).$$

Thus we can define $u \in C^\infty_{(0,1)}(\mathbb{C}^N)$ by $u|\Omega'(z) := \overline{\partial} h_z$ for all $z \in \mathbb{C}^N$. Since $(\Omega'(z_j))_{j\in\mathbb{N}}$ is locally finite in the sense described above, for all $0 < y' < y$ there is $C_3 > 0$ not depending on x with

$$|u(\zeta)| \le C_3 \|x\|_{y'} \exp H_y(\overline{\zeta}), \quad \zeta \in \mathbb{C}^N.$$

These bounds imply L^2-estimates, i.e. for all $0 < y' < y$ there are $C_4 > 0$ not depending on x with

$$\int_{\mathbb{C}^N} |u(\zeta)|^2 \exp(-2(H_y(\overline{\zeta}))\, d\lambda(\zeta) \le C_4 \|x\|^2_{y'}.$$

Since $\overline{\partial} u|\Omega'(z) = \overline{\partial}\,\overline{\partial} h_z = 0$ for each $z \in \mathbb{C}^N$, we get by Lemma 1.7 some $g \in W^2_H$ (even $g \in C^\infty(\mathbb{C}^N)$) with $\overline{\partial} g = u$. Moreover, we may assume that this g is chosen in such a way that

$$\int_{\mathbb{C}^N} |g(\zeta)|^2 \exp(-2(H_{y_2}(\overline{\zeta}) + 2\log(1 + |\zeta|^2)))\, d\lambda(\zeta)$$
$$\le 2 \int_{\mathbb{C}^N} |u(\zeta)|^2 \exp(-2(H_{y_2}(\overline{\zeta}))\, d\lambda(\zeta).$$

Then $a_z := h_z - g|\Omega'(z)$ is in $A(\Omega'(z))$ for each $z \in \mathbb{C}^N$, and for all $z, w \in \mathbb{C}^N$ we have

$$a_z - a_w = h'_{z,w} \quad \text{on } \Omega'(z) \cap \Omega'(w)$$

and thus

$$f'_z - \widehat{\mu} a_z = f'_w - \widehat{\mu} a_w \quad \text{on } \Omega'(z) \cap \Omega'(w).$$

Hence there is a unique $f \in A(\mathbb{C}^N)$ with $f = f'_z - \widehat{\mu} a_z$ on $\Omega'(z)$ for all $z \in \mathbb{C}^N$. Since f satisfies appropriate L^2-estimates on $\Omega'(z)$ and since $|f|^2$ is subharmonic on $\Omega'(z)$, $z \in \mathbb{C}^N$, the function f belongs to A^0_{H+L}, and moreover there is $C_5 > 0$ not depending on x with

$$|f(z)| \le C_5 \|x\|_{y_3} \exp(H_{y_1}(\overline{z}) + L(\overline{z})), \quad z \in \mathbb{C}^N.$$

Thus for each $x \in A^0_{H+L}(\widehat{\mu})$ and all $0 < y_3 < y_1$, we have constructed some $f \in A^0_{H+L}$ with $\varrho(f + \widehat{\mu} \cdot A^0_H) = x$ and

$$\|f + \widehat{\mu} \cdot A^0_H\|_{y_1} \le C_5 \|x\|_{y_3}.$$

This proves the assertion.

2.5. Corollary. *Let μ be as in Proposition 2.4. If $\overline{\partial} : W^2_H \to L^2_{H(0,1)}$ has a continuous linear right inverse or if for each $a \in V_\infty(\widehat{\mu})$ there is a plurisubharmonic function u_a on $\mathbb{C}^N$ such that for each $y > 0$ there are $x > 0$ and $C > 0$ with*

$$u_a(a) \ge 0 \quad \text{and} \quad u_a(z) \le C + H_y(\overline{z}) - H_x(\overline{a}) \quad \text{for all } z \in \mathbb{C}^N, a \in V_\infty(\widehat{\mu}),$$

then there is a continuous linear right inverse for the quotient map $A^0_{H+L} -$ $A^0_{H+L}/(\widehat{\mu}\cdot A^0_H)$.

Proof. We first note that in the hypothesis we may replace $V_\infty(\widehat{\mu})$ by $V(\widehat{\mu})$: If $V(\widehat{\mu})$ is bounded, we may choose $u_a :\equiv 0$ for all $a \in V(\widehat{\mu})$. Otherwise for each $a \in V(\widehat{\mu})$ we choose some $a' \in V_\infty(\widehat{\mu})$ with $|a-a'| = \operatorname{dist}(a, V_\infty(\widehat{\mu})$ and define $u_a(z) := u_{a'}(z-a+a')$ for $z \in \mathbb{C}^N$.

To prove the existence of a right inverse for $A^0_{H+L} \to A^0_{H+L}(\widehat{\mu})$, $f \mapsto$ $(f|\Omega(z) + I(\Omega(z)))_{z\in\mathbb{C}^N}$, we linearize the proof of the surjectivity of thi map in the proof of Proposition 2.4. This means that we will show that w can find preimages $f \in A^0_{H+L}$ of $x \in A^0_{H+L}(\widehat{\mu})$ with bounds as in 2.4, bu which depend on x in a linear way. To this end we only have to linearize th two choices which have been made in the proof of 2.4.

The first choice. For $z \in \mathbb{C}^N$, let P_z be the orthogonal projection o $A^2(\Omega(z))$ onto $I(\Omega(z))$. Then

$$R_z : E_{\Omega(z)} \to A^2(\Omega(z)), \quad R_z(x_{\Omega(z)}) := f_z - P_z(f_z), \quad \text{where } f_z \in x_{\Omega(z)},$$

is a continuous linear right inverse for the quotient map $A^2(\Omega(z)) \to E_{\Omega(z}$ and with an operator norm which does not exceed 1. In fact, in 2.4 we hav chosen $f_z = R_z(x_{\Omega(z)})$. Thus this choice has already been made in a linea way.

The second choice. In 2.4 for a given $x \in A^0_{H+L}(\widehat{\mu})$, a $\overline{\partial}$-closed $(0,1)$-forn $u \in L^2_{H\,(0,1)}$ has been constructed (which depends on x in a linear way). Pu

$$\Omega := \bigcup_z \Omega'(z),$$

where the union is taken over all $z \in \mathbb{C}^N$ for which $\Omega'(z)$ and $\bigcup\{\Omega'(z_j)$ $\Omega'(z_j)\cap\Omega(z) \neq \emptyset\}$ do not intersect $V(\widehat{\mu})$. By (2) and the definition of f and f'_{z_j}, we conclude that $\overline{\partial} h_z = 0$ for all such z.

Let $a \in \mathbb{C}^N\setminus\Omega$ but $a \notin V(\widehat{\mu})$. Since $\Omega'(a)$ is not contained in Ω, $\Omega'(a)\cap$ $V(\widehat{\mu}) \neq \emptyset$ or there is $j \in \mathbb{N}$ with $\Omega'(z_j)\cap\Omega(a) \neq \emptyset$ and $\Omega'(z_j)\cap V(\widehat{\mu}) \neq \emptyset$ Hence there is $a' \in \Omega'(a)\cap V(\widehat{\mu})$ or $a' \in \Omega'(z_j)\cap V(\widehat{\mu})$. We put $u_a(z) :=$ $u_{a'}(z-a+a')$ for all $z \in \mathbb{C}^N$. Then by the properties of the function r, th hypothesis of Proposition 1.8 is satisfied. Thus by Proposition 1.8, there i a continuous projection

$$P : W^2_H(\mathbb{C}^N, \Omega) \to A^0_H.$$

Now if g is in W^2_H with $\overline{\partial} g = u$, by the definition of u and Ω it follows tha $g \in W^2_H(\mathbb{C}^N,\Omega)$. If we put $\widetilde{g} := g - P(g)$, then $\widetilde{g}$ has the same properties a g (maybe with a larger constant C_5 and a smaller norm index y_3 which d not depend on x), but in addition it depends on u in a linear way.

2.6. Lemma. *Assume that for each $P \in A^0\setminus\{0\}$, the multiplication o erator $M_P : A^0_H \to A^0_H$, $M_P(f) = P\cdot f$, admits a continuous linear le*

inverse. If $\operatorname{int} G = \emptyset$, *there is $b \in S$ with $H(\overline{b}) = H(-\overline{b}) = 0$. In this case put $S' := \{ib\}$. Otherwise set $S' := S$. Then there is a family $(u_a)_{a\in\Gamma(S')}$ of plurisubharmonic functions on $\mathbb{C}^N$ such that the following holds: For each $y > 0$ there is $x > 0$ such that for all $z \in \mathbb{C}^N$ and $a \in \Gamma(S')$,*

$$0 \le u_a(a) \quad \text{and} \quad u_a(z) \le H_y(\overline{z}) - H_x(\overline{a}).$$

Proof. We proceed as in [20], Lemma 2.5. We first consider the case where $\operatorname{int} G \neq \emptyset$. For $x > 0$, we fix $G_x := (1+x)\operatorname{int} G$. Let $(a_j)_{j\in\mathbb{N}}$ be a dense sequence in $S' = S$. We fix $j \in \mathbb{N}$ and choose a hyperplane $a_j + W_j$ in $\mathbb{R}^{2N}$ which supports the convex set $\{z \in \mathbb{C}^N \mid H_1(\overline{z}) \le H_1(\overline{a}_j)\}$ in a_j. By the choice of $(G_x)_{x>0}$, the same hyperplane also supports $\{z \in \mathbb{C}^N \mid H_x(\overline{z}) \le H_x(\overline{a}_j)\}$ in a_j for all $x > 0$. The maximal $\mathbb{C}$-linear subspace $L_j := iW_j \cap W_j$ of the $\mathbb{R}$-linear space W_j has real dimension $2N-2$, hence complex dimension $N-1$, and has the property

$$\inf_{z\in a_j+L_j} H_x(\overline{z}) = H_x(\overline{a}_j) \quad \text{for all } x > 0.$$

We choose a $\mathbb{C}$-linear functional $l_j : \mathbb{C}^N \to \mathbb{C}$ with $\ker l_j = L_j$. Then for each $j \in \mathbb{N}$, we choose a sequence $(\lambda_{m,j})_{m\in\mathbb{N}}$ of positive numbers with $\lambda_{m+1,j} > 2\lambda_{m,j}$, $m \in \mathbb{N}$, such that the products

$$P_j(z) := \prod_{m=1}^{\infty} (1 - l_j(z)/l_j(\lambda_{m,j}a_j)), \quad z \in \mathbb{C}^N,\ j \in \mathbb{N},$$

and

$$P(z) = \prod_{j=1}^{\infty} P_j(z), \quad z \in \mathbb{C}^N,$$

define elements of A^0 (see Levin [11], Chap. I, Secs. 3 and 4). We fix $j \in \mathbb{N}$ again. We have

$$V(P_j) = \bigcup_{m\in\mathbb{N}} (\lambda_{m,j}a_j + L_j).$$

According to Proposition 2.4, we identify $A^0_H/(P_j\cdot A^0_H)$ and $A^0_H(P_j)$. Since $P_j(z+w) = P_j(z)$ for all $w \in L_j$ and since $\lambda_{m+1,j} \ge 2\lambda_{m,j}$, $m \in \mathbb{N}$, we may assume that the function r' of Lemma 2.2 is chosen in such a way that each ball $\Omega(z) = U(z, 16r'(z))$, $z \in \mathbb{C}^N$, meets at most one hyperplane $\lambda_{m,j}a_j + L_j$, $m \in \mathbb{N}$. By the hypothesis, the multiplication operator $M_P :$ $A^0_H \to A^0_H$ has a continuous linear left inverse $Q : A^0_H \to A^0_H$. Let $M_{P'_j} :$ $A^0_H \to A^0_H$ be the operator of multiplication by $P'_j := \prod_{j'\in\mathbb{N}, j'\neq j} P_{j'}$. Then $Q_j := Q\circ M_{P'_j}$ is a continuous linear left inverse of $M_{P_j} : A^0_H \to A^0_H$, and it induces a continuous linear right inverse $R_j : A^0_H(P_j) = A^0_H/(P_j\cdot A^0_H) \to A^0_H$ for the quotient map. Now for each $m \in \mathbb{N}$, let $f_{m,j} \in A^0_H(P_j)$ be given by $f_{m,j}|\Omega(z) = 1$ modulo $I(\Omega(z))$ if $\Omega(z)\cap(\lambda_{m,j}a_j + L_j) \neq \emptyset$, and $f_{m,j}|\Omega(z) = 0$

modulo $I(\Omega(z))$ otherwise, $z \in \mathbb{C}^N$. Then $u_{m,j} := \log|R_j(f_{m,j})|$ is plurisubharmonic on $\mathbb{C}^N$ with $u_{m,j} = 0$ on $\lambda_{m,j} a_j + L_j$. Let

$$\sigma_Q(y) := \sup\{x > 0 \mid \sup_{\|f\|_x \le 1} \|Q(f)\|_y < \infty\}, \quad y > 0,$$

be the characteristic of continuity of Q. By Proposition 2.4, for all $y > 0$ and all $0 < x < x' < \sigma_Q(y)$, there are $C, C' > 0$ such that for all $m \in \mathbb{N}$,

$$\sup_{z \in \mathbb{C}^N} (u_{m,j}(z) - H_y(\overline{z})) = \log\|R_j(f_{m,j})\|_y \le C' + \log\|f_{m,j}\|_{x'}$$
$$\le C + \sup_{z \in \lambda_{m,j} a_j + L_j} (-H_x(\overline{z})) = C - \lambda_{m,j} H_x(\overline{a}_j).$$

We substitute $z = \lambda_{m,j} w$. For the upper semicontinuous regularization u_j of $\limsup_{m\to\infty} \lambda_{m,j}^{-1} u_{m,j}(\lambda_{m,j}\cdot)$ we get $0 \le u_j(a_j)$ and

$$u_j(w) \le H_y(\overline{w}) - H_x(\overline{a}_j), \quad w \in \mathbb{C}^N.$$

We fix $a \in S' = S$. We choose a subsequence $(a_{j_k})_{k\in\mathbb{N}}$ converging to a and denote by u_a the upper semicontinuous regularization of $\limsup_{k\to\infty} u_{j_k}$. Then by the Hartogs Lemma, we have

$$0 \le u_a(a) \quad \text{and} \quad u_a(w) \le H_y(\overline{w}) - H_x(\overline{a}), \quad w \in \mathbb{C}^N.$$

Finally, for each $a \in \Gamma(S')\backslash\{0\}$ we put

$$u_a(w) := |a| u_{a/|a|}(w/|a|), \quad w \in \mathbb{C}^N,$$

and the proof is finished in the case where $\operatorname{int} G \ne \emptyset$.

In the other case, if $a \in S'$ we distinguish the cases $H(\overline{a}) = 0$ and $H(\overline{a}) > 0$. We will prove a little bit more than necessary.

Put $S'_1 := \{ib \mid b \in S, H(\overline{b}) = H(-\overline{b}) = H(\overline{ib}) = 0\}$ and $G_x := G + U(x)$. Then for each $x > 0$ and each $a \in S'_1$, we have $H_x(\overline{z}) = x|z|$ for all z in a neighborhood of a. Now the previous proof produces plurisubharmonic functions $(u_a)_{a\in\Gamma(S'_1)}$ with the desired properties.

Let $\varepsilon > 0$. Put $S'_2 := \{ib \mid b \in S,\ H(\overline{b}) = H(-\overline{b}) = 0 \text{ and } H(\overline{ib}) \ge \varepsilon\}$ and let G_x be the interior of the convex hull of $(1+x)G \cup U(x)$. Since $H_x(z) = \max\{(1+x)H(z), x|z|\}$, $z \in \mathbb{C}^N$, for sufficiently small $0 < x \le x_0$ we have $H_x(\overline{z}) = (1+x)H(\overline{z})$ for all z in some neighborhood of S'_2. Now after small changes, the previous proof produces plurisubharmonic functions $(u_a)_{a\in\Gamma(S'_2)}$ with the desired properties.

From [21] we recall the following notation:

Notation. We define

$$v_H(z) := \sup_u u(z), \quad z \in \mathbb{C}^N,$$

where the supremum is taken over all plurisubharmonic functions u with $u \le H$ such that $u \le \log|z| + O(1)$ as $z \to 0$. By [21], this function is plurisubharmonic, does not exceed H, and satisfies $v_H \le \log|z| + O(1)$ as $z \to 0$ (here we allow a plurisubharmonic function to be $\equiv -\infty$). By [21], there is a unique upper semicontinuous function $C_H : S \to [0,\infty[$ such that

$$P_H := \{z \in \mathbb{C}^N \mid v_H(z) = H(z)\} = \{\lambda a \mid a \in S, 1/C_H(a) \le \lambda < \infty\}.$$

2.7. Remark. If G is not pluripolar, i.e. if the $\mathbb{C}$-linear span of G equals $\mathbb{C}^N$, by [21] we have

$$H(z) = \lim_{\delta\downarrow 0} \delta v_H(z/\delta) = \sup_{\delta>0} \delta v_H(z/\delta), \quad z \in \mathbb{C}^N\backslash\{0\}.$$

The limit is uniform on closed subsets of $\mathbb{C}^N\backslash\{0\}$.

2.8. Lemma. *For $N = 1$ let $\Gamma_H \subset \mathbb{C}$ be the support of ΔH, i.e. H is harmonic precisely on $\mathbb{C}\backslash\Gamma_H$. If $u \le H$ is subharmonic on $\mathbb{C}$ and $u(0) < 0$, then $\{z \in \mathbb{C} \mid u(z) = H(z)\} \subset \Gamma_H$.*

Proof. Assume that there is $z \in \mathbb{C}\backslash\Gamma_H$ with $u(z) = H(z)$. Then $u - H$ is a nonpositive subharmonic function on $\mathbb{C}\backslash\Gamma_H$ which vanishes at z. Hence it vanishes on the component of $\mathbb{C}\backslash\Gamma_H$ which contains z. This contradicts $u(0) < 0 = H(0)$ since u is upper semicontinuous.

2.9. Lemma. *Assume that G is not pluripolar. Let $A \subset S$ be closed. The following are equivalent*:

(i) *There is $\delta > 0$ with $C_H(a) \ge \delta$ for all $a \in A$.*

(ii) *For each (some) $y > 0$ there is $\varepsilon > 0$ such that $u_{y,\varepsilon} = H$ on A, where $u_{y,\varepsilon}$ is the largest plurisubharmonic function on $\mathbb{C}^N$ with $u_{y,\varepsilon} \le H$ and with $u_{y,\varepsilon} \le H_y - \varepsilon$.*

(iii) *There is a plurisubharmonic function on $\mathbb{C}^N$ with $u \le H$, $u(0) < 0$ and $u = H$ on A.*

(iv) *For each $a \in \Gamma(A)$ there is a plurisubharmonic function u_a on $\mathbb{C}^N$ such that for each $y > 0$ there are $x > 0$ and $C > 0$ such that for all $z \in \mathbb{C}^N$ and $a \in \Gamma(A)$,*

$$u_a(a) \ge 0 \quad \text{and} \quad u_a(z) \le C + H_y(z) - H_x(a).$$

If G is pluripolar, we have (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv).

Proof. (i)$\Rightarrow$(ii). Let $y > 0$. By the hypothesis, there is $\delta > 0$ such that $v_H(a/\delta) = H(a/\delta)$ for all $a \in A$. Define $u := \delta v_H(\cdot/\delta)$. Then u is plurisubharmonic, $u \le H$ and, since u is upper semicontinuous and $u(0) = -\infty < 0$, we can choose $\varepsilon > 0$ so small that $u \le H_y - \varepsilon$. Thus $u_{y,\varepsilon} \ge u = H$ on A.

(ii)$\Rightarrow$(i). Assume that (ii) holds for some $y > 0$ and $\varepsilon > 0$. Since $u_{y,\varepsilon} \le H_y - \varepsilon$, we may choose $0 < r < 1$ with $u_{y,\varepsilon}(z) \le H(z) - \varepsilon/2$ for all $|z| = r$. According to Remark 2.7, we may choose $\delta > 0$ so small that $\delta v_H(z/\delta) \ge H(z) - \varepsilon/2$ for all $|z| = r$. We define $u := \delta v_H(\cdot/\delta)$ on $B(r)$ and $u :=$

$\max\{\delta v_H(\cdot/\delta), u_{y,\varepsilon}\}$ elsewhere. Then u is plurisubharmonic on $\mathbb{C}^N$ with $u \le H$ and $u \le \delta \log|z| + O(1)$ as $z \to 0$. Thus $v_H \ge u(\delta\cdot)/\delta$. Since $u(z) \ge u_{y,\varepsilon}(z)$ if $|z| = 1$, we obtain $H(a') \ge v_H(a') \ge u_{y,\varepsilon}(\delta a')/\delta = H(a')$ if $\delta a' \in A$. Then $\delta v_H(a/\delta) = H(a)$ for all $a \in A$. Hence $C_H(a) \ge \delta$ for all $a \in A$.

(ii)⇒(iii). Put $u := u_{y,\varepsilon}$ for some $y > 0$ and $\varepsilon > 0$ chosen according to (ii).

(iii)⇒(iv). For each $a \in \Gamma(A)\backslash\{0\}$ we put

$$u_a := u(\cdot/|a|)|a| - H(a).$$

Then u_a is plurisubharmonic and $u_a(a) = u(a/|a|)|a| - H(a) = 0$. Let $y > 0$ be arbitrary. Since $u \le H$ and $u(0) < 0$, there is $\varepsilon > 0$ with $u \le H_y - \varepsilon$. We choose $x > 0$ with $H_x(z) \le H(z) + \varepsilon|z|$ for all $z \in \mathbb{C}^N$. Then

$$u_a(z) \le H_y(z) - \varepsilon|a| - H(a) \le H_y(z) - H_x(a) \quad \text{for all } z \in \mathbb{C}^N,\ a \in \Gamma(A)\backslash\{0\}.$$

(iv)⇒(ii). Replacing u_a by the upper semicontinuous regularization of $\limsup_{\lambda\to\infty} \lambda^{-1} u_{\lambda a}(\lambda\cdot)$, we may assume $C = 0$ in (iv). Hence for each $y > 0$ there are $x > 0$ and $\varepsilon > 0$ with

$$u_a + H(a) \le H_y - (H_x(a) - H(a)) \le H_y - \varepsilon|a| \le H_y, \quad a \in \Gamma(A).$$

Thus for each $a \in A$ we have $u_a + H(a) \le H$, $u_a(a) + H(a) = H(a)$, and for each $y > 0$ there is $\varepsilon > 0$ with $u_a + H(a) \le H_y - \varepsilon$. This gives $u_{y,\varepsilon} = H$ on A.

2.10. Theorem. *For each convex compact set $G \subset \mathbb{C}^N$ containing the origin in its relative interior, the following are equivalent:*

(i) *Each differential operator $P(D) : A(G) \to A(G)$, $P \in A^0\backslash\{0\}$, admits a solution operator.*

(ii) *There is $\delta > 0$ with $C_H \ge \delta$ on S, i.e. $v_H = H$ outside a compact neighborhood of the origin* (*i.e. in a "neighborhood of infinity"*).

(iii) *There is a plurisubharmonic function u on $\mathbb{C}^N$ with $u \le H$, $u(0) < 0$ and with $u = H$ in a neighborhood of infinity.*

(iv) *There is a family $(u_a)_{a\in\mathbb{C}^N}$ of plurisubharmonic functions on $\mathbb{C}^N$ such that the following holds: For each $y > 0$ there is $x > 0$ such that for all $z, a \in \mathbb{C}^N$,*

$$0 \le u_a(a) \quad \text{and} \quad u_a(z) \le H_y(\overline{z}) - H_x(\overline{a}).$$

(v) *The continuous operator $\overline{\partial} : W_H^2 \to L^2_{H(0,1)}$ has a continuous linear right inverse* (see Lemma 1.7).

Each of these equivalent conditions implies that the interior of G is non-void.

Proof. First we prove that (i) and (iv) each imply that $\operatorname{int} G$ is nonvoid. Assume that (i) holds and that $\operatorname{int} G = \emptyset$. Then there is $b \in S$ with $H(\overline{b}) = H(\overline{-b}) = 0$. We put $a := ib$. By Lemma 2.6, there is a plurisubharmonic function u on $\mathbb{C}^N$ with $u(a) \ge 0$ and such that for all $y > 0$ there is $x > 0$ with

$$(3) \qquad u(z) \le H_y(\overline{z}) - H_x(\overline{a}) \quad \text{for all } z \in \mathbb{C}^N.$$

The function $\widetilde{H} : \mathbb{C} \to \mathbb{R}_+$, $\zeta \mapsto H(\overline{\zeta b})$, is the support function of a compact convex subset of $\mathbb{C}$. Since $\widetilde{H}(-1) = \widetilde{H}(1) = 0$, this set is contained in $i\mathbb{R}$. Hence $\widetilde{H}$ is harmonic in the upper (and lower) halfplane. For $\widetilde{u} : \zeta \mapsto u(\zeta b)$, we deduce from (3) that $\widetilde{u} \le \widetilde{H}$ and $\widetilde{u}(i) = \widetilde{H}(i)$. By Lemma 2.8, this is a contradiction to $\widetilde{u}(0) \le -H_x(\overline{a}) < 0$.

If (iv) holds, by the reasoning of Lemma 2.9(iv)⇒(ii), we may assume that $C = 0$. Hence as above we get a contradiction if we assume that $\operatorname{int} G = \emptyset$.

(i)⇒(iv). Since $\operatorname{int} G \ne \emptyset$, (iv) follows from Lemma 2.6.

(iv)⇒(v). By Langenbruch [8], Thm. 1.3 and Rem. 1.11, there is a continuous linear projection $P : W_H^2 \to A_H^0$. Hence by Lemma 1.7, a continuous linear right inverse $R : L^2_{H(0,1)} \to W_H^2$ for $\overline{\partial} : W_H^2 \to L^2_{H(0,1)}$ is given by

$$R(g) := f - P(f) \quad \text{whenever } f \in W_H^2 \text{ with } \overline{\partial} f = g.$$

(v)⇒(i): Corollary 2.5.

(iv)⇒(ii). Since the interior of G is nonvoid, this holds by Lemma 2.9.

(ii)⇒(iii)⇒(iv): Lemma 2.9.

In the case of one complex variable we get a complete result for a given single convolution operator:

2.11. Theorem. *For $N = 1$, let $G, K \subset \mathbb{C}$ be compact and convex. Let G contain the origin in its relative interior. If $\mu \in A(K)'$ defines a surjective convolution operator $T_\mu : A(G + K) \to A(G)$, then the following are equivalent* (see Proposition 1.5):

(i) *$T_\mu : A(G+K) \to A(G)$ admits a solution operator.*

(ii) *There is $\delta > 0$ with $C_H(\overline{a}) \ge \delta$ for all $a \in A := S \cap V_\infty(\widehat{\mu})$.*

Proof. (i)⇒(ii). Following an idea of Korobeĭnik and Melikhov [5], we make a reduction to the case of a differential operator (see also [17], Lemma 8). We choose a canonical product $P \in A^0\backslash\{0\}$ with $V_\infty(P) = V_\infty(\widehat{\mu})$ and such that $g := \widehat{\mu}/P$ is an entire function. g has the same indicator as $\widehat{\mu}$ (see Levin [11], III, Thm. 5). By the hypothesis and by Duality 1.4, the multiplication operator $M_{\widehat{\mu}} : A_H^0 \to A_{H+L}^0$ has a continuous linear left inverse L. Hence the operator $LM_g : A_H^0 \to A_H^0$ is a continuous linear left inverse for $M_P : A_H^0 \to A_H^0$. As in the proof of Lemma 2.6 (with $S' := S \cap V_\infty(P)$), we obtain subharmonic functions u_a on $\mathbb{C}$, $a \in V_\infty(P) = V_\infty(\widehat{\mu})$, such that for each $y > 0$ there is $x > 0$ with

$$u_a(a) \ge 0 \quad \text{and} \quad u_a(z) \le H_y(\overline{z}) - H_x(\overline{a}) \quad \text{for all } z \in \mathbb{C},\ a \in V_\infty(\widehat{\mu}).$$

By Lemma 2.9, applied with $A := \{\overline{a} \mid a \in S'\}$, we get (ii) in the case where G is not polar. If G is polar, i.e. if $G = \{0\}$, then $H \equiv 0$ is harmonic. The reasoning at the beginning of the proof of Theorem 2.10 shows that the assumption $V_\infty(\widehat{\mu}) \neq \emptyset$ leads to a contradiction. Thus $V_\infty(\widehat{\mu}) = \emptyset$ and (ii) holds trivially.

(ii)⇒(i). We first consider the case $G = \{0\}$. In this case $v_H \equiv -\infty$ and thus $C_H \equiv 0$. Hence (ii) implies that $V_\infty(\widehat{\mu}) = \emptyset$, i.e. $V(\widehat{\mu})$ consists of at most finitely many points. By Hadamard's factorization theorem, there are $w \in \mathbb{C}$ and a nonzero polynomial P with $\widehat{\mu}(z) = P(z)e^{zw}$, $z \in \mathbb{C}$. Since $T_\mu : A(K) \to A(\{0\})$, $T_\mu(f) = P(D)f(\cdot + w)$, is surjective, we obtain $K = \{w\}$. This shows that $\widehat{\mu}$ is slowly decreasing. Thus (i) holds by Corollary 2.5 (see Proposition 1.5(a)).

Now let G be nonpolar. By Proposition 1.5(b), $\widehat{\mu}$ is slowly decreasing on the support $\Gamma_{\overline{H}}$ of $\Delta\overline{H}$. By Lemma 2.8, we have $\Gamma(P_{\overline{H}}) \subset \Gamma_{\overline{H}}$. By the hypothesis, $V_\infty(\widehat{\mu}) \subset \Gamma(P_{\overline{H}})$. Hence $\widehat{\mu}$ is slowly decreasing on $V_\infty(\widehat{\mu})$. By Proposition 1.5(c), also $(\widehat{\mu} \cdot A(\mathbb{C})) \cap A^0_{H+L} = \widehat{\mu} \cdot A^0_H$.

Furthermore, by the hypothesis and by Lemma 2.9, there are subharmonic functions u_a on $\mathbb{C}$, $a \in V_\infty(\widehat{\mu})$, such that for each $y > 0$ there is $x > 0$ with

$$u_a(a) \geq 0 \quad \text{and} \quad u_a(z) \leq H_y(\overline{z}) - H_x(\overline{a}) \quad \text{for all } z \in \mathbb{C},\ a \in V_\infty(\widehat{\mu}).$$

Thus all the hypotheses of Corollary 2.5 are satisfied. Hence T_μ admits a solution operator.

2.12. COROLLARY. *For $N = 1$ let $K \subset \mathbb{C}$ be convex and compact and $\mu \in A(K)'$.*

(a) *If $G = \{0\}$, the only convolution operators $T_\mu : A(K) \to A(\{0\})$ which admit a solution operator are those for which $K = \{w\}$ and $\widehat{\mu}(z) = P(z)e^{zw}$, $z \in \mathbb{C}$, for some $w \in \mathbb{C}$ and some nonzero polynomial P.* (The result for $K = \{0\}$ is essentially contained in Meise and Taylor [13].)

(b) *Let $G = [a,b] \subset \mathbb{R}$ be a nontrivial compact interval, and suppose the convolution operator $T_\mu : A(G+K) \to A(G)$ is surjective. Then $T_\mu : A(G+K) \to A(G)$ admits a solution operator if and only if $V_\infty(\widehat{\mu}) \subset \mathbb{R}i$.* (The result for $K = \{0\}$ is also contained in Langenbruch [9].)

(c) *If G is a compact convex polygon, let $A \subset S$ be the (finite) set of outer normals to the faces of G. If the convolution operator $T_\mu : A(G+K) \to A(G)$ is surjective, then it admits a solution operator if and only if $V_\infty(\widehat{\mu}) \subset \bigcup_{a\in A} \mathbb{R}_+\overline{a}$.*

Proof. (a) follows from 2.11. (c) implies (b).

(c) By Theorem 2.11, we only have to prove $\Gamma(P_{\overline{H}}) = \bigcup_{a\in A} \mathbb{R}_+\overline{a}$. By Lemma 2.8, the inclusion "⊂" holds. The other inclusion holds by Lemma 2.9

and for instance by [19], Lemma 2.9 (which also works in the present situation).

Remark. We recall that until now no characterization of the surjective convolution operators $T_\mu : A([-1,1] + K) \to A([-1,1])$ is known.

An immediate consequence of Theorem 2.11 is the following.

2.13. THEOREM. *For $N = 1$, the statements of Theorem 2.10 are also equivalent to*

(vi) *Each surjective convolution operator $T_\mu : A(G+K) \to A(G)$ admits a solution operator.*

NOTATION AND REMARK. If G is not pluripolar, there is a largest plurisubharmonic function $g_G : \mathbb{C}^N \to \mathbb{R}_+$ with $g_G = 0$ on G and such that $g_G(z) - \log(1+|z|)$ is bounded on $\mathbb{C}^N$, namely

$$g_G(z) := \sup_u u(z), \quad z \in \mathbb{C}^N,$$

where the supremum is taken over all plurisubharmonic function u on $\mathbb{C}^N$ with $u \leq 0$ on G and such that $u(z) - \log(1+|z|)$ is bounded above on $\mathbb{C}^N$. The function g_G is called the *pluricomplex Green function* of G with pole at infinity (see Klimek [4] and [21]). Let H_x denote the support function of the (convex) level set $G_x := \{z \in \mathbb{C}^N \mid g(z) \leq x\}$, $x > 0$. It is shown in [21] that a lower bound for $C_H|A$ is equivalent to a lower bound for a certain quantity $D_G|A$ which measures the rate of approximation of G by the level sets G_x, $x > 0$, in the directions of A, namely

$$D_G(a) := \lim_{x\downarrow 0} \frac{H_x(a) - H(a)}{x} \in [0,\infty[, \quad a \in S.$$

In [21] we prove that there is $\delta > 0$ with $\delta C_H \leq D_G \leq C_H$. This gives:

2.14. THEOREM. *If $N = 1$ and G is not polar* (*i.e.* $\#G > 1$), *the assertions of Theorem 2.10 are also equivalent to*

(vii) *For each* (*some*) *biholomorphic mapping $\psi : \{z \in \mathbb{C} \mid |z| > 1\} \to \mathbb{C}\backslash G$ with $\psi(\infty) = \infty$, there is $\delta > 0$ such that $|\psi'(z)| \geq \delta$ for all $|z| > 1$.*

(viii) *There is $\delta > 0$ such that $G + \delta x U(1) \subset G_x$ for all $x > 0$.*

Proof. By [21], the assertions (viii) and Proposition 1.9(ii) are equivalent. An application of the Koebe distortion theorem as in [16], Lemma 3.4, shows that (viii) and (vii) are equivalent (Korobeĭnik and Melikhov [5], Thm. 4.3).

Remark. If $\mu \in A(K)'$ for some compact and convex set $K \subset \mathbb{C}^N$, in the present paper we considered for each convex compact set $G \subset \mathbb{C}^N$ the convolution operator $T_\mu : A(G+K) \to A(G)$. Let $0 \in \operatorname{int} G$. It has been proved by Krivosheev [7] (see also [18], Thm. 3.9) that $T_\mu : A(G+K)$ – $A(G)$ is surjective if and only if $T_\mu : A(\operatorname{int} G + K) \to A(\operatorname{int} G)$ is surjective

It is an obvious question whether there is a solution operator $A(G)$ – $A(G+K)$ if and only if there is a solution operator $A(\operatorname{int} G) \to A(\operatorname{int} G+K)$ We do not know the answer in general. If $N = 1$, in many "concrete situations the answer is yes, because well known theorems of function theor give at the same time the same answer for both cases, i.e., for G and int G In particular, all examples which have been given in [16] for domains int (are in an obvious way also examples for compact sets G.

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DEPARTMENT OF MECHANICS
AND MATHEMATICS
ROSTOV STATE UNIVERSITY
ZORGE ST. 5
344104 ROSTOV-NA-DONU, RUSSIA

MATHEMATISCHES INSTITUT
HEINRICH-HEINE-UNIVERSITÄT DÜSSELDORF
UNIVERSITÄTSSTRASSE 1
40225 DÜSSELDORF, GERMANY

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