Solution operators for convolution equations on the germs of analytic functions on compact convex sets in $\mathbb{C}^N$

by

S. N. MELIKHOV (Rostov-na-Donu) and SIEGFRIED MOMM (Düsseldorf)

Abstract. If $G \subset \mathbb{C}^N$ is compact and convex it is known for a long time that the nonzero constant coefficients linear partial differential operators (of finite or infinite order) are surjective on the space of all analytic functions on $G$. We consider the question whether solutions of the inhomogeneous equation can be given in terms of a continuous linear operator. For instance we characterize those sets $G$ for which this is always the case.

Introduction. For a given compact convex set $G \subset \mathbb{C}^N$, let $A(G)$ denote the space of all germs of analytic functions on $G$. This space is endowed with its natural inductive limit topology. If $K \subset \mathbb{C}^N$ is another compact convex set, for each analytic functional $\mu \in A(\mathbb{C}^N)^* \setminus \{0\}$ carried by $K$, a continuous linear operator is given by

$$T_\mu : A(G + K) \to A(G), \quad T_\mu(h)(x) = \langle \mu, h(z + \cdot) \rangle, \quad z \in G.$$

If $K = \{0\}$, the convolution operator $T_\mu$ is a partial differential operator on $A(G)$ (of finite or infinite order) and can be written as $T_\mu(f) = \sum_{\alpha \in \mathbb{N}^N} a_\alpha f^{(\alpha)}, \quad f \in A(G)$. The coefficients are determined by the entire function $\tilde{\mu}(z) = \mu(e^{zw}) = \sum_{\alpha \in \mathbb{N}^N} a_\alpha z^\alpha$. In this case $T_\mu$ is surjective. If $K \neq \{0\}$, a characterization of surjective operators $T_\mu$ is known when $G$ has a nonempty interior.

In the present paper, we investigate whether a given surjective operator $T_\mu : A(G + K) \to A(G)$ admits a continuous linear right inverse $R : A(G) \to A(G + K)$, i.e. we investigate whether it is possible to find solutions $R(f) \in A(G + K)$ of the convolution equation $T_\mu(R(f)) = f$ which depend on $f \in A(G)$ in a continuous and linear way.


The first named author thanks for the support by the Russian Foundation of Fundamental Research.
To formulate our result, we assume that the origin of \( C^N \) is contained in the relative interior of the convex compact set \( G \). Let \( H : C^N \to [0, \infty] \) be the support function of \( G \) and denote by \( v_H : C^N \to [-\infty, \infty] \) the extremal plurisubharmonic function introduced in [21]: \( v_H \) is the largest plurisubharmonic function on \( C^N \) with \( v_H \leq H \) and \( v_H(z) \leq \log |z| + O(1) \) as \( z \to 0 \). If \( P_H \subseteq C^N \) denotes the set of all \( z \in C^N \) for which \( v_H(z) = H(z) \), then there is an upper semicontinuous function \( C_H : \{ z \in C^N : |z| = 1 \} \to [0, \infty] \) such that \( P_H = \{ a \in S : 1/C_H(a) \leq \lambda < \infty \} \).

**Theorem I.** The following statements are equivalent:

(i) Each nonzero partial differential operator \( T_\mu : A(G) \to A(G) \) admits a continuous linear right inverse.

(ii) There is \( \delta > 0 \) such that \( C_H \geq \delta \) on \( S \).

(iii) There is a neighborhood of infinity (i.e., the complement of a compact set) on which \( v_H = H \). In particular, the interior of \( G \) is not empty.

(iv) There is a plurisubharmonic function \( u \leq H \) on \( C^N \) with \( u(0) < 0 \) and \( u = H \) on a neighborhood of infinity.

If \( N = 1 \), we can also characterize whether a single convolution operator \( T_\mu : A(G + K) \to A(G) \) admits a continuous linear right inverse. For this sake let \( A_\mu \subseteq S \) be the set of all accumulation points of \( (a/|a|)_{a \in C|a|=0} \).

**Theorem II.** Let \( N = 1, K \) a compact convex set in \( C \) and \( \mu \) a nonzero analytic functional on \( A(C) \) which is carried by \( K \). If the convolution operator \( T_\mu : A(G + K) \to A(G) \) is surjective, the following are equivalent:

(i) \( T_\mu : A(G + K) \to A(G) \) admits a continuous linear right inverse.

(ii) There is some \( \delta > 0 \) with \( C_H \geq \delta \) on \( A_\mu \).

In particular, the equivalent conditions of Theorem I imply that for every compact convex set \( K \subseteq C \) all surjective convolution operators \( T_\mu : A(G + K) \to A(G) \) admit continuous linear right inverses.

By [21], when \( G \) is not pluripolar and \( g_G : C^N \to [0, \infty] \) is the pluricomplex Green function of \( C^N \setminus G \) with pole at infinity, the function \( C_H \) may be replaced by a function \( D_G \) which measures the growth of \( g_G \) at \( \partial G \): We put \( G_\delta := \{ z \in C^N : g_G(z) \leq \delta \} \), we denote the support function of \( G_\delta \) by \( H_\delta \), and define \( D_G(a) := \lim_{x \to 0} \frac{H_\delta(a) - H(a)}{x} \in [0, \infty], \quad \forall a \in S \).

By [21], Theorem I implies:

**Theorem III.** For \( N = 1 \), let \( G \) be a compact convex set in \( C \) with \( \# G > 1 \). The following are equivalent:

(i) Each nonzero differential operator \( T_\mu : A(G) \to A(G) \) admits a continuous linear right inverse.

(ii) For each (some) biholomorphic mapping \( \psi : \{ z \in C : |z| > 1 \} \to C \setminus G \) with \( \psi(\infty) = \infty \), there is \( \delta > 0 \) with \( \psi(z) \geq \delta \).

(iii) There is \( \delta > 0 \) such that \( G + \delta u \subseteq C_\delta \) for all \( u > 0 \), where \( U \) denotes the unit disc in \( C \).

For the evaluation of these conditions, we refer to [21] and to classical results from function theory. For example, the equivalent conditions of Theorem III hold if the boundary of \( G \) is of class \( C^\lambda \) for some \( \lambda > 1 \). They do not hold if \( \partial G \) has a corner. The results of the present paper extend results of [16], [17] and [20], where the dual case of open convex \( G \subseteq C^N \) was investigated. There are two special cases for which different but equivalent versions of Theorem II have been obtained earlier or simultaneously, respectively: If \( K = G = \{ 0 \} \) it follows from Meise and Taylor [13] that only the nonzero differential operators of finite order admit continuous linear right inverses. If \( K = \{ 0 \} \) and \( G = [-1, 1] \subseteq R \), Langenbruch [9] proves that a nonzero operator \( T_\mu \) admits a continuous linear right inverse if and only if \( \mu \subseteq [-1, 1] \).

For the proof of our results, we extend the technique which has been developed in [20] using ideas of Meise and Taylor [14]. Doing this we extend some results of Langenbruch and Momrm [10] to the case of several variables. Crucial for this extension is the application of a slightly improved version of a result of Langenbruch [8] on the existence of a continuous linear right inverse for the \( \delta \)-operator on weighted \( L^p \) spaces.

1. **Preliminaries.** Throughout this paper, for all \( z, w \in C^N \) and \( r > 0 \), we will use the following abbreviations:

   \( w, z \) := \( \sum_{i=1}^{N} w_i z_i, \mid z \mid := (\sum z_i^2)^{1/2}, \)

   \( U(z, r) := \{ w \in C^N : \mid w - z \mid < r \}, \quad U_0 (r) := U_0 (0, r) = \{ w \in C^N : \mid w - 0 \mid < r \}, \quad B_0 (r) = B_0 (0, r) = \{ 0 \in C^N : \mid 0 \mid < r \}, \quad S : = \partial G(1), \quad \overline{R}_+ = [0, \infty]. \)

If \( A \subseteq C^N \), we write \( G(A) := \{ \tau \in H^1 \mid \tau \geq 0, \quad \forall \tau \in A \} \). By \( \text{int} G \) (resp. \( \overline{G} \)) we denote the interior (resp. closure) of a set \( G \).


**Notation.** For the sequel, we fix a compact convex set \( G \subseteq C^N \) which contains the origin in its relative interior. Let \( H \) be its support function, i.e.

\( H(z) = \sup_{u \in G} \Re \langle w, z \rangle \in [0, \infty], \quad z \in C^N. \)

If \( K \subseteq C^N \) is convex and compact, the support function of \( K \) will always be denoted by \( L \). Sometimes it is useful to consider \( \overline{H} \) (resp. \( \overline{L} \)) defined by \( \overline{H}(z) := H(z), \quad z \in C^N \) (resp. \( \overline{L}(z) := L(z) \)).
1.1. Function spaces. For each open set $D \subset \mathbb{C}^N$, we denote by $A(D)$ (resp. $A^{\infty}(D)$) the space of all (bounded) analytic functions on $D$. Let $K \subset \mathbb{C}^N$ be convex and compact. Let $G_{x}$, $x > 0$, be a family of bounded convex domains of $\mathbb{C}^N$ with $G = \bigcap_{x > 0} G_{x}$ and such that $G_x \subset G_y$ for all $0 < x < y$. We put $G + K := \{z + w \mid z \in G, w \in K\}$ and denote by $A(G + K)$ the space of all germs of analytic functions on $G + K$, i.e.,

$$A(G + K) := \bigcup_{x > 0} A^{\infty}(G_{x} + K),$$

endowed with the usual inductive limit topology defined by the norms

$$|f|_x := \sup_{z \in G_x + K} |f(z)|, \quad f \in A^{\infty}(G_{x} + K), \quad x > 0.$$ 

If $H_{x}$ is the support function of $G_{x}$, $x > 0$, we denote by $A^{0}_{H_{x} + L}$ the Fréchet space of all entire functions $f$ on $\mathbb{C}^N$ with

$$\|f\|_x := \sup_{z \in \mathbb{C}^N} |f(z)| \exp(-H_x(\bar{z}) - L(z)) < \infty \quad \text{for all} \ x > 0.$$ 

If $L = 0$ we write $A^{0}_{H} = A^{0}_{H_{x} + L}$. If $H = 0$ we write $A^{0} = A^{0}_{H_{x}}$.

1.2. Remark. If $\Omega \subset \mathbb{C}^N$ is a bounded convex domain with $0 \in \Omega$ and with support function $\omega$, let $A^{0}_{\omega}$ denote the Hilbert space of all entire functions on $\mathbb{C}^N$ with

$$\int_{\mathbb{C}^N} |f(z)|^2 \exp(-2\omega(z)) \, d\lambda(z) < \infty.$$ 

It may be well known and can be found in Taylor [25], Thm. 3, that the linear span of the exponentials $\{\exp(\cdot, \bar{w}) \mid w \in \Omega\}$ is dense in $A^{0}_{\omega}$. This shows that also for each nonpluriharmonic set $K \subset \Omega$, the linear span of $\{\exp(\cdot, \bar{w}) \mid w \in K\}$ is dense in $A^{0}_{\omega}$. Otherwise by the Hahn–Banach Theorem, there would be a functional $\nu \in A^{0}_{\omega} \setminus \{0\}$ such that the analytic function

$$\mathcal{D}(w) := \nu(\exp(\cdot, \bar{w})), \quad w \in \Omega,$$

is not identically zero but vanishes on $K$. This would prove that $K$ is pluripolar.

1.3. Convolution operators. Let $K \subset \mathbb{C}^N$ be convex and compact. We fix an analytic functional $\mu \in A(K)'$. Then

$$T_{\mu}(f)(z) := \mu(f(z + \cdot)), \quad z \in G, \quad f \in A(G + K),$$

defines a continuous linear operator $T_{\mu} : A(G + K) \to A(G)$. By the Laplace transform $\tilde{\mu}(z) := \mu(e^{z})$, $z \in \mathbb{C}^N$ (see Hörmander [8], Thm. 4.5.3), we may identify $A(K)'$ and $A^{0}_{K}$. If $K = \{0\}$, the series $P(z) := \tilde{\mu}(z) = \sum_{\alpha \in \mathbb{N}^N} a_{\alpha} z^{\alpha}$ converges in the topology of $A^{0}_{K} = A^{0}$ and so does $\mu = \sum_{\alpha \in \mathbb{N}^N} a_{\alpha} \delta_{0}(\alpha)$ in $A(K)'$, where $\delta_{0}$ is the functional of evaluation at 0. In this case

$$P(D)f := \sum_{\alpha \in \mathbb{N}^N} a_{\alpha} f^{(\alpha)} = T_{\mu}(f), \quad f \in A(G).$$

$P(D) : A(G) \to A(G)$ is called a differential operator. We will say that $T_{\mu} : A(G + K) \to A(G)$ admits a solution operator if there is a continuous linear map $R : A(G) \to A(G + K)$ with $T_{\mu} \circ R = \text{id}_{A(G)}$.

The following well known results about the Laplace transform and the duality theory of Fréchet spaces can be found in Hörmander [3], Thm. 4.5.3, and in Meise and Vogt [15], respectively.

1.4. Duality. Let $K \subset \mathbb{C}^N$ be convex and compact. The Laplace transform, given by $F(\nu)(z) := (\nu(e^{z}))$, defines by restriction a Fréchet space isomorphism $F : A(G + K) \to A^{0}_{H_{x} + L}$. Moreover, for all $0 < x < x_1$ there is $C > 0$ with

(i) $||F(\nu)||_{x_1} \leq ||\nu||_{x}$ for all $\nu \in A(G + K)'$ and

(ii) $||F^{-1}(f)||_{x_1} \leq C||f||_{x}$ for all $f \in A^{0}_{H_{x} + L}$,

where $|| \cdot ||_x$ denotes the dual norm of $|| \cdot ||_x$. Let $\mu \in A(K)' \setminus \{0\}$ be such that $T_{\mu} : A(G + K) \to A(G)$ is surjective. Identifying $A(G + K)'$ and $A^{0}_{H_{x} + L}$ with $A^{0}_{H_{x} + L}$ and $A^{0}_{H_{x}}$, respectively, the transposed map $T^{\ast}_{\mu} : A(G) \to A(G + K)'$ is the multiplication operator $M_{\tilde{\mu}} : A^{0}_{H_{x}} \to A^{0}_{H_{x} + L}$, $M_{\tilde{\mu}}(f) = \tilde{\mu} \cdot f$. By duality theory for Fréchet–Schwartz spaces, the following holds: $T^{\ast}_{\mu}$ is surjective if and only if $\tilde{\mu} : A^{0}_{H} \to A^{0}_{H_{x} + L}$ is a closed subspace of $A^{0}_{H_{x} + L}$ (the latter being true by hypothesis). $T_{\mu}$ has a solution operator on $A(G)$ if and only if the quotient map $\pi : A^{0}_{H_{x} + L} \to A^{0}_{H_{x} + L} : \tilde{\mu} \cdot f$ has a continuous linear right inverse.

For the following notion compare Ehrenpreis [1], and see Sigurdsson [24] for further references. In Proposition 1.6 we collect well known results on the surjectivity of convolution operators $T_{\mu} : A(G + K) \to A(G)$. These results have a long history. For this history, in particular concerning much older results in the case $N = 1$, we refer to the literature cited in the proof of Proposition 1.6.

1.5. Definition. Let $K \subset \mathbb{C}^N$ be convex and compact and let $\mu \in A(K)'$. If $A \subset S$ is closed, $\mu$ and $\tilde{\mu}$ will be called slowly decreasing (or of regular growth) on the cone $T(A)$ if the following holds: For each $\varepsilon > 0$ there is $R > 0$ such that for all $z \in T(A)$ with $|z| \geq R$ there is $w \in B(z, \varepsilon |z|)$ with $|\tilde{\mu}(w)| \geq \exp(L(|w|) - \varepsilon |w|)$. If $A = S$ we simply say that $\mu$ and $\tilde{\mu}$ are slowly decreasing.

1.6. Proposition. Let $K \subset \mathbb{C}^N$ be convex and compact and let $\mu \in A(K)'$. 

(a) If $\tilde{\mu}$ is slowly decreasing (i.e. on $C_N$), then $T_\mu : A(G + K) \to A(G)$ is surjective and $(\tilde{\mu} \cdot A(C_N)) \cap A_{H_{+L}}^0 = \tilde{\mu} \cdot A_H$. If $K = \{0\}$, then each $\mu \in A(K) \setminus \{0\}$ is slowly decreasing.

(b) Let $\Gamma_H$ denote the support of $(dH)^N$. If $T_\mu : A(G + K) \to A(G)$ is surjective, then $\tilde{\mu}$ is slowly decreasing on the cone $\Gamma_H$. If $\text{int } G \neq \emptyset$, then $T_\mu : A(G + K) \to A(G)$ is surjective if and only if $\tilde{\mu}$ is slowly decreasing on the cone $\Gamma_H$. In this case again, $(\tilde{\mu} \cdot A(C_N)) \cap A_{H_{+L}} = \tilde{\mu} \cdot A_H$.

(c) Let $N = 1$. If $T_\mu : A(G + K) \to A(G)$ is surjective, then $(\mu \cdot A(C)) \cap A_{H_{+L}} = \tilde{\mu} \cdot A_H$.

**Proof.** (a) The assertion for arbitrary $K \subset C_N$ compact and convex follows from Morozhakov [22] (see also [18]). The assertion for $K = \{0\}$ is true by Martinez [12], Thm. 7 and Lemma 15.

(b) This is essentially contained in Krivosheev [7]. An explicit reference is [18], Prop. 2.3 and Thm. 3.9.

(c) For each $z > 0$, we consider the inductive limit space $A_{H_{+L}}$ which consists of all entire functions $f$ on $C$ with

$$\sup_{x \in C} |f(z)| \exp(-H_x(z) - L(z)) \cdot |z|^n < \infty$$

for some $n \in N$.

By Krasicovich-Ternovski [6], Thm. 4.4, for each $x > 0$ the set $\tilde{\mu} \cdot C[z]$ is dense in $(\tilde{\mu} \cdot A(C)) \cap A_{H_{+L}}$. (From [6], Thm. 4.4(1), it follows that the rational functions which are constructed in [6], Thm. 4.4, are in fact polynomials in the present situation.) Since $C[z] \subset A_H^0$, also $\tilde{\mu} \cdot A_H$ is dense in $(\tilde{\mu} \cdot A(C)) \cap A_{H_{+L}}$ for all $x > 0$. Thus $\tilde{\mu} \cdot A_H$ is dense in $(\tilde{\mu} \cdot A(C)) \cap A_{H_{+L}} = (\tilde{\mu} \cdot A(C)) \cap \text{proj}_{z=0} A_{H_{+L}}$. Since $T_\mu$ is surjective, by duality theory, the subspace $\tilde{\mu} \cdot A_H$ is closed in $A_{H_{+L}}$. Thus the assertion follows.

**Notation.** We consider the Fréchet space

$$L_H^2 := \left\{ f \in L_{loc}^2(C_N) \mid \|f\|_z := \left( \int_{C_N} |f(z)|^2 \exp(-2H_x(z)) \, d\lambda(z) \right)^{1/2} < \infty \right\}$$

for each $z > 0$.

By $L^2_{H,(0,1)}$, we denote the corresponding Fréchet space of all $H$-closed $(0,1)$-forms with coefficients in $L_H^2$. If $\Omega \subset C_N$ is open, we consider the Fréchet space

$$W^2_{\Omega}(C_N, \Omega) := \{ f \in L_{loc}^2(C_N) \mid f \in L^2_H, \tilde{\nu}f \in L^2_{H,(0,1)} \text{ and } f|\Omega \in A(\Omega) \}$$

endowed with the norms $(\|f\|_z^2 \cdot \|f\|_{z}^2)^{1/2} = x > 0$. By the mean value property of analytic functions, we have $A_H^0 = W^2_{\Omega}(C_N, C_N)$. Finally, we define $W^2_H := W^2_{\Omega}(C_N, \emptyset)$.

1.7. **Lemma.** The continuous linear map

$$\tilde{\nu} : W^2_H \to L^2_{H,(0,1)}, \quad f \mapsto \tilde{\nu}f,$

is surjective with kernel $A_H^0$. Moreover, for every $g \in L^2_{H,(0,1)}$, $x > 0$, and $q > 1$ there is $f \in W^2_H$ with $\tilde{\nu}f = g$ and

$$\int_{C_N} |f|^q \exp(-2H_x - 2 \log(1 + |z|^2)) \, d\lambda \leq q \int_{C_N} |g|^q \exp(-2H_x) \, d\lambda.$$

**Proof.** As in Meise and Taylor [13], Prop. 2.1, we apply Hörmander [3], 4.4.2, together with the Mittag-Leffler Lemma (which can be applied in view of Remark 1.2). The quantitative remark holds by the proof of the Mittag-Leffler Lemma.

We will apply the following slight extension of a result of Langenbruch [8]. To avoid technical definitions, we state it only for the situation which will be considered in this paper.

1.8. **Proposition.** Let $\Omega \subset C_N$ be open and assume that for each $a \in C_N \setminus \Omega$ there is a plurisubharmonic function $u_a$ on $C_N$ with $u_a(a) \geq 0$ satisfying the following condition: For each $y > 0$ there are $x > 0$ and $C > 0$ with

$$u_a(z) \leq C + H_y(z) - H_x(a)$$

for all $z \in C_N$, $a \in C_N \setminus \Omega$.

Then there is a continuous linear projection $P : W^2_H(C_N, \Omega) \to A^0_H = W^2_H(C_N, C_N)$.

**Proof.** If for each $a \in C_N$ there is a plurisubharmonic function $u_a$ on $C_N$ with $u_a(a) \geq 0$ and such that for each $y > 0$ there are $x > 0$ and $C > 0$ with

$$u_a(z) \leq C + H_y(z) - H_x(a)$$

for all $z, a \in C_N$, then by Langenbruch [8], Thm. 1.3 and Remark 1.11 (applied with $r(z) := 1, z \in C_N$), there would be a continuous linear projection $P : W^2_H = W^2_H(C_N, \emptyset) \to A^0_H$. If we put formally $u_a := 0$ for all $a \in \Omega$, then the proof shows that our assertion is true. (Note that in the proof in [8], Thm. 1.3, the absence of the upper bounds for $u_a$ for $a \in \Omega$ does not affect the results of [8], Lemma 1.5, on the projections $\pi_k, \ k = N, \ldots, 1$, which are defined for all compactly supported $(0,k)$-forms $f$ with coefficients in $L^2_{loc}(C_N)$ such that $\tilde{\nu}f$ is a $(0,k+1)$-form with coefficients in $L^2_{loc}(C_N)$.) Since $\tilde{\nu}f|\Omega \equiv 0$ for all $f \in A(\Omega)$, a small straightforward modification of the proof in [8] gives the desired estimate for $P := \pi_0 : W^2_H(C_N, \Omega) \to A^0_H$,

$$\pi_0(f) := f - \sum_{m \in \mathbb{N}} r_{m0}(\pi_1(h_m f)),$$

where we use the notation of [8].
2. Solution operators. From Hörmander [2], Lemma 3.2, we recall the following:

**2.1. Lemma.** For \( \zeta \in \mathbb{C}^N \) and \( r > 0 \), let \( g, P \) be analytic in \( U(\zeta, 4r) \) such that \( g/P \) is also analytic in \( U(\zeta, 4r) \). Then

\[
|g(\zeta)/P(\zeta)| \leq \sup_{|\zeta - w| < 4r} |g(w)| \, \sup_{|\zeta - w| < r} |P(w)|/(\sup_{|\zeta - w| < r} |P(w)|)^2.
\]

**Notation.** If \( F \) is an entire function, we put \( V(F) := \{ z \in \mathbb{C}^N : F(z) = 0 \} \). Its tangent cone at infinity is defined by

\[
V_\infty(F) := \{ a + t a_j : t \geq 0, a = \lim_{j \to \infty} a_j/|a_j| \}
\]

for some sequence \( (a_j)_{j \in \mathbb{N}} \) in \( V(F) \) with \( \lim_{j \to \infty} |a_j| = \infty \).

We note that \( \text{dist}(a, V_\infty(F)) = o(|a|) \) as \( a \to \infty \) in \( V(F) \). This assertion is void (as is \( V_\infty(F) \)) if \( F \) is bounded.

**2.2. Lemma.** Let \( K \subset \mathbb{C}^N \) be convex and compact and let \( \mu \in \mathbb{A}(K) \) be slowly decreasing on \( V_\infty(K) \). Then there is a locally bounded function \( r : \mathbb{C}^N \to [1, \infty] \) with \( r(z) = o(|z|) \) as \( z \to \infty \). Suppose that for each \( \varepsilon > 0 \) there is \( R > 0 \) such that for all \( z \in \mathbb{C}^N \) with \( |z| \geq R \) we have \( U(z, 1+\varepsilon) \cap \Omega = \emptyset \) and \( \|\mu\| \geq 1/\varepsilon \).

Put \( r'(z) := \sup\{|z-w|+2r(w) : w \in \mathbb{C}^N, |z-w| \leq r(z)+r(w)\} \), \( z \in \mathbb{C}^N \). Then \( 2r \leq r' \) and \( r'(z) = o(|z|) \) as \( z \to \infty \). If \( U(z, 2r(z)) \cap U(w, r(w)) = \emptyset \), then also \( U(w, 2r(w)) \subset U(z, r'(z)) \).

**Proof.** Put \( A := S \cap V_\infty(K) \). By Definition 1.5, for each \( j \in \mathbb{N} \) there is \( R_j > 0 \) such that for each \( z \in V_\infty(K) \) with \( |z| \geq R_j \) there is \( w \in U(z, |z|/j) \) with \( \|\mu(w)\| \geq \exp(1/|z|) \). We may assume that \( R_j \geq j, R_{j+1} > R_j \) and that \( V(\mu) \cap U(R_j) \subset F(1/4+1/j) \) for all \( j \in \mathbb{N} \).

We put \( r(z) := |z|/j \) if \( R_j \leq |z| < R_{j+1} \), for some \( j \in \mathbb{N} \), and \( r(z) := 1 \) if \( |z| < R_j \). Direct computation shows that the functions \( r \) and \( r' \) have the desired properties.

**2.3. Auxiliary spaces.** Let \( K \subset \mathbb{C}^N \) be convex and compact and let \( \mu \in \mathbb{A}(K) \) be slowly decreasing on \( V_\infty(K) \). For each open set \( \Omega \subset \mathbb{C}^N \), let \( A^2(\Omega) \) be the Hilbert space of all square integrable functions in \( \mathbb{A}(\Omega) \). Let \( I(\Omega) \) be its closed subspace \( I(\Omega) = (\mu \cdot \mathbb{A}(\Omega)) \subset A^2(\Omega) \). We put \( E_\Omega := A^2(\Omega)/I(\Omega) \) and for \( x_\Omega \in E(\Omega) \),

\[
|\xi|_\Omega := \inf_{\xi \in x_\Omega} |\xi|_2 = \inf_{\xi \in x_\Omega} \left( \int_\Omega |\xi|^2 \, d\lambda \right)^{1/2}.
\]

We choose \( r' : [1, \infty] \to [1, \infty] \) according to 2.2, and set \( \vec{r} := 16r' \). For each \( z \in \mathbb{C}^N \), we write \( \Omega(z) := U(z, r(z)) \). We consider the Fréchet space

\[
A^0_{H+L}(\vec{r}) := \left\{ x = (f_\Omega(x) + I(\Omega(x)))_{x \in \mathbb{C}^N} \mid \|x\|_y < \infty \text{ for all } y > 0, \right. \]

\[
\left. f_\Omega(x) - f_\Omega(w) \in I(\Omega(z) \cap \Omega(w)) \text{ whenever } \Omega(z) \cap \Omega(w) \neq \emptyset \right\},
\]

where

\[
\|x\|_y := \sup_{x \in \mathbb{C}^N} |x_{\Omega(z)}| \exp(-H_y(z) - L(z)).
\]

We note that \( E_\Omega(z) = 0 \) if \( \Omega(z) \cap V(\vec{r}) = \emptyset \).

**2.4. Proposition.** Let \( K \subset \mathbb{C}^N \) be convex and compact and let \( \mu \in \mathbb{A}(K) \) be slowly decreasing on \( V_\infty(K) \) such that \( \mu \cdot A(K) \cap A^0_{H+L} = \mu \cdot A^0_{H} \).

Then the linear map

\[
\theta : A^0_{H+L}(\vec{r}) \to A^0_{H+L}(\vec{r}), \quad \theta(f + \mu \cdot A^0_{H}) := (f + \mu \cdot A^0_H)_{x \in \mathbb{C}^N},
\]

is an isomorphism of Fréchet spaces. To be more precise, for all \( 0 < y_2 < y_1 \), there is \( C > 0 \) with

\[
(a) \left\| g(f + \mu \cdot A^0_H) \right\|_{y_2} \leq C \left\| f + \mu \cdot A^0_H \right\|_{y_1} \text{ for all } f \in A^0_{H+L},
\]

\[
(b) \left\| \theta^{-1} (x) \right\|_{y_1} \leq C \|x\|_{y_2} \text{ for all } x \in A^0_{H+L}(\vec{r}).
\]

**Proof.** As in [20], we roughly follow the proof of Meise and Taylor [14], Thm. 12. By direct computation, we see that the map is well defined and continuous in such a way that (a) holds. We are going to prove that \( \theta \) is surjective and that \( \theta^{-1} \) is continuous and satisfies (b).

Let \( z = (x_{\Omega(z)})_{x \in \mathbb{C}^N} \in A^0_{H+L}(\vec{r}) \). We fix \( 0 < y_3 < y_2 < y_1 \). For each \( x \in \mathbb{C}^N \), let \( f_x \in A^2(\Omega(z)) \) be unique with \( f_x + I(\Omega(z)) = x_{\Omega(z)} \) and minimal norm, i.e.,

\[
\|f_x\|_2 = \inf_{f \in E_{\Omega(z)}} \|f\|_2 = |x_{\Omega(z)}|_{\Omega(z)}.
\]

Since \( x_{\Omega(z)} \in A^0_{H+L}(\vec{r}) \), for all \( y > 0 \) we obtain

\[
\|x\|_y \leq \|x\|_y \exp(H_y(z) + L(z)), \quad z \in \mathbb{C}^N.
\]

Since \( |f_x|_2 \) is subharmoinic, for all \( z \in \mathbb{C}^N \) we get

\[
(1) \quad |f_x|_2 \leq \left( \text{vol}_{2\pi} U(z, r(z)/2) \right)^{-1/2} |f_x|_2 \quad \text{if } z \in U(z, \vec{r}/2).
\]

By the definition of \( A^0_{H+L}(\vec{r}) \), for all \( z, w \in \mathbb{C}^N \) with \( \Omega(z) \cap \Omega(w) \neq \emptyset \), there is \( h_{z,w} \in A(\Omega(z) \cap \Omega(w)) \) with

\[
f_x - f_w = h_{z,w} \text{ on } \Omega(z) \cap \Omega(w).
\]

Now, for each \( z \in \mathbb{C}^N \), we put \( \Omega'(z) := U(z, r(z)/20) \subset U(z, \vec{r}/20) \). If \( \Omega'(z) \cap V(\vec{r}) \neq \emptyset \), we denote by \( f'_x \) the restriction of \( f_x \) to \( \Omega'(z) \). If \( \Omega'(z) \cap V(\vec{r}) = \emptyset \), then \( f'_x \) is a constant function.
Thus we can define \( u \in C_{0,1}^\infty(\mathbb{C}^N) \) by \( u|_{\Omega'(z)} := \overline{\partial} h_z \) for all \( z \in \mathbb{C}^N \). Since \((\Omega'(z))_{z \in \mathbb{C}^N}\) is locally finite in the sense described above, for all \( 0 < y' < y \) there is \( C_3 > 0 \) not depending on \( z \) with
\[
|u(\zeta)| \leq C_3 \|x\|_{\tilde{p}} \exp H_y(\zeta), \quad \zeta \in \mathbb{C}^N.
\]
These bounds imply \( L^2 \)-estimates, i.e. for all \( 0 < y' < y \) there are \( C_4 > 0 \) not depending on \( x \) with
\[
\int_{\mathbb{C}^N} |u(\zeta)|^2 \exp(-2(H_y(\zeta))) \, d\lambda(\zeta) \leq C_4 \|x\|_{\tilde{p}}^2.
\]
Since \( \overline{\partial} h_z|_{\Omega'(z)} = \overline{\partial} h_z = 0 \) for each \( z \in \mathbb{C}^N \), we get by Lemma 1.7 some \( g \in W^2_{\tilde{p}}(\text{even g } \in C^\infty(\mathbb{C}^N)) \) with \( \overline{\partial} g = u \). Moreover, we may assume that this \( g \) is chosen in such a way that
\[
\int_{\mathbb{C}^N} |g(\zeta)|^2 \exp(-2(H_{y_2}(\zeta)) + 2 \log(1 + |\zeta|^2)) \, d\lambda(\zeta) \leq 2 \int_{\mathbb{C}^N} |u(\zeta)|^2 \exp(-2(H_{y_2}(\zeta))) \, d\lambda(\zeta).
\]
Then \( a_{\alpha} := h_z - g|_{\Omega'(z)} \) is in \( A(\Omega'(z)) \) for each \( z \in \mathbb{C}^N \), and for all \( z, w \in \mathbb{C}^N \) we have
\[
a_{\alpha} - a_{\omega} = h_{z,w} \quad \text{on } \Omega'(z) \cap \Omega'(w)
\]
and thus
\[
f' - \mu a_{\alpha} = f' - \mu a_{\omega} \quad \text{on } \Omega'(z) \cap \Omega'(w).
\]
Hence there is a unique \( f \in A(\mathbb{C}^N) \) with \( f = f'_z - \mu a_{\alpha} \) on \( \Omega'(z) \) for all \( z \in \mathbb{C}^N \). Since \( f \) satisfies appropriate \( L^2 \)-estimates on \( \Omega'(z) \) and since \( |f|^2 \) is subharmonic on \( \Omega'(z) \), \( z \in \mathbb{C}^N \), the function \( f \) belongs to \( A_{0,1}^H \) and moreover there is \( C_5 > 0 \) not depending on \( z \) with
\[
|f(z)| \leq C_5 \|x\|_{y_2} \exp(H_{y_2}(z) \pm L(\xi)), \quad z \in \mathbb{C}^N.
\]
Thus for each \( z \in A_{\beta}^{H_{y_2} + L}(\mu) \) and all \( 0 < y_2 < y_1 \), we have constructed some \( f \in A_{\beta}^{H_{y_2} + L}(\mu) \) with \( \phi(f + \mu \cdot A_{0,1}^H) = x \) and
\[
\|f + \mu \cdot A_{0,1}^H\|_{y_1} \leq C_6 \|x\|_{y_2}.
\]
This proves the assertion.

2.5. Corollary. Let \( \mu \) be as in Proposition 2.4. If \( \overline{\partial} : L^2_{\tilde{p}} \rightarrow L^2_{\tilde{p}(0,1)} \) has a continuous linear right inverse or if for each \( a \in V_{\infty}(\mu) \) there is a plurisubharmonic function \( u_a \) on \( \mathbb{C}^N \) such that for each \( y > 0 \) there are \( x > 0 \) and \( C > 0 \) with
\[
u_a(x) \geq 0 \quad \text{and} \quad u_a(z) \leq C + H_y(x) - H_x(\overline{\partial}), \quad \text{for all } z \in \mathbb{C}^N, \ a \in V_{\infty}(\mu),
\]
then there is a continuous linear right inverse for the quotient map \( A_{H+L}^0 - A_{H+L}^0 / (\mathcal{G} - A_{H+L}^0) \).

**Proof.** We first note that in the hypothesis we may replace \( V_{\infty}(\mathcal{B}) \) by \( V(\mathcal{B}) \). If \( V(\mathcal{B}) \) is bounded, we may choose \( u_0 := 0 \) for all \( a \in V(\mathcal{B}) \). Otherwise for each \( a \in V(\mathcal{B}) \) we choose some \( a' \in V_{\infty}(\mathcal{B}) \) with \( |a - a'| = \text{dist}(a, V_{\infty}(\mathcal{B})) \) and define \( u_a(z) := u_{a'}(z - a + a') \) for \( z \in \mathbb{C}^N \).

To prove the existence of a right inverse for \( A_{H+L}^0 \rightarrow A_{H+L}^0 / (\mathcal{G} - A_{H+L}^0) \), let \( f \in I(\mathcal{G}(z) - I(\mathcal{G}(z))) \) be any continuous linear function on \( \mathbb{C}^N \). For each \( a \in V(\mathcal{B}) \), write \( f_a(z) := f_a(z - a + a') \) for \( z \in \mathbb{C}^N \) and define \( u_a(z) := u_{a'}(z - a + a') \) for \( z \in \mathbb{C}^N \).

The first choice. For \( z \in \mathbb{C}^N \), let \( P_a \) be the orthogonal projection of \( A^0(\mathcal{G}(z)) \) onto \( I(\mathcal{G}(z)) \) for \( P_a(x) := P_a(x_{\mathcal{G}(z)}) \). Then \( R_a = (f_a - I(\mathcal{G}(z))) \) is a continuous linear function on \( \mathbb{C}^N \) for each \( a \in V(\mathcal{B}) \).

The second choice. For each \( a \in V(\mathcal{B}) \), define \( u_a(z) := u_{a'}(z - a + a') \) for \( z \in \mathbb{C}^N \). Then \( u_a(z) \) is a continuous linear function on \( \mathbb{C}^N \) for each \( a \in V(\mathcal{B}) \).

**Proof of Theorem 1.8.** We first consider the case where \( G = \emptyset \). Let \( G := \{ 1 + \mathbb{Z} \} \) be any dense sequence in \( S' = S \) and choose a hyperplane \( H \subset \mathbb{C}^N \) such that \( H \cap \mathbb{Z} = \emptyset \). Then \( H \cap (1 + \mathbb{Z}) = \emptyset \). Let \( \lambda \in \mathbb{C}^N \) and \( \lambda \not\in H \) be such that \( \lambda \notin H \). Then \( H \cap \mathbb{Z} = \emptyset \) for all \( \lambda \not\in H \).

We choose a \( \mathcal{C} \)-linear functional \( l_j : \mathbb{C}^N \rightarrow \mathbb{C}^N \) with \( k(l_j) = k_j \). Then for each \( j \in \mathbb{N} \), we choose a sequence \( \{ \lambda_{m_j} \}_m \in \mathbb{N} \) of positive numbers with \( \lambda_{m_j} > 2 \lambda_{m_j} \). Then define \( P_j(z) := \prod_{m=1}^{\infty} (1 - l_j(z)/l_j(\lambda_{m_j}a_j)) \), \( z \in \mathbb{C}^N \), \( j \in \mathbb{N} \), and

\[
P(z) := \prod_{j=1}^{\infty} P_j(z), \quad z \in \mathbb{C}^N,
\]
define elements of \( A^0(\mathcal{G}(z)) \) for each \( z \in \mathbb{C}^N \).
modulo $I(\Omega(z))$ otherwise, $z \in \mathbb{C}^N$. Then $u_{m,j} := \log |R_j(f_{m,j})|$ is plurisubharmonic on $\mathbb{C}^N$ with $u_{m,j} = 0$ on $\lambda_{m,j} \omega_j + L_j$. Let

$$\sigma_Q(y) := \sup \{ x > 0 \mid \sup_{|f| \leq 1} |Q(f)|_y < \infty \}, \quad y > 0,$$

be the characteristic of continuity of $Q$. By Proposition 2.4, for all $y > 0$ and all $0 < x < y' < \sigma_Q(y)$, there are $C, C' > 0$ such that for all $m \in \mathbb{N}$,

$$\sup_{z \in \mathbb{C}^N} \{ u_{m,j}(z) - H_j(z) \} = \log \|R_j(f_{m,j})\|_y \leq C' + \log \|f_{m,j}\|_x \leq C + \sup_{z \in \lambda_{m,j} \omega_j + L_j} (-H_j(z)) = C - \lambda_{m,j} H_\omega(\omega_j).$$

We substitute $z = \lambda_{m,j} w$. For the upper semicontinuous regularization $u_j$ of $\limsup_{m \to \infty} \lambda_{m,j}^{-1} u_{m,j}(\lambda_{m,j} \cdot)$ we get $0 \leq u_j(a_j)$ and

$$u_j(w) \leq H_\omega(\omega) - H_\omega(\omega_j), \quad w \in \mathbb{C}^N.$$ 

We fix $a \in S' = S$. We choose a sequence $(a_{j_k})_{k \in \mathbb{N}}$ converging to $a$ and denote by $u_a$ the upper semicontinuous regularization of $\limsup_{t \to 0} u_{j_k}$. Then by the Hartogs Lemma, we have

$$0 \leq u_a(\omega) \quad \text{and} \quad u_a(w) \leq H_\omega(\omega) - H_\omega(\omega_j), \quad w \in \mathbb{C}^N.$$ 

Finally, for each $a \in \Gamma(S') \setminus \{0\}$ we put

$$u_a(w) := \frac{a_{j_k}}{u_{j_k}(\omega | w | a)}, \quad w \in \mathbb{C}^N,$$

and the proof is finished in the case where int $G \neq \emptyset$.

In the other case, if $a \in S'$ we distinguish the cases $H(\omega) = 0$ and $H(\omega) > 0$. We will prove a little bit more than necessary.

Put $S'_1 := \{ b \in S' \mid H(\overline{b}) = H(-\overline{b}) = H(\overline{b}) = 0 \}$ and $S'_2 := G + U(x)$. Then for each $x > 0$ and each $a \in S'_1$, we have $H_\omega(z) = x |z|$ for all $z$ in a neighborhood of $a$. Now the previous proof produces plurisubharmonic functions $(u_a)_{a \in \Gamma(S')}^\prime$ with the desired properties.

Let $\epsilon > 0$. Put $S'_2 := \{ b \in S' \mid H(\overline{b}) = H(-\overline{b}) = 0 \}$ and let $G \omega$ be the interior of the convex hull of $(1 + x)G \cup U(x)$. Since $H_\omega(z) = \max \{(1 + x)H(z), |z| \}$, $z \in \mathbb{C}^N$, for sufficiently small $0 < x \leq x_0$ we have $H_\omega(z) = (1 + x_0)H(\omega)$ for all $z$ in some neighborhood of $S'_1$. Now after small changes, the previous proof produces plurisubharmonic functions $(u_a)_{a \in \Gamma(S')}^\prime$ with the desired properties.

From [21] we recall the following notation:

**NOTATION.** We define

$$u_H(z) := \sup_{u \leq H} u(z), \quad z \in \mathbb{C}^N,$$

where the supremum is taken over all plurisubharmonic functions $u$ with $u \leq H$ such that $u \leq \log |z| + O(1)$ as $z \to 0$. By [21], this function is plurisubharmonic, does not exceed $H$, and satisfies $u_H \leq \log |z| + O(1)$ as $z \to 0$ (here we allow a plurisubharmonic function to be $\equiv -\infty$). By [21], there is a unique upper semicontinuous function $C^*_H : S \to [0, \infty]$ such that

$$P_H := \{ z \in \mathbb{C}^N \mid u_H(z) = H(z) \} = \{ \lambda \alpha \mid \alpha \in S, C^*_H(\alpha) \leq \lambda < \infty \}.$$

**2.7. Remark.** If $G$ is not pluripolar, i.e. if the $\mathbb{C}$-linear span of $G$ equals $\mathbb{C}^N$, by [21] we have

$$H_H(z) = \lim_{\delta \to 0} \frac{1}{H(\delta)} \delta H_H(z/\delta) = \sup_{\delta > 0} \frac{1}{\delta} \delta u_H(z/\delta), \quad z \in \mathbb{C}^N \setminus \{0\}.$$ 

The limit is uniform on closed subsets of $\mathbb{C}^N \setminus \{0\}$.

**2.8. Lemma.** For $N = 1$ let $\Gamma_H \subset \mathbb{C}$ be the support of $\Delta H$, i.e. $H$ is harmonic precisely on $\mathbb{C} \setminus \Gamma_H$. If $u \leq H$ is subharmonic on $\mathbb{C}$ and $u(0) < 0$, then $\{ z \in \mathbb{C} \mid u(z) = H(z) \} \subset \Gamma_H$.

**Proof.** Assume that there is $z \in \mathbb{C} \setminus \Gamma_H$ with $u(z) = H(z)$. Then $u - H$ is a nonpositive subharmonic function on $\mathbb{C} \setminus \Gamma_H$ which vanishes at $z$. Hence it vanishes on the component of $\mathbb{C} \setminus \Gamma_H$ which contains $z$. This contradicts $u(0) < 0 = H(0)$ since $u$ is upper semicontinuous.

**2.9. Lemma.** Assume that $G$ is not pluripolar. Let $A \subset S$ be closed. The following are equivalent:

(i) There is $\delta > 0$ with $C_H(a) \geq \delta$ for all $a \in A$.

(ii) For each (some) $y > 0$ there is $\varepsilon > 0$ such that $u_{y,\varepsilon} = H \circ a_\varepsilon$ on $A$.

(iii) There is a plurisubharmonic function on $\mathbb{C}^N$ with $u \leq H$.

(iv) For each $a \in \Gamma(A)$ there is a plurisubharmonic function $u_a$ on $\mathbb{C}^N$ such that for each $y > 0$ there are $x > 0$ and $C > 0$ such that for all $z \in \mathbb{C}^N$ and $a \in \Gamma(A)$,

$$u_a(\omega) \geq 0 \quad \text{and} \quad u_a(z) \leq C + H_H(z) - H_\omega(\omega).$$

If $G$ is pluripolar, we have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

**Proof.** (i) $\Rightarrow$ (ii). Let $y > 0$. By the hypothesis, there is $\delta > 0$ such that $u_H(a/\delta) = H(a/\delta)$ for all $a \in A$. Define $u := \delta u_H(\delta)$. Then $u$ is plurisubharmonic, $u \leq H$ and, since $u$ is upper semicontinuous and $u(0) = -\infty < 0$, we can choose $\varepsilon > 0$ so small that $u \leq H - \varepsilon$. Thus $u_{y,\varepsilon} \geq u = H$.

(ii) $\Rightarrow$ (i). Assume that (ii) holds for some $y > 0$ and $\varepsilon > 0$. Since $u_{y,\varepsilon} \leq H_\omega - \varepsilon$, we may choose $0 < r < 1$ with $u_{y,\varepsilon}(x) \leq H(z) - \varepsilon/2$ for all $|z| = r$. According to Remark 2.7, we may choose $\delta > 0$ so small that $\delta u_H(\delta) \geq H(z) - \varepsilon/2$ for all $|z| = r$. We define $u := \delta u_H(\delta)$ on $B(r)$ and $u :=$
max \{\delta v_H(z/|z|, u_{y,\varepsilon})\} elsewhere. Then \(u\) is plurisubharmonic on \(\mathbb{C}^N\) with \(u \leq H\) and \(u \leq \delta \log |z| - O(1)\) as \(z \to 0\). Thus \(v_H \geq u(\cdot - \delta)/\delta\). Since \(u(z) \geq u_{y,\varepsilon}(z)\) if \(|z| = 1\), we obtain \(H\left(\alpha'\right) \geq v_H\left(\alpha'\right) \geq u_{y,\varepsilon}(\alpha')/\delta = H(\alpha')\) if \(\alpha' \in A\). Then \(\delta v_H(a/\delta) = H(a)\) for all \(a \in A\). Hence \(C_H(a) \geq \delta\) for all \(a \in A\).

(ii) \Rightarrow (iii). Put \(u := u_{y,\varepsilon}\) for some \(y > 0\) and \(\varepsilon > 0\) chosen according to (ii).

(iii) \Rightarrow (iv). For each \(a \in \Gamma(A) \setminus \{0\}\) we put

\[ u_a := u(|a|/|a|) - H(a). \]

Then \(u_a\) is plurisubharmonic and \(u_a(a) = u(a/|a|)|a| - H(a) = 0\). Let \(y > 0\) be arbitrary. Since \(u \leq H\) and \(u(0) < 0\), there is \(\varepsilon > 0\) with \(u \leq H_y - \varepsilon\). We choose \(z > 0\) with \(H_a(z) \leq H(z) + \varepsilon|z|\) for all \(z \in \mathbb{C}^N\). Then \(u_a(z) \leq H_a(z) - \varepsilon|a| - H(a) \leq H_a(z) - H_a(a)\) for all \(z \in \mathbb{C}^N, a \in \Gamma(A) \setminus \{0\}\).

(iv) \Rightarrow (v). Replacing \(u_a\) by the upper semicontinuous regularization of \(\sup_{\lambda \to \infty} \lambda^{-1} u_{\lambda a}(\lambda)\), we may assume \(C = 0\) in (iv). Hence for each \(y > 0\) there are \(z > 0\) and \(\varepsilon > 0\) with

\[ u_a + H(a) \leq H_y - (H_a(a) - H(a)) \leq H_y - \varepsilon|a| \leq H_y, \quad a \in \Gamma(A). \]

Thus for each \(a \in A\) we have \(u_a + H(a) \leq H\), \(u_a(a) + H(a) = H(a)\), and for each \(y > 0\) there is \(\varepsilon > 0\) with \(u_a + H(a) \leq H_y - \varepsilon\). This gives \(u_{y,\varepsilon} = H\) on \(A\).

2.10. Theorem. For each convex compact set \(G \subset \mathbb{C}^N\) containing the origin in its relative interior, the following are equivalent:

(i) Each differential operator \(\mathcal{P}(D) : A(G) \to A(G), P \in A^0 \setminus \{0\}\), admits a solution operator.

(ii) There is \(\delta > 0\) with \(C_H \geq \delta\) on \(S\), i.e. \(u_H \geq H\) outside a compact neighborhood of the origin (i.e. in a "neighborhood of infinity")

(iii) There is a plurisubharmonic function \(u\) on \(\mathbb{C}^N\) with \(u \leq H\), \(u(0) < 0\) and with \(u = H\) in a neighborhood of infinity.

(iv) There is a family \(\{u_a\}_{a \in \mathbb{C}^N}\) of plurisubharmonic functions on \(\mathbb{C}^N\) such that the following holds: For each \(y > 0\) there is \(\varepsilon > 0\) such that for all \(z, a \in \mathbb{C}^N\),

\[ 0 \leq u_y(a) \quad \text{and} \quad u_y(z) \leq H_y(\varepsilon) - H_y(\varepsilon). \]

(v) The continuous operator \(\bar{\delta} : W_H \to L^2_{H(0,1)}\) has a continuous linear right inverse (see Lemma 1.7).

Each of these equivalent conditions implies that the interior of \(G\) is nonvoid.

Proof. First we prove that (i) and (iv) each imply that \(\text{int} G\) is nonvoid. Assume that (i) holds and that \(\text{int} G = \emptyset\). Then there is \(b \in S\) with \(H(b) = H(-b) = 0\). We put \(a := \bar{b}\). By Lemma 2.8, there is a plurisubharmonic function \(u\) on \(\mathbb{C}^N\) with \(u(a) \geq 0\) and such that for all \(y > 0\) there is \(z > 0\) with

\[ u(z) \leq H_y(z) - H_y(\varepsilon) \quad \text{for all} \quad z \in \mathbb{C}^N. \]

The function \(\bar{\delta} : \mathbb{C} \to \mathbb{R}, \zeta \mapsto H(\zeta)\), is the support function of a compact convex subset of \(\mathbb{C}\). Since \(\bar{\delta}(1) = H(1) = 0\), this set is contained in \(t\mathbb{R}\). Hence \(\bar{\delta}\) is harmonic in the upper (and lower) halfplane. For \(\bar{\delta} : \zeta \mapsto u(\zeta(\bar{b}))\), we deduce from (3) that \(\bar{\delta} \leq \bar{H} = \bar{H}(i)\). By Lemma 2.8, this is a contradiction to \(\bar{\delta}(0) - \bar{H}(0) < 0\).

If (iv) holds, by the reasoning of Lemma 2.9 (iv) \Rightarrow (ii), we may assume that \(C = 0\). Hence as above we get a contradiction if we assume that \(\text{int} G = \emptyset\).

(i) \Rightarrow (iv). Since \(\text{int} G \neq \emptyset\), (iv) follows from Lemma 2.6.

(iv) \Rightarrow (v). By Langenbruch [8], Thm. 1.3 and Rem. 1.11, there is a continuous linear projection \(P : W_H \to A^0_H\). Hence by Lemma 1.7, a continuous linear right inverse \(R : L^2_{H(0,1)} \to W_H\) for \(\bar{\delta} : W_H \to L^2_{H(0,1)}\) is given by

\[ R(g) := f - P(f) \quad \text{whenever} \quad f \in W_H \text{ with } \bar{\delta} f = g. \]

(v) \Rightarrow (i): Corollary 2.5.

(i) \Rightarrow (ii). Since the interior of \(G\) is nonvoid, this holds by Lemma 2.9.

In the case of one complex variable we get a complete result for a given single convolution operator:

2.11. Theorem. For \(N = 1\), let \(G, K \subset C\) be compact and convex. Let \(G\) contain the origin in its relative interior. If \(\mu \in \mathcal{A}(K)^\prime\) defines a surjective convolution operator \(T_\mu : A(G + K) \to A(G)\), then the following are equivalent (see Proposition 1.5):

(i) \(T_\mu : A(G + K) \to A(G)\) admits a solution operator.

(ii) There is \(\delta > 0\) with \(C_H \geq \delta\) for all \(a \in A := S \cap V_{\infty}(\mu)\).

Proof. (i) \Rightarrow (ii). Following an idea of Korobčnik and Melikov [5], we make a reduction to the case of a differential operator (see also [17], Lemma 8). We choose a canonical product \(P \in A^0 \setminus \{0\}\) with \(V_{\infty}(P) = V_{\infty}(\mu)\) and such that \(g := \bar{\mu}/P\) is an entire function. \(g\) has the same indicator as \(\bar{\mu}\) (see Levin [11], III, Thm. 5). By the hypothesis and by Duality 1.4, the multiplication operator \(M_\mu : A^0_H \to A^0_{H + \mu}\) has a continuous linear right inverse \(L\). Hence the operator \(LM_\mu : A^0_H \to A^0_{H + \mu}\) is a continuous linear left inverse for \(M_\mu : A^0_H \to A^0_H\). As in the proof of Lemma 2.6 (with \(S' := S \cap V_{\infty}(P)\)), we obtain subharmonic functions \(u_a\) on \(\mathbb{C}, a \in V_{\infty}(\mu) \subset V_{\infty}(\bar{\mu})\), such that for each \(y > 0\) there is \(x > 0\) with

\[ u_a(a) \geq 0 \quad \text{and} \quad u_a(x) \leq H_y(\varepsilon) - H_x(\varepsilon) \quad \text{for all} \quad z \in \mathbb{C}, \quad a \in V_{\infty}(\bar{\mu}). \]
By Lemma 2.9, applied with $A := \{ \bar{a} \mid a \in S' \}$, we get (ii) in the case where $G$ is not polar. If $G$ is polar, i.e. if $G = \{ 0 \}$, then $H \equiv 0$ is harmonic. The reasoning at the beginning of the proof of Theorem 2.10 shows that the assumption $V_\infty (\bar{\mu}) \neq 0$ leads to a contradiction. Thus $V_\infty (\bar{\mu}) = 0$ and (ii) holds trivially.

(ii) $\Rightarrow$ (i). We first consider the case $G = \{ 0 \}$. In this case $\nu_H \equiv -\infty$ and thus $C_G \equiv 0$. Hence (ii) implies that $V_\infty (\bar{\mu}) = 0$, i.e. $V(\bar{\mu})$ consists of at most finitely many points. By Hadamard's factorization theorem, there are $w \in C$ and a nonzero polynomial $P$ with $\bar{\mu}(z) = P(z)e^{zw}$, $z \in C$. Since $T_w : A(K) \to A(\{ 0 \})$, $T_w(f) = P(D)f'(-w)$, is surjective, we obtain $K = \{ w \}$. This shows that $\bar{\mu}$ is slowly decreasing. Thus (i) holds by Corollary 2.5 (see Proposition 1.5(a)).

Now let $G$ be nonpolar. By Proposition 1.5(b), $\bar{\mu}$ is slowly decreasing on the support $\Gamma_H$ of $\Delta H$. By Lemma 2.8, we have $\Gamma(\bar{\mu}) \subseteq \Gamma_H$. By the hypothesis, $V_\infty (\bar{\mu}) \subseteq \Gamma(\bar{\mu})$. Hence $\bar{\mu}$ is slowly decreasing on $V_\infty (\bar{\mu})$. By Proposition 1.5(c), also $\bar{\mu}(A(\bar{\mu})) \cap A_H \nabla \bar{\mu} = \bar{\mu} \cdot A_H$.

Furthermore, by the hypothesis and by Lemma 2.9, there are subharmonic functions $u_A$ on $C$, $a \in V_\infty (\bar{\mu})$, such that for each $y > 0$ there is $z > 0$ with

$$u_A(a) \geq 0 \text{ and } u_A(z) \leq H_y(z) - H_y(a) \quad \text{for all } z \in C, \ a \in V_\infty (\bar{\mu}).$$

Thus all the hypotheses of Corollary 2.5 are satisfied. Hence $T_w$ admits a solution operator.

2.12. COROLLARY. For $N = 1$ let $K \subseteq C$ be convex and compact and $\mu \in A(K)'$.

(a) If $G = \{ 0 \}$, the only convolution operators $T_w : A(K) \to A(\{ 0 \})$, which admit a solution operator are those for which $K = \{ w \}$ and $\bar{\mu}(z) = P(z)e^{zw}$, $z \in C$, for some $w \in C$ and some nonzero polynomial $P$. (The result for $K = \{ 0 \}$ is essentially contained in Meise and Taylor [13].)

(b) Let $G = [a, b]$ be a nontrivial compact interval, and suppose the convolution operator $T_{\mu} : A(G + K) \to A(G)$ is surjective. Then $T_{\mu} : A(G + K) \to A(G)$ admits a solution operator if and only if $V_\infty (\bar{\mu}) \subseteq \mathbb{R}_+ x$. (The result for $K = \{ 0 \}$ is also contained in Langenbruch [9].)

(c) If $G$ is a compact convex polygon, let $A \subseteq S$ be the finite set of out normals to the faces of $G$. If the convolution operator $T_{\mu} : A(G + K) \to A(G)$ is surjective, then it admits a solution operator if and only if $V_\infty (\bar{\mu}) \subseteq \bigcup_{a \in A} \mathbb{R}_+ x$.

Proof. (a) follows from 2.11. (c) implies (b).

(c) By Theorem 2.11, we only have to prove $\Gamma(\bar{\mu}) = \bigcup_{a \in A} \mathbb{R}_+ x$. By Lemma 2.8, the inclusion $\mathbb{C}^N$ holds. The other inclusion holds by Lemma 2.1.
proved by Krivosheev [7] (see also [18], Thm. 3.0) that $T_\nu : A(G + K) - A(G)$ is surjective if and only if $T_\nu : A(\text{int } G + K) \rightarrow A(\text{int } G)$ is surjective.

It is an obvious question whether there is a solution operator $A(G) - A(G + K)$ if and only if there is a solution operator $A(\text{int } G) \rightarrow A(\text{int } G + K).$ We do not know the answer in general. If $N = 1$, in many "concrete situations the answer is yes, because well known theorems of function theor give at the same time the same answer for both cases, i.e., for $G$ and $\text{int } G.$ In particular, all examples which have been given in [16] for domains $\text{int } G$ are in an obvious way also examples for compact sets $G.$

References