Remark. Studies analogous to the one developed here can be made by replacing the function $(1+x^2)^k$ in the definition of the space $\mathcal{H}_{\mu,k}$, $k \in \mathbb{Z}$, $k < 0$, by other functions. For example, if we put the function $e^{-kx}$ instead of $(1+x^2)^k$ our procedure permits defining the Hankel convolution in the spaces of E. L. Koh and A. H. Zemanian [KZ].

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# On rank one elements


by

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**Abstract.** Without the "scarcity lemma", two kinds of "rank one elements" are identified in semisimple Banach algebras.


Suppose $A$ is a complex Banach algebra, with identity 1 (usually not zero), and invertible group $A^{-1}$: then the *radical* of $A$ can be defined ([5], Theorem 7.2.3) as the set

(0.1) $$\mathrm{Rad}(A) = \{a \in A : 1 + Aa \subseteq A^{-1}\}.$$

It is familiar that this is a closed two-sided ideal; also,

(0.2) $$1 + Aa \subseteq A^{-1} \Rightarrow 1 + A^{-1}a \subseteq A^{-1} \Rightarrow A^{-1} + a \subseteq A^{-1} \Rightarrow 1 + (A^{-1} + A^{-1})a \subseteq A^{-1};$$

since of course $A^{-1} + A^{-1} = A$ this gives an alternative description of $\mathrm{Rad}(A)$, and also provides an elementary instance of the "scarcity lemma" ([1], Theorem 7.1.7). We recall the *spectrum* and the *non-zero spectrum*,

(0.3) $$\sigma_A(a) = \sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin A^{-1}\} \quad \text{and} \quad \sigma'(a) = \sigma(a) \setminus \{0\};$$

thus

(0.4) $$a \in \mathrm{Rad}(A) \Leftrightarrow \sigma'(xa) = \emptyset \quad \text{for every } x \in A,$$

or equivalently, for every $x \in A^{-1}$. We call the algebra $A$ *semisimple* iff $\mathrm{Rad}(A) = \{0\}$, or equivalently, if

(0.5) $$\#\sigma'(xa) = 0 \text{ for every } x \in A \Rightarrow a = 0,$$

and *semiprime* iff

(0.6) $$aAa = \{0\} \Rightarrow a = 0;$$

since the left hand side of (0.6) implies that $a \in \mathrm{Rad}(A)$ it is clear that a semisimple algebra is always semiprime. We observe

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**1.** Lemma. *If $A$ is semisimple and if*

$$a \in A^{-1} \Rightarrow \#\sigma'(a) = 1, \tag{1.1}$$

*then $A = \mathbb{C}1$ is one-dimensional.*

Proof. Begin by noticing that (1.1) can be rewritten as

$$a \in A \Rightarrow \#\sigma(a) = 1, \tag{1.2}$$

for (1.1) is the same as (1.2) with $A$ replaced by $A^{-1}$; but now if $a \in A$ and $|\lambda| > \|a\|$ then the spectrum of $a - \lambda \in A^{-1}$ is a singleton. Now if $a \in A$ has spectrum $\{\lambda\}$ and $x \in A^{-1}$ is arbitrary then $x(a-\lambda)$ is not invertible and hence has spectrum $\{0\}$. But this says $a - \lambda \in \mathrm{Rad}(A) = \{0\}$. ∎

By the same argument the Hirschfeld/Johnson criterion [6] for $A$ to be finite-dimensional can be rewritten as

$$a \in A^{-1} \Rightarrow \#\sigma(a) < \infty. \tag{1.3}$$

We are ready for two definitions of "rank one element":

**2.** Definition. We call the element $a \in A$ *spatially of rank one* iff

$$aAa \subseteq \mathbb{C}a, \tag{2.1}$$

and *spectrally of rank one* iff

$$x \in A \Rightarrow \#\sigma'(xa) \le 1. \tag{2.2}$$

It is clear that both kinds of "rank one" elements form two-sided ideals of the multiplicative semigroup $A$. Evidently, if $a \in A$ is spatially of rank one, there is ([10], Definition 2.2) a linear functional $\tau_a$ on $A$, uniquely determined if $a \ne 0$, for which $axa = \tau_a(x)a$; by the closed graph theorem $\tau_a$ is also continuous. When $a = 0$ we shall take $\tau_a = 0$. By an abuse of language we include 0 among the rank one elements. We recall ([3], Proposition 30.6) that under condition (2.1), $a \in A$ generates a *minimal left ideal* of semiprime $A$:

$$\{0\} \ne AJ \subseteq J \subseteq Aa \Rightarrow Aa \subseteq J. \tag{2.3}$$

Indeed, if $0 \ne ba \in J$ then by (0.6) there exists $c \in A$ with $bacba \ne 0$, and by (2.1), $0 \ne \lambda \in \mathbb{C}$ with $\lambda a = acba \in J$. It is clear that

$$a \text{ spatially of rank one} \Rightarrow a \text{ spectrally of rank one}; \tag{2.4}$$

the converse fails ([9], p. 214) for $0 \ne a \in \mathrm{Rad}(A)$ when $A$ is semiprime and not semisimple. What we want to demonstrate here is that the converse of (2.4) holds when $A$ is semisimple; this is easy to see when $A = BL(X, X)$, for if there are $x_1$ and $x_2$ in $X$ for which $(Tx_1, Tx_2)$ is linearly independent then there are linear functionals $g_j \in X^\dagger$ for which $g_i(Tx_j) = \delta_{ij}$; but now if $S = \lambda_1 g_1 \odot x_1 + \lambda_2 g_2 \odot x_2$ then $\lambda_1$ and $\lambda_2$ are eigenvalues of $ST$. We claim that the converse of (2.4) also holds when $a = a^2$ is idempotent:



**3.** Lemma. *If $A$ is semisimple and $p = p^2 \in A$ is spectrally of rank one then it is spatially of rank one.*

Proof. Note first that if $B = pAp + \mathbb{C}(1-p) \subseteq A$ and $b \in pAp$ then

$$\sigma'_{pAp}(b) = \sigma'_B(b) \quad \text{and} \quad \partial\sigma_B(b) \subseteq \sigma_A(b) \subseteq \sigma_B(b); \tag{3.1}$$

thus $p$ also satisfies condition (2.2) in the algebra $pAp$. Also, $pAp$ is semisimple if $A$ is; indeed,

$$\mathrm{Rad}(pAp) \subseteq \mathrm{Rad}(A), \tag{3.2}$$

since if $px'p$ is inverse to $p - pxpap$ in $pAp$ then $px'p + (1-p)xpapx'p + (1-p)$ is inverse to $1 - xpap$ in $A$ (think of triangular $2\times 2$ matrices). Thus Lemma 1 applies. ∎

The converse to (2.3) holds ([3], Proposition 30.6): if $J \subseteq A$ is a minimal left ideal then $J = Ap$ with $p = p^2$ of rank 1. Indeed, since $J^2 \ne \{0\}$ there is $a \in J$ with $Ja \ne \{0\}$ and hence $Ja = J$, and thus $p \in J$ with $pa = a$. This says $1 - p \in a_{-1}(0) = \{x \in A : xa = 0\}$; now

$$p - p^2 \in J \cap a_{-1}(0) \subseteq J \not\subseteq a_{-1}(0)$$

says $p - p^2 = 0$, and then $\{0\} \ne Ap \subseteq J$ says $Ap = J$. Finally for the rank one condition every non-zero element of $pAp$ has a left inverse: if $0 \ne pbp$ then $\{0\} \ne Apbp \subseteq J$ giving $Apbp = J$ and hence $c \in A$ for which $cpbp = p$ and hence $pcp \cdot pbp = p$.

Lemma 3 gives the general result:

**4.** Theorem. *If $A$ is semisimple and $a \in A$ then*

$$a \text{ spectrally of rank one} \Rightarrow a \text{ spatially of rank one}. \tag{4.1}$$

Proof. Suppose $a \in A$ is spectrally of rank one; then 0 is certainly not an accumulation point of the spectrum of $a$, and hence ([5], Theorem 7.5.3) we have a "support projection" $p \in A$ for which

$$p = p^2 = ca = ac \quad \text{with} \quad \sigma(a(1-p)) \subseteq \{0\} \tag{4.2}$$

($1 - p$ is the spectral projection induced by $a$ at $0 \in \mathbb{C}$). We claim

$$pAp = \mathbb{C}p \quad \text{and} \quad a = ap; \tag{4.3}$$

this then gives $aAa = \mathbb{C}a$ by the ideal property of rank one elements. The first part of (4.3) is clear from Lemma 3: if $x \in A$ then

$$\#\sigma'(xp) = \#\sigma'(xca) \le 1.$$

For the second part of (4.3), we claim that every element of $Aa(1-p) \subseteq Aa$ is quasinilpotent, so that $a(1-p) \in \mathrm{Rad}(A) = \{0\}$. Indeed, if $b \in Aa(1-p)$ has a non-zero point of spectrum then its support projection $q$ as in (4.2) has $0 \neq q \neq 1-p$; but then elements $\lambda q + \mu(1-q) \in Aa(1-p)$ would have two-point spectrum $\{\lambda, \mu\}$. ∎

Theorem 4 is not original: a stronger version is given by Mouton and Raubenheimer ([9], Theorem 2.2), in which (2.2) is replaced by the weaker condition

$$x \in A^{-1} \Rightarrow \#\sigma'(xa) \leq 1. \tag{4.4}$$

Of course, this is equivalent to (2.2) by the scarcity lemma ([1], Theorem 7.1.7; [9], Lemma 2.7). A similar version of the special case of Theorem 4 in which $A = BL(X, X)$ is given by Jafarian and Sourour ([7], Theorem 1). The condition (2.2) is the definition of "rank one" adopted by Aupetit and Mouton [2], who go on to characterise the socle and its "inessential hull" in a similar fashion.

Rank one elements are the cornerstone of Mouton's proof [8] of Aupetit's perturbation theorem ([1], Theorem 5.7.4):

**5.** THEOREM. *If $A$ is semisimple, if $a \in A$ and if $d \in A$ is of rank one, then*

$$\operatorname{acc}\sigma(a+d) \subseteq \eta\sigma(a) \quad \textit{and} \quad \operatorname{acc}\sigma(a) \subseteq \eta\sigma(a+d). \tag{5.1}$$

Proof. Here $\eta K$ denotes the "connected hull" ([5], Definition 7.10.1) of a compact set $K \subseteq \mathbb{C}$: the complement of $\eta K$ is the unbounded component of the complement of $K$. Since $dxd = \tau_d(x)d$ for $x \in A$ it follows that

$$a \in A^{-1} \Rightarrow \sigma(a+d) = \{\lambda \in \mathbb{C} : \tau_d((a-\lambda)^{-1}) = 1\}. \tag{5.2}$$

Now the function $f = \tau_d((a-z)^{-1}) - 1$ is holomorphic and not identically zero on the connected set $\sigma(a+d) \setminus \eta\sigma(a)$, and therefore ([4], Theorem 3.7) its zero set $f^{-1}(0)$ has no accumulation points (is this the tip of the iceberg of the scarcity lemma?); hence $\sigma(a+d) \setminus \eta\sigma(a) \subseteq \operatorname{iso}\sigma(a+d)$. This gives the first part of (5.1), and hence also the second. ∎

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