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A convolution operation for a distributional Hankel transformation

by

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Abstract. We investigate the Hankel transformation and the Hankel convolution on new spaces of generalized functions.

1. Introduction. The Hankel integral transformation is usually defined by

$$(h_\mu\phi)(y) = \int_0^\infty \sqrt{xy}\, J_\mu(xy)\phi(x)\,dx, \quad y \in (0,\infty),$$

where J_μ denotes the Bessel function of the first kind and order μ. Throughout this paper we will assume that μ is greater than $-1/2$.

The Hankel transformation has been investigated over several spaces of generalized functions by employing various procedures ([Z1], [Z2], [KZ] and [KL], amongst others). A. H. Zemanian [Z1] defined h_μ in distribution spaces by using the adjoint method. He introduced the space $\mathcal{H}_\mu$ of all complex-valued functions ϕ on $I = (0,\infty)$ such that

$$\eta^\mu_{k,m}(\phi) = \sup_{0<x<\infty} \left|(1+x^2)^k \left(\frac{1}{x}D\right)^m (x^{-\mu-1/2}\phi(x))\right| < \infty$$

for every $m, k \in \mathbb{N}$. The space $\mathcal{H}_\mu$ is endowed with the topology induced by the family $\{\eta^\mu_{k,m}\}_{k,m\in\mathbb{N}}$ of seminorms. Thus $\mathcal{H}_\mu$ is a Fréchet space. The Hankel transformation is an automorphism of $\mathcal{H}_\mu$ [Z3, Theorem 5.4-1]. The generalized Hankel transform $h'_\mu f$ of $f \in \mathcal{H}'_\mu$, where $\mathcal{H}'_\mu$ is the dual space of $\mathcal{H}_\mu$, is defined by

$$\langle h'_\mu f, \phi\rangle = \langle f, h_\mu\phi\rangle, \quad \phi \in \mathcal{H}_\mu.$$

Also, in order to study the Hankel transformation of distributions of rapid growth, A. H. Zemanian [Z2] introduced the function space β_μ. For

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every $a > 0$ the space $\beta_{\mu,a}$ consists of those functions ϕ in $\mathcal{H}_\mu$ such that $\phi(x) = 0$ for every $x \geq a$. It is equipped with the topology induced on it by $\mathcal{H}_\mu$. Thus it is a Fréchet space. It is clear that if $0 < a < b$, then $\beta_{\mu,a}$ is contained in $\beta_{\mu,b}$ and the topology of $\beta_{\mu,a}$ is the same as the one induced on it by $\beta_{\mu,b}$. The space β_μ is the inductive limit of the family $\{\beta_{\mu,a}\}_{a>0}$. It is a dense subspace of $\mathcal{H}_\mu$. In [Z2] the behaviour of the Hankel transformation on β_μ is investigated.

I. I. Hirschman [Hi] and D. T. Haimo [H] studied a convolution for a Hankel type transformation closely connected with h_μ. From their results by straightforward manipulations one can deduce analogous results for the Hankel transformation h_μ. Firstly, the Hankel convolution was studied over the space $L_{\mu,1}$ of measurable functions $\phi(x)$, $x \in (0,\infty)$, such that $\int_0^\infty x^{\mu+1/2}|\phi(x)|\,dx < \infty$. If $\phi, \varphi \in L_{\mu,1}$, the *Hankel convolution* is defined by

$$(\phi \,\sharp\, \varphi)(x) = \int_0^\infty \varphi(y)(\tau_x\phi)(y)\,dy, \quad x \in (0,\infty),$$

where τ_x, $x \in (0,\infty)$, denotes the *Hankel translation operator* given by

$$(\tau_x\phi)(y) = \int_0^\infty \phi(z)D_\mu(x,y,z)\,dz, \quad x,y \in (0,\infty),$$

and, for $x, y, z \in (0,\infty)$,

$$D_\mu(x,y,z) = \begin{cases} \frac{(xyz)^{1/2-\mu}[z^2-(x-y)^2]^{\mu-1/2}[(x+y)^2-z^2]^{\mu-1/2}}{2^{3\mu-1}\sqrt{\pi}\,\Gamma(\mu+1/2)}, & |x-y| < z < x+y, \\ 0, & z < |x-y| \text{ or } x+y < z. \end{cases}$$

The function D_μ has the following useful property:

$$\text{(1.1)} \quad \int_0^\infty x^{\mu+1/2}D_\mu(x,y,z)\,dz = \frac{1}{2^\mu\Gamma(\mu+1)}(xy)^{\mu+1/2}, \quad x,y \in (0,\infty).$$

The Hankel convolution has been investigated on the spaces β'_μ and $\mathcal{H}'_\mu$ of generalized functions in a series of papers by J. J. Betancor and I. Marrero ([BM1]–[BM4]). After characterizing the space $\mathcal{O}$ of multipliers of $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$ [BM1, Theorem 2.3], they introduce the space $\mathcal{O}'_{\mu,\sharp} = h'_\mu(x^{\mu+1/2}\mathcal{O}) \subset \mathcal{H}'_\mu$ of convolution operators in $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$. If $f \in \mathcal{H}'_\mu$ and $g \in \mathcal{O}'_{\mu,\sharp}$, then the Hankel convolution $f \,\sharp\, g$ is the element of $\mathcal{H}'_\mu$ defined by

$$\langle f \,\sharp\, g, \phi\rangle = \langle f(x), \langle g(y), (\tau_x\phi)(y)\rangle\rangle, \quad \phi \in \mathcal{H}_\mu.$$

The space $\mathcal{O}'_{\mu,\sharp}$ is a subspace of $\mathcal{H}'_\mu$ that is closed under $\sharp$-convolution. The main property of $\sharp$-convolution is the following interchange formula [BM3, (1.3)]. If $f \in \mathcal{H}'_\mu$ and $g \in \mathcal{O}'_{\mu,\sharp}$ then

$$\text{(1.2)} \quad h'_\mu(f \,\sharp\, g) = x^{-\mu-1/2}h'_\mu(f)h'_\mu(g).$$

In this paper, inspired by the studies of B. J. González and E. R. Negrin ([GN1] and [GN2]) on convolution and Fourier transform, we investigate the Hankel convolution in a new subspace of $\mathcal{H}'_\mu$. For $k \in \mathbb{Z}$, $k < 0$, we consider a Fréchet space $\mathcal{H}_{\mu,k}$ of functions such that

$$\mathcal{O}'_{\mu,\sharp} \subset \mathcal{H}'_{\mu,k} \subset \mathcal{H}'_\mu.$$

In Section 2 we define the Hankel transform on $\mathcal{H}'_{\mu,k}$ by using the kernel method. The Hankel convolution is defined and analyzed on $\mathcal{H}'_{\mu,k}$ in Section 3. We establish that the Hankel convolution is a closed operation in $\mathcal{H}'_{\mu,k}$. Moreover, the generalized Hankel transformation satisfies the interchange formula (1.2) when f and g are in $\mathcal{H}'_{\mu,k}$. The main results are summarized in the following

THEOREM. *Let f, g be in $\mathcal{H}'_{\mu,k}$ and let $k \in \mathbb{Z}$, $k < 0$. The Hankel convolution $f \,\sharp\, g$ defined by*

$$\langle f \,\sharp\, g, \phi\rangle = \langle f(x), \langle g(y), (\tau_x\phi)(y)\rangle\rangle, \quad \phi \in \mathcal{H}_\mu,$$

is an element of $\mathcal{H}'_{\mu,k}$. Moreover, if $f, g, h \in \mathcal{H}'_{\mu,k}$ then:

(a) $h'_\mu(f \,\sharp\, g)(y) = h'_\mu(f)(y)h'_\mu(g)(y)y^{-\mu-1/2}$, $y \in I$.
(b) $f \,\sharp\, g = g \,\sharp\, f$.
(c) $f \,\sharp\, (g \,\sharp\, h) = (f \,\sharp\, g) \,\sharp\, h$.
(d) *The functional δ_μ defined by*

$$\langle \delta_\mu, \phi\rangle = 2^\mu\Gamma(\mu+1)\lim_{x\to 0+} x^{-\mu-1/2}\phi(x), \quad \phi \in \mathcal{H}_{\mu,k},$$

is in $\mathcal{H}'_{\mu,k}$ and $\delta_\mu \,\sharp\, f = f \,\sharp\, \delta_\mu = f$.
(e) $S_\mu(f \,\sharp\, g) = (S_\mu f) \,\sharp\, g = f \,\sharp\, (S_\mu g)$.

Throughout this paper, I denotes the real interval $(0,\infty)$. We represent by C always a suitable positive constant (not necessarily the same at each occurrence). We denote by S_μ the Bessel operator $x^{-\mu-1/2}Dx^{2\mu+1}Dx^{-\mu-1/2}$.

2. The generalized Hankel transformation. In this section we investigate the Hankel transformation on a certain space of generalized functions by using the kernel method. The techniques and arguments employed here are usual in other studies on distributional integral transforms ([DP], [KZ], [KL] and [Z1], amongst others). Therefore the proofs of some of our results will be just outlined.

Let $k \in \mathbb{Z}$, $k < 0$. We introduce the space $A_{\mu,k}$ of complex-valued smooth functions $\phi(x)$, $x \in (0,\infty)$, such that

$$\gamma^m_{\mu,k}(\phi) = \sup_{0<x<\infty}|(1+x^2)^k x^{-\mu-1/2}S^m_\mu\phi(x)| < \infty$$

for every $m \in \mathbb{N}$. The space $A_{\mu,k}$ is endowed with the topology generated by the family $\{\gamma_{\mu,k}^m\}_{m\in\mathbb{N}}$ of seminorms. It is not hard to prove that $A_{\mu,k}$ is a complete space. Hence $A_{\mu,k}$ is a Fréchet space.

From [KZ, (9)] it is immediately deduced that β_μ is contained in $A_{\mu,k}$. We denote by $\mathcal{H}_{\mu,k}$ the closure of β_μ in $A_{\mu,k}$. Thus $\mathcal{H}_{\mu,k}$ is also a Fréchet space. The space $\mathcal{H}_{\mu,k}$ does not coincide with $A_{\mu,k}$. In fact, let $\phi_k(x) = x^{\mu+1/2}(1+x^2)^{-k}$, $x \in I$. By [KZ, (9)] one has for every $m \in \mathbb{N}$,

$$S_\mu^m \phi(x) = x^{\mu+1/2}\sum_{j=0}^{m} b_{j,m}x^{2j}\left(\frac{1}{x}D\right)^{m+j}[x^{-\mu-1/2}\phi(x)],$$

where $b_{j,m}$, $j = 0,\ldots,m$, are suitable real numbers. Thus,

$$x^{-\mu-1/2}S_\mu^m\phi_k(x)$$
$$= \sum_{j=0}^{m} b_{j,m}2^{m+j}(-k)(-k-1)\ldots(-k-m-j+1)(1+x^2)^{-k-m-j}, \quad x \in I.$$

Hence $\gamma_{\mu,k}^m(\phi_k) < \infty$, $m \in \mathbb{N}$, and $\phi_k \in A_{\mu,k}$. On the other hand, if ϕ_k is in $\mathcal{H}_{\mu,k}$, then there exists a sequence $(\phi_{k,n})_{n\in\mathbb{N}} \subset \beta_\mu$ with $\phi_{k,n} \to \phi_k$ in $A_{\mu,k}$ as $n \to \infty$. In particular,

$$\sup_{0<x<\infty}|(1+x^2)^k x^{-\mu-1/2}(\phi_k(x)-\phi_{k,n}(x))| \to 0 \quad \text{as } n \to \infty.$$

Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{0<x<\infty}|(1+x^2)^k x^{-\mu-1/2}(\phi_k(x)-\phi_{k,n_0}(x))| < 1/2.$$

Then

$$\begin{aligned}1 &= |(1+x^2)^k x^{-\mu-1/2}\phi_k(x)|\\ &\le |(1+x^2)^k x^{-\mu-1/2}(\phi_k(x)-\phi_{k,n_0}(x))|\\ &\quad + |(1+x^2)^k x^{-\mu-1/2}\phi_{k,n_0}(x)| < 1/2\end{aligned}$$

for $x \ge C$, with some $C > 0$, because $\phi_{k,n_0} \in \beta_\mu$, which is a contradiction. Therefore $\phi_k \notin \mathcal{H}_{\mu,k}$.

In the following lemma we give a sufficient condition in order that an element in $A_{\mu,k}$ belongs to $\mathcal{H}_{\mu,k}$, which will be useful in the sequel.

LEMMA 2.1. *Let $\phi \in A_{\mu,k}$. If for each $m \in \mathbb{N}$,*

$$\sup_{0<x<\infty}\left|x^m\left(\frac{1}{x}D\right)^m(x^{-\mu-1/2}\phi(x))\right| < \infty$$

then $\phi \in \mathcal{H}_{\mu,k}$.

Proof. Let λ be a smooth function on I such that

$$\lambda(x) = \begin{cases}1, & x \in (-\infty,1),\\ 0, & x \in (2,\infty).\end{cases}$$

Define, for every $n \in \mathbb{N}-\{0\}$, $\lambda_n(x) = \lambda(x-n+1)$, $x \in I$, and $\phi_n(x) = \lambda_n(x)\phi(x)$, $x \in I$. By hypothesis $\phi_n \in \beta_\mu$, $n \in \mathbb{N}$. Moreover, by invoking again [KZ, (9)] we have for every $m \in \mathbb{N}$, $n \in \mathbb{N}-\{0\}$ and $x \in I$,

$$\begin{aligned}&x^{-\mu-1/2}S_\mu^m[\phi_n(x)-\phi(x)]\\ &\quad= \sum_{j=0}^{m} b_{j,m}x^{2j}\left(\frac{1}{x}D\right)^{m+j}[x^{-\mu-1/2}(\phi_n(x)-\phi(x))]\\ &\quad= \sum_{j=0}^{m} b_{j,m}\sum_{i=0}^{m+j}\binom{m+j}{i}x^i\\ &\qquad\times\left(\frac{1}{x}D\right)^{i}[x^{-\mu-1/2}\phi(x)]x^{2j-i}\left(\frac{1}{x}D\right)^{m+j-i}(\lambda_n(x)-1),\end{aligned}$$

where $b_{j,m}$, $j = 0,\ldots,m$, are suitable real numbers.

Also, for each $l \in \mathbb{N}$ and $n \in \mathbb{N}-\{0\}$,

$$\left(\frac{1}{x}D\right)^l(\lambda_n(x)-1) = \sum_{s=0}^{l} c_s x^{-2l+s}D^s(\lambda_n(x)-1), \quad x \in I,$$

where c_s, $s = 0,\ldots,l$, are certain real numbers.

Hence there exists $C > 0$ such that for each $n \in \mathbb{N}-\{0\}$ and $x \in I$,

$$\begin{aligned}&|(1+x^2)^k x^{-\mu-1/2}S_\mu^m[\phi_n(x)-\phi(x)]|\\ &\quad\le C\sum_{j=0}^{m}\sum_{i=0}^{m+j}\sum_{s=0}^{m+j-i}(1+x^2)^k x^{2m+i+s}|D^s(\lambda_n(x)-1)|.\end{aligned}$$

Let $\varepsilon > 0$. There exists $M > 0$ such that

$$|(1+x^2)^k x^{-\mu-1/2}S_\mu^m[\phi_n(x)-\phi(x)]| < \varepsilon, \quad x \ge M,\ n \in \mathbb{N}-\{0\}.$$

Also, as $\lambda_n(x) = 1$ for $x \in (0,n)$ and $n \in \mathbb{N}-\{0\}$, there exists $n_0 \in \mathbb{N}-\{0\}$ such that

$$|(1+x^2)^k x^{-\mu-1/2}S_\mu^m[\phi_n(x)-\phi(x)]| < \varepsilon, \quad x \in (0,M),\ n \in \mathbb{N},\ n \ge n_0.$$

Therefore, for every $n \in \mathbb{N}$, $n \ge n_0$,

$$\sup_{0<x<\infty}|(1+x^2)^k x^{-\mu-1/2}S_\mu^m[\phi_n(x)-\phi(x)]| < \varepsilon.$$

Thus we conclude that $\phi_n \to \phi$ in $A_{\mu,k}$ as $n \to \infty$. Hence, $\phi \in \mathcal{H}_{\mu,k}$. ∎

An immediate consequence of Lemma 2.1 is that the space $\mathcal{H}_\mu$ is contained in $\mathcal{H}_{\mu,k}$.

A first application of Lemma 2.1 is the following.

PROPOSITION 2.1. *Let $y \in I$ and $k \in \mathbb{Z}$, $k < 0$. The function $\phi_y(x) = \sqrt{xy}\,J_\mu(xy)$, $x \in I$, is in $\mathcal{H}_{\mu,k}$.*

Proof. Let $m \in \mathbb{N}$. By [Z3, Lemma 5.4-1(5)] we have

$$S^m_{\mu,x}(\sqrt{xy}\,J_\mu(xy)) = (-y^2)^m\sqrt{xy}\,J_\mu(xy), \quad x \in I.$$

Hence, since $z^{-\mu}J_\mu(z)$ is a bounded function on I, there exists $C > 0$ such that

$$\sup_{0<x<\infty} |(1+x^2)^k x^{-\mu-1/2} S^m_{\mu,x}\phi_y(x)| \le C y^{2m+\mu+1/2}.$$

Then $\phi_y \in A_{\mu,k}$.

Moreover, according to [Z3, Ch. 5, (6)], for every $m \in \mathbb{N}$,

$$x^m\left(\frac{1}{x}D\right)^m (x^{-\mu-1/2}\phi_y(x)) = (-1)^m y^{\mu+1/2+m}(xy)^{-\mu}J_{\mu+m}(xy), \quad x \in I.$$

Hence, for every $m \in \mathbb{N}$,

$$\sup_{0<x<\infty}\left|x^m\left(\frac{1}{x}D\right)^m (x^{-\mu-1/2}\phi_y(x))\right| < \infty,$$

because $z^{-\mu}J_{\mu+m}(z)$ is a bounded function on $(0,\infty)$, and from Lemma 2.1 we deduce that $\phi_y \in \mathcal{H}_{\mu,k}$. ■

The Bessel operator S_μ defines a continuous linear mapping from $\mathcal{H}_{\mu,k}$ into itself.

PROPOSITION 2.2. *Let $k \in \mathbb{Z}$, $k < 0$, and let P be a polynomial. Then the mapping $\phi \mapsto P(S_\mu)\phi$ is linear and continuous from $\mathcal{H}_{\mu,k}$ into itself.*

Proof. It is sufficient to show that S_μ defines a continuous mapping from $\mathcal{H}_{\mu,k}$ into itself. Let $\phi \in \mathcal{H}_{\mu,k}$. There exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ in β_μ such that $\phi_n \to \phi$ in $\mathcal{H}_{\mu,k}$ as $n \to \infty$. Then it is clear that $(S_\mu\phi_n)_{n\in\mathbb{N}} \subset \beta_\mu$. Moreover, since for every $m \in \mathbb{N}$ and $\phi \in A_{\mu,k}$,

$$\gamma^m_{\mu,k}(S_\mu\phi) = \gamma^{m+1}_{\mu,k}(\phi),$$

it follows that $S_\mu\phi_n \to S_\mu\phi$ in $\mathcal{H}_{\mu,k}$ as $n \to \infty$, and the mapping $\phi \mapsto S_\mu\phi$ is continuous. ■

As usual, we denote by $\mathcal{H}'_{\mu,k}$ the dual space of $\mathcal{H}_{\mu,k}$. The space $\mathcal{O}'_{\mu,\sharp}$ considered in [BM3] is contained in $\mathcal{H}'_{\mu,k}$ because $\mathcal{H}_{\mu,k} \subset \mathcal{O}_{\mu,\sharp} = \bigcup_{k\in\mathbb{Z},k<0}\mathcal{H}_{\mu,k}$. Moreover, from [KZ, (9)] it immediately follows that if $\phi_n \to 0$ in $\mathcal{H}_\mu$ as $n \to \infty$, then $\phi_n \to 0$ in $\mathcal{H}_{\mu,k}$ as $n \to \infty$. Hence, $\mathcal{H}'_{\mu,k}$ is contained in $\mathcal{H}'_\mu$.

We now introduce a new function space that will be denoted by $\mathcal{X}_{\mu,k}$; it consists of all those locally integrable functions on $(0,\infty)$ such that

$$\int_0^\infty (1+x^2)^{-k}x^{\mu+1/2}|f(x)|\,dx < \infty.$$

It is easy to see that $\mathcal{X}_{\mu,k} \subset \mathcal{H}'_{\mu,k}$. In the next section we will refer again to $\mathcal{X}_{\mu,k}$.

An immediate consequence of Proposition 2.2 is the following.

PROPOSITION 2.3. *Let $k \in \mathbb{Z}$, $k < 0$, and let P be a polynomial. Then the mapping $f \mapsto P(S_\mu)f$ is linear and continuous from $\mathcal{H}'_{\mu,k}$ into itself when in $\mathcal{H}'_{\mu,k}$ we consider either the weak* or the strong topology.*

For every $f \in \mathcal{H}'_{\mu,k}$ we define the generalized Hankel transform $h'_\mu f$ by

$$(h'_\mu f)(y) = \langle f(x), \sqrt{xy}\,J_\mu(xy)\rangle, \quad x \in I.$$

Note that by Proposition 2.1 the definition is allowable.

We now establish some properties of the generalized Hankel transformation.

PROPOSITION 2.4. *Let $k \in \mathbb{Z}$, $k < 0$, and let P be a polynomial. Then for every $f \in \mathcal{H}'_{\mu,k}$ we have*

$$h'_\mu(P(S_\mu)f)(y) = P(-y^2)h'_\mu(f)(y), \quad y \in I.$$

Proof. It is sufficient to take into account that $S_\mu\sqrt{z}\,J_\mu(z) = -\sqrt{z}\,J_\mu(z)$ (cf. [Z3, Ch. 5, (6), (7)]. ■

PROPOSITION 2.5. *Let $k \in \mathbb{Z}$, $k < 0$, and $f \in \mathcal{H}'_{\mu,k}$. There exist $C > 0$ and $r \in \mathbb{N}$ such that*

$$|(h'_\mu f)(y)| \le C\begin{cases} y^{\mu+1/2}, & y \in (0,1),\\ y^{\mu+1/2+2r}, & y \in (1,\infty).\end{cases}$$

Proof. This result follows immediately from [Z3, Theorem 1.8-1] by taking into account [Z3, Ch. 5, (6), (7)]. ■

PROPOSITION 2.6. *Let $k \in \mathbb{Z}$, $k < 0$, and $f \in \mathcal{H}'_{\mu,k}$. Then $h'_\mu f$ is $-2k-1$ times differentiable.*

Proof. Firstly we prove that $h'_\mu f$ is continuous in I. For every $y \in I$ and $0 < |h| < y$ we have

$$(h'_\mu f)(y+h) - (h'_\mu f)(y) = \langle f(x), \sqrt{x(y+h)}\,J_\mu(x(y+h))\rangle - \langle f(x), \sqrt{xy}\,J_\mu(xy)\rangle.$$

Hence, the continuity of f in $y \in I$ will be established when we show that

(2.1) $$\sqrt{x(y+h)}\,J_\mu(x(y+h)) \to \sqrt{xy}\,J_\mu(xy) \quad \text{in } \mathcal{H}_{\mu,k} \text{ as } h \to 0.$$

To prove (2.1), let $y \in I$ and $m \in \mathbb{N}$. We can write

$$x^{-\mu-1/2}S^m_{\mu,x}[\sqrt{x(y+h)}\,J_\mu(x(y+h)) - \sqrt{xy}\,J_\mu(xy)]$$
$$= (-1)^k[(y+h)^{2m+\mu+1/2}(x(y+h))^{-\mu}J_\mu(x(y+h)) - y^{2m+\mu+1/2}(xy)^{-\mu}J_\mu(xy)]$$

for $x \in I$ and $0 < |h| < y$.

Assume that $\varepsilon > 0$. Since $z^{-\mu}J_\mu(z)$ is bounded on I there exists $M > 0$ such that for $x \ge M$ and $0 < |h| < y$,

(2.2) $$(1+x^2)^k|x^{-\mu-1/2}S^m_{\mu,x}[\sqrt{x(y+h)}\,J_\mu(x(y+h)) - \sqrt{xy}\,J_\mu(xy)]| < \varepsilon.$$

Moreover, by taking into account the mean value we can find $h_0 > 0$ such that for every $0 < x < M$ and $0 < |h| < h_0$,

$$(2.3)\quad (1+x^2)^k|x^{-\mu-1/2}S^m_{\mu,x}[\sqrt{x(y+h)}\,J_\mu(x(y+h)) - \sqrt{xy}\,J_\mu(xy)]| < \varepsilon.$$

By combining (2.2) and (2.3) we conclude that

$$\sup_{0<x<\infty}|(1+x^2)^k x^{-\mu-1/2}S^m_{\mu,x}[\sqrt{x(y+h)}\,J_\mu(x(y+h)) - \sqrt{xy}\,J_\mu(xy)]| < \varepsilon$$

provided that $0 < |h| < h_0$. Thus (2.1) is established.

We now prove that $h'_\mu f$ is differentiable provided that $k \in \mathbb{Z}$, $k \le -1$. Let $0 < y < \infty$. For each $0 < |h| < y$, one has

$$\frac{(h'_\mu f)(y+h) - (h'_\mu f)(y)}{h} = \left\langle f(x), \frac{\sqrt{x(y+h)}\,J_\mu(x(y+h)) - \sqrt{xy}\,J_\mu(xy)}{h}\right\rangle.$$

It will be established that

$$I_h(x) = \frac{\sqrt{x(y+h)}\,J_\mu(x(y+h)) - \sqrt{xy}\,J_\mu(xy)}{h} - \frac{\partial}{\partial y}[\sqrt{xy}\,J_\mu(xy)] \to 0$$

in $A_{\mu,k}$ as $h \to 0^+$.

For every $0 < |h| < y$ and $0 < x < \infty$ we can write

$$I_h(x) = \frac{1}{h}\int_y^{y+h}\int_y^u \frac{\partial^2}{\partial\varrho^2}[\sqrt{x\varrho}\,J_\mu(x\varrho)]\,d\varrho\,du.$$

Let $m \in \mathbb{N}$. For every $x \in I$ and $0 < |h| < y$ from [Z3, Ch. 5, (6), (7)] we infer that

$$x^{-\mu-1/2}S^m_{\mu,x}I_h(x) = (-1)^m\frac{1}{h}\int_y^{y+h}\int_y^u \frac{\partial^2}{\partial\varrho^2}[\varrho^{2m+\mu+1/2}(x\varrho)^{-\mu}J_\mu(x\varrho)]\,d\varrho\,du.$$

Since $z^{1/2}J_\mu(z)$ is bounded on I there exists $C > 0$ such that

$$\left|\frac{\partial^2}{\partial\varrho^2}[\varrho^{2m+\mu+1/2}(x\varrho)^{-\mu}J_\mu(x\varrho)]\right| \le C(\varrho^{2m-2} + x\varrho^{2m-1} + x^2\varrho^{2m}),\qquad x,\varrho \in I.$$

Hence, for $x \in I$ and $0 < |h| < y$,

$$|(1+x^2)^k x^{-\mu-1/2}S^m_{\mu,x}I_h(x)| \le C(1+x^2)^{k+1}\frac{1}{h}\int_y^{y+h}\int_y^u \varrho^{2m-2}(1+\varrho+\varrho^2)\,d\varrho\,du.$$

Then

$$\sup_{0<x<\infty}|(1+x^2)^k x^{-\mu-1/2}S^m_{\mu,x}I_h(x)| \to 0 \quad \text{as } h \to 0,$$

provided that $k \le -1$.

Therefore, $h'_\mu f$ is differentiable when $k \in \mathbb{Z}$, $k \le -1$.

The proof of the general case follows by using similar arguments. ∎

PROPOSITION 2.7. *Let $k \in \mathbb{Z}$, $k < 0$, and $f \in \mathcal{H}'_{\mu,k}$. Then*

$$\langle h'_\mu f, \phi\rangle = \langle f, h_\mu\phi\rangle,\qquad \phi \in \mathcal{H}_\mu.$$

Proof. Proceed as in the proof of [KZ, Theorem 3], replacing the function e^{-ax} $(a > 0)$ by $(1+x^2)^k$. ∎

Proposition 2.7 yields a uniqueness result for the generalized Hankel transform on $\mathcal{H}'_{\mu,k}$.

PROPOSITION 2.8. *Let $k \in \mathbb{Z}$, $k < 0$, and $f, g \in \mathcal{H}'_{\mu,k}$. If $h'_\mu f = h'_\mu g$ then $f = g$.*

Proof. Let $\phi \in \mathcal{H}_{\mu,k}$. There exists a sequence $(\phi_n)_{n\in\mathbb{N}} \subset \beta_\mu$ such that $\phi_n \to \phi$ in $A_{\mu,k}$ as $n \to \infty$. Then, since $f, g \in \mathcal{H}'_{\mu,k}$, one has $\langle f, \phi_n\rangle \to \langle f, \phi\rangle$ and $\langle g, \phi_n\rangle \to \langle g, \phi\rangle$ as $n \to \infty$. Moreover, by [Z3, Theorem 5.4-1] and Proposition 2.7,

$$\langle f, \phi_n\rangle = \langle h'_\mu f, h_\mu\phi_n\rangle = \langle h'_\mu g, h_\mu\phi_n\rangle = \langle g, \phi_n\rangle,\qquad n \in \mathbb{N},$$

and the proof is complete. ∎

Note that from Propositions 2.7 and 2.8 it follows that each generalized function f in $\mathcal{H}'_{\mu,k}$ is uniquely determined by its Hankel transform $h'_\mu f$.

3. The Hankel convolution on $\mathcal{H}'_{\mu,k}$. We now define the Hankel convolution on the spaces $\mathcal{H}'_{\mu,k}$. First we analyze the Hankel translation τ_x, $x \in I$, on $\mathcal{H}'_{\mu,k}$.

Our first result, which will be very useful in the sequel, establishes that the operators S_μ and τ_x, $x \in I$, commute.

LEMMA 3.1. *Let $m \in \mathbb{N}$ and $k \in \mathbb{Z}$, $k < 0$. Then for every $\phi \in \mathcal{H}_{\mu,k}$,*

$$S^m_{\mu,x}(\tau_x\phi)(y) = \tau_x(S^m_\mu\phi)(y),\qquad x, y \in I.$$

Proof. Let $\phi \in \mathcal{H}_{\mu,k}$. We have

$$(3.1)\qquad (\tau_x\phi)(y) = \int_{|x-y|}^{x+y}\phi(z)D_\mu(x,y,z)\,dz,\qquad x, y \in I.$$

Let $r > 0$. Consider a smooth function λ on $(0,\infty)$ such that $\lambda(x) = 1$ for $x \in (0, 2r)$ and $\lambda(x) = 0$ for $x \in (2r+1, \infty)$. Now we prove that $\lambda\phi \in \beta_\mu$.

In fact, consider the vector space

$$M_\mu = \left\{\phi \in C^\infty(0,\infty) : \gamma_m(\phi) = \sup_{0<x<\infty}\left|\left(\frac{1}{x}D\right)^m[x^{-\mu-1/2}\phi(x)]\right| < \infty,\ m \in \mathbb{N}\right\}.$$

Following usual techniques it is proved that M_μ endowed with the topology generated by the family $\{\gamma_m\}_{m\in\mathbb{N}}$ of seminorms is a Fréchet space. Moreover, if $(\phi_n)_{n\in\mathbb{N}} \subset \beta_\mu \subset M_\mu$ is such that ϕ_n converges to ϕ in $A_{\mu,k}$ as $n\to\infty$, according to [S, Ch. IV, Proposition 2] and by using the Leibniz formula we can find $C>0$ such that

$$\gamma_m(\lambda(\phi_p-\phi_q)) = \sup_{0<x<\infty}\left|\left(\frac{1}{x}D\right)^m [x^{-\mu-1/2}\lambda(x)(\phi_p(x)-\phi_q(x))]\right|$$
$$\le C\max_{0\le n\le m}\sup_{0<x<\infty}|x^{-\mu-1/2}S^n_{\mu,x}(\phi_p-\phi_q)(x)|,\qquad p,q\in\mathbb{N}.$$

Hence $(\lambda\phi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in M_μ. Then there exists $\psi\in M_\mu$ such that $\lambda\phi_n\to\psi$ in M_μ as $n\to\infty$. Thus, since $\lambda(x)\phi_n(x)\to\psi(x)$ and $(1+x^2)^k x^{-\mu-1/2}\phi_n(x)\to(1+x^2)^k x^{-\mu-1/2}\phi(x)$ for $x\in I$ as $n\to\infty$, it follows that $\lambda\phi=\psi$ and we conclude that $\lambda\phi\in\beta_\mu$.

On the other hand, as $\phi(y)=(\phi\lambda)(y)$ for $0<y<2r$, from (3.1) we deduce that $(\tau_x\phi)(y)=(\tau_x\phi\lambda)(y)$ for $x,y\in(0,r)$. Hence, by invoking [MB3, (1.3)] we conclude that for every $m\in\mathbb{N}$,

$$S^m_{\mu,x}(\tau_x\phi)(y) = S^m_{\mu,x}(\tau_x\phi\lambda)(y)$$
$$=\tau_x(S^m_\mu(\phi\lambda))(y) = \tau_x(S^m_\mu\phi)(y),\qquad 0<x,y<r.$$

Thus, since $r>0$ is arbitrary, the result is established. ∎

LEMMA 3.2. *Let $k\in\mathbb{Z}$, $k<0$. For every $x\in I$ the mapping $\phi\mapsto\tau_x\phi$ is linear and continuous from $\mathcal{H}_{\mu,k}$ into itself.*

Proof. Let $x\in I$. By [BM2, Corollary 3.3], $\tau_x\phi\in\beta_\mu$ for every $\phi\in\beta_\mu$.

Let ϕ be in $\mathcal{H}_{\mu,k}$. There exists a sequence $(\phi_n)_{n\in\mathbb{N}}\subset\beta_\mu$ such that $\phi_n\to\phi$ in $A_{\mu,k}$ as $n\to\infty$. According to Lemma 3.1 and [GN1, Lemma 2.1], for every $m,n\in\mathbb{N}$, we can write for $x,y\in I$,

$$|(1+y^2)^k y^{-\mu-1/2}S^m_{\mu,y}[(\tau_x\phi_n)(y)-(\tau_x\phi)(y)]|$$
$$=|(1+y^2)^k y^{-\mu-1/2}\tau_x[S^m_{\mu,y}(\phi_n-\phi)](y)|$$
$$\le(1+y^2)^k y^{-\mu-1/2}\int_0^{x+y}|S^m_{\mu,z}(\phi_n-\phi)(z)|D_\mu(x,y,z)\,dz$$
$$\le(1+y^2)^k(1+(x+y)^2)^{-k}y^{-\mu-1/2}$$
$$\times\int_0^{x+y}(1+z^2)^k z^{-\mu-1/2}|S^m_{\mu,z}(\phi_n-\phi)(z)|z^{\mu+1/2}D_\mu(x,y,z)\,dz$$
$$\le 4^{-k}(1+x^2)^k y^{-\mu-1/2}$$
$$\times\sup_{0<z<\infty}(1+z^2)^k z^{-\mu-1/2}|S^m_{\mu,z}(\phi_n-\phi)(z)|\int_0^{x+y}z^{\mu+1/2}D_\mu(x,y,z)\,dz.$$

Then from (1.1) it follows that

$$\text{(3.2)}\qquad \sup_{0<y<\infty}|(1+y^2)^k y^{-\mu-1/2}S^m_{\mu,y}[(\tau_x\phi_n)(y)-(\tau_x\phi)(y)]|$$
$$\le\frac{1}{2^\mu\Gamma(\mu+1)}4^{-k}(1+x^2)^k x^{\mu+1/2}$$
$$\times\sup_{0<z<\infty}(1+z^2)^k z^{-\mu-1/2}|S^m_{\mu,z}(\phi_n-\phi)(z)|,\qquad m,n\in\mathbb{N}.$$

Hence $\tau_x\phi_n\to\tau_x\phi$ in $A_{\mu,k}$ as $n\to\infty$ and $\tau_x\phi\in\mathcal{H}_{\mu,k}$.

Also an inequality analogous to (3.2) proves that the Hankel translation τ_x defines a continuous mapping from $\mathcal{H}_{\mu,k}$. ∎

The last lemma allows us to define the Hankel convolution of a distribution in $\mathcal{H}'_{\mu,k}$ and a function in $\mathcal{H}_{\mu,k}$. If $f\in\mathcal{H}'_{\mu,k}$ and $\phi\in\mathcal{H}_{\mu,k}$ then the Hankel convolution $f\,\sharp\,\phi$ is defined by

$$\text{(3.3)}\qquad (f\,\sharp\,\phi)(x)=\langle f,\tau_x\phi\rangle,\qquad x\in I.$$

Note that if $f\in\mathcal{X}_{\mu,k}$ then for every $\phi\in\mathcal{H}_{\mu,k}$,

$$(f\,\sharp\,\phi)(x)=\int_0^\infty f(y)(\tau_x\phi)(y)\,dy,\qquad 0<x<\infty.$$

In this sense the classical $\sharp$-convolution can be seen as a special case of the distributional $\sharp$-convolution (3.3).

Before defining the $\sharp$-convolution of two elements of $\mathcal{H}'_{\mu,k}$ we will prove that the distributions in $\mathcal{H}'_{\mu,k}$ define convolution operators in $\mathcal{H}_{\mu,k}$.

LEMMA 3.3. *Let $k\in\mathbb{Z}$, $k<0$, and $f\in\mathcal{H}'_{\mu,k}$. Then the mapping $\phi\mapsto f\sharp\phi$ is linear and continuous from $\mathcal{H}_{\mu,k}$ into itself.*

Proof. We divide the proof in several steps.

CLAIM (a). *For every $\phi\in\beta_\mu$ and $m\in\mathbb{N}$,*

$$\text{(3.4)}\qquad S^m_{\mu,x}\langle f(y),(\tau_x\phi)(y)\rangle=\langle f(y),\tau_x(S^m_{\mu,x}\phi)(y)\rangle,\qquad x\in I.$$

Let $\phi\in\beta_\mu$. According to [BM3, (1.3)] one has

$$(\tau_x\phi)(y)=h_\mu[t^{-\mu-1/2}(xt)^{1/2}J_\mu(xt)h_\mu(\phi)(t)](y),\qquad x,y\in I.$$

We are going to establish that

$$\text{(3.5)}\qquad S_{\mu,x}\langle f(y),(\tau_x\phi)(y)\rangle=\langle f(y),S_{\mu,x}(\tau_x\phi)(y)\rangle,\qquad x\in I.$$

Firstly it must be seen that for every $x\in I$,

$$(3.6)\qquad \left\langle f(y), \frac{h_\mu[((x+h)t)^{-\mu}J_\mu((x+h)t)h_\mu(\phi)(t)](y)}{h} - \frac{h_\mu[(xt)^{-\mu}J_\mu(xt)h_\mu(\phi)(t)](y)}{h}\right\rangle \to \langle f(y), D_x h_\mu[(xt)^{-\mu}J_\mu(xt)h_\mu(\phi)(t)](y)\rangle \quad \text{as } h\to 0.$$

Let $x \in I$ and $0<|h|<x$. We have

$$\begin{aligned} I_h(y) &= \frac{h_\mu[((x+h)t)^{-\mu}J_\mu((x+h)t)h_\mu(\phi)(t)](y) - h_\mu[(xt)^{-\mu}J_\mu(xt)h_\mu(\phi)(t)](y)}{h} \\ &\quad - D_x h_\mu[(xt)^{-\mu}J_\mu(xt)h_\mu(\phi)(t)](y) \\ &= \frac{1}{h}\int\limits_x^{x+h}\int\limits_x^{u} \frac{\partial^2}{\partial\eta^2} h_\mu[(\eta t)^{-\mu}J_\mu(\eta t)h_\mu(\phi)(t)](y)\,d\eta\,du, \qquad y\in I. \end{aligned}$$

Then for $m\in\mathbb{N}$ and $y\in I$ one has

$$S^m_{\mu,y} I_h(y) = \frac{1}{h}\int\limits_x^{x+h}\int\limits_x^{u}\frac{\partial^2}{\partial\eta^2}h_\mu[(\eta t)^{-\mu}J_\mu(\eta t)h_\mu(S^m_\mu\phi)(t)](y)\,d\eta.$$

By taking into account that $z^{-\mu}J_\mu(z)$ is a bounded function on I and that $\frac{d}{dz}(z^{-\mu}J_\mu(z)) = z^{-\mu}J_{\mu+1}(z)$, $z\in I$, we can conclude that

$$(3.7)\qquad (1+y^2)^k y^{-\mu-1/2} S^m_{\mu,y} I_h(y) \to 0$$

as $h\to 0$ uniformly in $y\in(0,\infty)$.

Since $f\in\mathcal{H}'_{\mu,k}$, (3.6) follows from (3.7).

By proceeding in a similar way we can prove that

$$\frac{d}{dx}\left\langle f(y), x^{2\mu+1}\frac{d}{dx}x^{-\mu-1/2}(\tau_x\phi)(y)\right\rangle = \left\langle f(y), \frac{d}{dx}x^{2\mu+1}\frac{d}{dx}x^{-\mu-1/2}(\tau_x\phi)(y)\right\rangle, \qquad x\in I.$$

Thus (3.5) is established.

From Lemma 3.1 and (3.5), (3.4) is immediately deduced.

Claim (b). *The mapping $\phi\mapsto f\,\sharp\,\phi$ is continuous from β_μ into $A_{\mu,k}$ when we consider on β_μ the topology induced by $A_{\mu,k}$.*

Since $f\in\mathcal{H}'_{\mu,k}$, according to [Z3, Theorem 1.8-1] there exist $C>0$ and $r\in\mathbb{N}$ such that

$$(3.8)\qquad |\langle f,\phi\rangle| \le C\max_{0\le m\le r}\sup_{0<y<\infty}|(1+y^2)^k y^{-\mu-1/2}S^m_{\mu,y}\phi(y)|, \qquad \phi\in\mathcal{H}_{\mu,k}.$$

Let $\phi\in\beta_\mu$ and $m\in\mathbb{N}$. By combining (3.4) and (3.8) it follows that

$$|S^m_{\mu,x}\langle f,\tau_x\phi\rangle| \le C\max_{0\le m\le r}\sup_{0<y<\infty}|(1+y^2)^k y^{-\mu-1/2}\tau_x(S^{m+n}_\mu\phi)(y)|, \qquad 0<x<\infty.$$

Hence, by proceeding as in the proof of (3.2) we obtain

$$\sup_{0<x<\infty}|(1+x^2)^k x^{-\mu-1/2}S^m_{\mu,x}\langle f,\tau_x\phi\rangle| \le 4^{-k}C\max_{0\le n\le r}\sup_{0<y<\infty}|(1+y^2)^k y^{-\mu-1/2}S^{m+n}_{\mu,y}\phi(y)|$$

and our claim is proved.

Claim (c). *For every $\phi\in\beta_\mu$ one has $f\,\sharp\,\phi\in\mathcal{H}_{\mu,k}$.*

Let $\phi\in\beta_\mu$. By (b), $f\,\sharp\,\phi$ is in $A_{\mu,k}$. To see that $f\,\sharp\,\phi\in\mathcal{H}_{\mu,k}$ we will use Lemma 2.1. Let $m\in\mathbb{N}$. By invoking again [BM3, (1.2)] we see that

$$\begin{aligned} \left(\frac{1}{x}D\right)^m &[x^{-\mu-1/2}(f\,\sharp\,\phi)(x)] \\ &= \left\langle f(y), \left(\frac{1}{x}D\right)^m h_\mu[(xt)^{-\mu}J_\mu(xt)h_\mu(\phi)(t)](y)\right\rangle \\ &= \langle f(y), h_\mu[(-t^2)^m(xt)^{-\mu-m}J_{\mu+m}(xt)h_\mu(\phi)(t)](y)\rangle, \qquad x\in I. \end{aligned}$$

Then by [Z3, Lemma 5.4-1] for some $C>0$ and $r\in\mathbb{N}$ one has, for $x\in I$,

$$\begin{aligned} &\left|x^m\left(\frac{1}{x}D\right)^m[x^{-\mu-1/2}(f\,\sharp\,\phi)(x)]\right| \\ &\le C\max_{0\le n\le r}\sup_{0<y<\infty}|(1+y^2)^k y^{-\mu-1/2} \\ &\qquad\times S^n_{\mu,y}h_\mu[(xt)^{-\mu-m}J_{\mu+m}(xt)h_\mu(S^m_\mu\phi)(t)](y)| \\ &= C\max_{0\le n\le r}\sup_{0<y<\infty}|(1+y^2)^k y^{-\mu-1/2} \\ &\qquad\times h_\mu[(xt)^{-\mu-m}J_{\mu+m}(xt)h_\mu(S^{n+m}_\mu\phi)(t)](y)| \\ &\le C\max_{0\le n\le r}\sup_{0<y<\infty}(1+y^2)^k\int\limits_0^\infty|(ty)^{-\mu}J_\mu(ty)| \\ &\qquad\times|(xt)^{-\mu-m}J_{\mu+m}(xt)|t^{\mu+1/2}|h_\mu(S^{n+m}_\mu\phi)(t)|\,dt \le C, \end{aligned}$$

because $z^{-\mu}J_\mu(z)$ is bounded on I.

We now finish the proof by taking into account the above claims.

The space β_μ is dense in $\mathcal{H}_{\mu,k}$. Hence, the mapping

$$\beta_\mu\to\mathcal{H}_{\mu,k}, \qquad \phi\mapsto f\,\sharp\,\phi,$$

can be continuously extended to $\mathcal{H}_{\mu,k}$. Denote by T the extended mapping. It is well known that if $\phi \in \mathcal{H}_{\mu,k}$, then

$$T\phi = \lim_{n\to\infty} f \sharp \phi_n,$$

where the limit is understood in $\mathcal{H}_{\mu,k}$ and $(\phi_n)_{n\in\mathbb{N}}$ is a sequence in β_μ such that $\phi_n \to \phi$ in $\mathcal{H}_{\mu,k}$ as $n \to \infty$. It is easy to see that convergence in $\mathcal{H}_{\mu,k}$ implies pointwise convergence on $(0,\infty)$. Moreover, by (3.2),

$$\begin{aligned}
&|(f \sharp \phi)(x) - (f \sharp \phi_n)(x)| \\
&\quad \le C \max_{0\le l\le r} \sup_{0<y<\infty} |(1+y^2)^k y^{-\mu-1/2} S^l_{\mu,y}[(\tau_x\phi)(y) - (\tau_x\phi_n)(y)]| \\
&\quad \le C4^{-k}((1+x^2)^k x^{\mu+1/2} \\
&\qquad \times \max_{0\le l\le r} \sup_{0<y<\infty} |(1+y^2)^k y^{-\mu-1/2} S^l_{\mu,y}(\phi-\phi_n)(y)|, \quad x\in I,
\end{aligned}$$

with $C>0$ and $r\in\mathbb{N}$.

Therefore $(f\sharp\phi_n)(x) \to (f\sharp\phi)(x)$ as $n\to\infty$ for every $x\in(0,\infty)$. Then we conclude that $(T\phi)(x) = (f\sharp\phi)(x)$, $x\in I$, and $f\sharp\phi\in\mathcal{H}_{\mu,k}$. Thus the proof is complete. ■

We can now define $\sharp$-convolution in $\mathcal{H}'_{\mu,k}$ as follows. If $f,g\in\mathcal{H}'_{\mu,k}$ then we define the Hankel convolution $f\sharp g$ by

$$\langle f\sharp g,\phi\rangle = \langle f(x), \langle g(y), (\tau_x\phi)(y)\rangle\rangle, \quad \phi\in\mathcal{H}_{\mu,k}.$$

By Lemma 3.3, $f\sharp g\in\mathcal{H}'_{\mu,k}$. Hence the Hankel convolution is a closed operation in $\mathcal{H}'_{\mu,k}$.

The main algebraic properties of $\sharp$-convolution are established in the following

THEOREM 3.1. *Let $k\in\mathbb{Z}$, $k<0$. If $f,g,h\in\mathcal{H}'_{\mu,k}$ then:*

(a) $h'_\mu(f\sharp g)(y) = h'_\mu(f)(y)h'_\mu(g)(y)y^{-\mu-1/2}$, $y\in I$.

(b) $f\sharp g = g\sharp f$.

(c) $f\sharp(g\sharp h) = (f\sharp g)\sharp h$.

(d) *The functional δ_μ defined by*

$$\langle\delta_\mu,\phi\rangle = 2^\mu\Gamma(\mu+1)\lim_{x\to0+} x^{-\mu-1/2}\phi(x), \quad \phi\in\mathcal{H}_{\mu,k},$$

is in $\mathcal{H}'_{\mu,k}$ and $\delta_\mu\sharp f = f\sharp\delta_\mu = f$.

(e) $S_\mu(f\sharp g) = (S_\mu f)\sharp g = f\sharp(S_\mu g)$.

Proof. (a) For every $y\in I$ according to [W, p. 367 and p. 411] we have

$$\begin{aligned}
h'_\mu(f\sharp g)(y) &= \langle f\sharp g, \phi_y(x)\rangle = \langle f(t), \langle g(x), \tau_t(\phi_y)(x)\rangle\rangle \\
&= y^{-\mu-1/2}\langle f(t), \langle g(x), \sqrt{ty}\,J_\mu(ty)\sqrt{xy}\,J_\mu(xy)\rangle\rangle \\
&= y^{-\mu-1/2} h'_\mu(f)(y) h'_\mu(g)(y),
\end{aligned}$$

where $\phi_y(x) = \sqrt{xy}\,J_\mu(xy)$, $x,y\in I$.

(b) By using (a) it follows that

$$h'_\mu(f\sharp g)(y) = h'_\mu(f)(y)h'_\mu(g)(y)y^{-\mu-1/2} = h'_\mu(g\sharp f)(y), \quad y\in I.$$

Hence according to Proposition 2.8, $f\sharp g = g\sharp f$.

(c) This property is also an immediate consequence of Proposition 2.8 and the above property (a).

(d) Let $\phi\in\mathcal{H}_{\mu,k}$. There exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ in β_μ such that $\phi_n\to\phi$ in $\mathcal{H}_{\mu,k}$ as $n\to\infty$. Setting $l_n = \lim_{x\to0+} x^{-\mu-1/2}\phi_n(x)$ for each $n\in\mathbb{N}$, we have

$$\begin{aligned}
|l_n - l_s| &= \lim_{x\to0+}(1+x^2)^k x^{-\mu-1/2}|\phi_n(x)-\phi_s(x)| \\
&\le \sup_{0<x<\infty}(1+x^2)^k x^{-\mu-1/2}|\phi_n(x)-\phi_s(x)|, \quad n,s\in\mathbb{N}.
\end{aligned}$$

Hence there exists $l\in\mathbb{C}$ such that $l_n\to l$ as $n\to\infty$. Moreover, it is not hard to see that the limit $\lim_{x\to0+} x^{-\mu-1/2}\phi(x)$ exists and is equal to l.

Also we can write

$$\begin{aligned}
|\langle\delta_\mu,\phi\rangle| &= 2^\mu\Gamma(\mu+1)\lim_{x\to0+}|x^{-\mu-1/2}\phi(x)| \\
&\le 2^\mu\Gamma(\mu+1)\sup_{0<x<\infty}(1+x^2)^k x^{-\mu-1/2}|\phi(x)|, \quad \phi\in\mathcal{H}_{\mu,k}.
\end{aligned}$$

Hence $\delta_\mu\in\mathcal{H}'_{\mu,k}$.

On the other hand,

$$\begin{aligned}
h'_\mu(\delta_\mu)(y) &= \langle\delta_\mu(x), \sqrt{xy}\,J_\mu(xy)\rangle \\
&= 2^\mu\Gamma(\mu+1)\lim_{x\to0+} x^{-\mu-1/2}\sqrt{xy}\,J_\mu(xy) = y^{\mu+1/2}, \quad y\in I.
\end{aligned}$$

Therefore by invoking (a) we obtain $h'_\mu(f\sharp\delta_\mu) = h'_\mu(f)$. Then Proposition 2.8 shows that $f\sharp\delta_\mu = f$.

(e) This property follows from Propositions 2.8 and 2.4 by using again (a). ■

The following continuity property of $\sharp$-convolution is an immediate consequence of Lemma 3.3.

PROPOSITION 3.1. *Let $k\in\mathbb{Z}$, $k<0$. Assume that $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{H}'_{\mu,k}$ that converges to $f\in\mathcal{H}'_{\mu,k}$ in the weak* topology (respectively, in the strong topology) of $\mathcal{H}'_{\mu,k}$. Then for every $g\in\mathcal{H}'_{\mu,k}$,*

$$f_n\sharp g\to f\sharp g \quad \text{as } n\to\infty$$

in the weak topology (respectively, in the strong topology) of $\mathcal{H}'_{\mu,k}$.*

Remark. Studies analogous to the one developed here can be made by replacing the function $(1+x^2)^k$ in the definition of the space $\mathcal{H}_{\mu,k}$, $k \in \mathbb{Z}$, $k < 0$, by other functions. For example, if we put the function e^{-kx} instead of $(1+x^2)^k$ our procedure permits defining the Hankel convolution in the spaces of E. L. Koh and A. H. Zemanian [KZ].

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On rank one elements

by

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Abstract. Without the "scarcity lemma", two kinds of "rank one elements" are identified in semisimple Banach algebras.

Suppose A is a complex Banach algebra, with identity 1 (usually not zero), and invertible group A^{-1}: then the *radical* of A can be defined ([5], Theorem 7.2.3) as the set

$$\mathrm{Rad}(A) = \{a \in A : 1 + Aa \subseteq A^{-1}\}. \tag{0.1}$$

It is familiar that this is a closed two-sided ideal; also,

$$\begin{aligned} 1 + Aa \subseteq A^{-1} &\Rightarrow 1 + A^{-1}a \subseteq A^{-1} \Rightarrow A^{-1} + a \subseteq A^{-1} \\ &\Rightarrow 1 + (A^{-1} + A^{-1})a \subseteq A^{-1}; \end{aligned} \tag{0.2}$$

since of course $A^{-1} + A^{-1} = A$ this gives an alternative description of $\mathrm{Rad}(A)$, and also provides an elementary instance of the "scarcity lemma" ([1], Theorem 7.1.7). We recall the *spectrum* and the *non-zero spectrum*,

$$\sigma_A(a) = \sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin A^{-1}\} \quad \text{and} \quad \sigma'(a) = \sigma(a) \setminus \{0\}; \tag{0.3}$$

thus

$$a \in \mathrm{Rad}(A) \Leftrightarrow \sigma'(xa) = \emptyset \quad \text{for every } x \in A, \tag{0.4}$$

or equivalently, for every $x \in A^{-1}$. We call the algebra A *semisimple* iff $\mathrm{Rad}(A) = \{0\}$, or equivalently, if

$$\#\sigma'(xa) = 0 \text{ for every } x \in A \Rightarrow a = 0, \tag{0.5}$$

and *semiprime* iff

$$aAa = \{0\} \Rightarrow a = 0; \tag{0.6}$$

since the left hand side of (0.6) implies that $a \in \mathrm{Rad}(A)$ it is clear that a semisimple algebra is always semiprime. We observe

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