Extension of operators from weak*-closed subspaces of $\ell_1$ into $C(K)$ spaces

by

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Abstract. It is proved that every operator from a weak*-closed subspace of $\ell_1$ into a space $C(K)$ of continuous functions on a compact Hausdorff space $K$ can be extended to an operator from $\ell_1$ to $C(K)$.

1. Introduction. This work is part of an effort to characterize those subspaces $E$ of a Banach space $X$ for which the pair $(E, X)$ has the following

**Extension Property** (E.P., in short). Every (bounded, linear) operator $T$ from $E$ into any $C(K)$ space $Y$ has an extension $\mathbf{T} : X \to Y$.

There is a quantitative version of the E.P.: for any $\lambda \geq 1$ we say that the pair $(E, X)$ has the $\lambda$-E.P. if for every $T : E \to Y$ there is an extension $\mathbf{T} : X \to Y$ with $\|\mathbf{T}\| \leq \lambda \|T\|$. It is easy to see that if $(E, X)$ has the E.P., then it has the $\lambda$-E.P. for some $\lambda$.

It is known [Zip] that for each $1 < p < \infty$ and every subspace $E$ of $\ell_p$, $(E, \ell_p)$ has the 1-E.P., while for $F \subseteq c_0$, $(F, c_0)$ has the $(1+\varepsilon)$-E.P. for every $\varepsilon > 0$ [LP]. However, there is a subspace $F$ of $c_0$ for which $(F, c_0)$ does not have the 1-E.P. [JZ2]. If $E$ itself is a $C(K)$ space then, clearly, $(E, X)$ has the E.P. if and only if $E$ is complemented in $X$. It follows from [Aml] that $C(K)$ has a subspace $E$ for which $(E, C(K))$ does not have the E.P. if $K$ is any compact metric space whose with derived set is nonempty (which is equivalent [BePe] to saying that $C(K)$ is not isomorphic to $c_0$).

Since every separable Banach space is a quotient of $\ell_1$, the following fact demonstrates the important role of the space $\ell_1$ in extension problems.

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Proposition 1.1. Let $E$ be a subspace of a Banach space $X$ and let $Q$ be an operator from $Z$ onto $X$ so that $\|Q\| = 1$ and $Q$ is bounded on $Z$. If $\langle Q^{-1}E, Z \rangle$ has the $\lambda$-E.P. then $(E, X)$ has the $\lambda/\delta$-E.P.

**Proof.** Let $T$ be an operator from $E$ into any $C(K)$ space $Y$. Consider the operator $S = TQ : Q^{-1}E \to Z$. If $S : Z \to Y$ extends $S$ then since $S$ vanishes on ker $Q$, $S$ induces an operator $\overline{S}$ from $X \sim Z/\ker Q$ into $Y$ so that $\overline{S}Q = S$ and $\|\overline{S}\| \leq \|S\|/\delta$.

An immediate consequence of Proposition 1.1 is that $\ell_1$ contains a subspace $F$ for which $(F, \ell_1)$ does not have the E.P. Indeed, if $E$ denotes an uncomplemented subspace of $C[0,1]$ which is isomorphic to $C[0,1]$ ([Am]) and if $\epsilon : \ell_1 \to C[0,1]$ is a quotient map and $F = Q^{-1}E$, then $(F, \ell_1)$ does not have the E.P. The main purpose of this paper is to prove the following.

**Theorem.** Let $\{X_n\}_{n=1}^{\infty}$ be finite-dimensional and let $E$ be a weakly-closed subspace of $X = (\sum X_n)_1$, regarded as the dual of $X_\ast = (\sum X_n^\ast)_\infty$. Then $(E, X)$ has the E.P. Moreover, if $E$ has the approximation property, then $(E, X)$ has the $(1 + \epsilon)$-E.P. for every $\epsilon > 0$.

**Remark.** Under the hypotheses of the Theorem, we do not know whether $(E, X)$ has the $(1 + \epsilon)$-E.P. for every $\epsilon > 0$ when $E$ fails the approximation property. The proof of Theorem yields only that $(E, X)$ has the $(3 + \epsilon)$-E.P. for all $\epsilon > 0$.

We know very little about the extension problem for general pairs $(E, X)$. However, the Theorem makes the following small contribution in the general case.

**Corollary 1.1.** Let $E$ be a subspace of the separable space $X$. Assume that there is a weakly-closed subspace $F$ of $\ell_1$ such that $X/E$ is isomorphic to $\ell_1/F$. Then $(E, X)$ has the E.P.

**Proof.** Let $Q : \ell_1 \to X$ and $S : X \to X/E$ be quotient maps. Theorem 2 of [LR] implies that there is an automorphism of $\ell_1$ which maps $Q^{-1}E = \ker(SQ)$ onto $F$. Since $(F, \ell_1)$ has the E.P. by our Theorem, so does the pair $(Q^{-1}E, \ell_1)$. It follows from Proposition 1.1 that $(E, X)$ has the E.P. 

We use standard Banach space theory notation and terminology, as may be found in [LT1], [LT2].

2. Preliminaries. Let $E$ be a subspace of $X$, $\lambda \geq 1$, and $0 < \epsilon < 1$. Given an operator $S : E \to Y$ we say that the operator $T : X \to Y$ is a $(\lambda, \epsilon)$-approximate extension of $S$ if $\|T\| \leq \lambda\|S\|$ and $\|S - T\|_E \leq \epsilon\|S\|$.

Our first observation is that the existence of approximate extensions implies the existence of extensions.

**Lemma 2.1.** Let $E$ be a subspace of $X$ and assume that each operator $S : E \to Y$ has a $(\lambda, \epsilon)$-approximate extension. Then the pair $(E, X)$ has the $\mu$-E.P. with $\mu \leq \lambda(1 - \epsilon)^{-1}$.

**Proof.** Put $S_1 = S$ and let $T_1$ be a $(\lambda, \epsilon)$-approximate extension of $S_1$. Then $\|T_1\| \leq \lambda\|S_1\| = \lambda\|S\|$ and $\|S_1 - T_1\|_E \leq \epsilon\|S\|$. Construct by induction sequences of operators $\{S_n\}_{n=1}^{\infty}$ from $E$ into $Y$ and $\{T_n\}_{n=1}^{\infty}$ from $X$ into $Y$ such that for each $n \geq 1$, $S_{n+1} = S_n - T_n\|E\|$ and $T_{n+1}$ is a $(\lambda, \epsilon)$-approximate extension of $S_{n+1}$. Then, by definition, $\|T_n\| \leq \lambda\|S_n\|$ and $\|S_{n+1}\| \leq \epsilon\|S_n\|$ for every $n \geq 1$. It follows that $\|S - \sum S_n\| \leq \epsilon\|S\|$ and $\|T_n\| \leq \lambda\|S_n\|/\epsilon$ for all $n \geq 1$. Hence the operator $T = \sum T_n$ extends $S$ and $\|T\| \leq \lambda(1 - \epsilon)^{-1}\|S\|$.

Given a finite-dimensional decomposition (FDD, in short) $\{Z_n\}_{n=1}^{\infty}$ of a space $Z$, we will be interested in subspaces of $Z$ with FDD's which are particularly well-positioned with respect to $\{Z_n\}_{n=1}^{\infty}$.

**Definition.** Let $F \subset Z$ and let $\{F_n\}_{n=1}^{\infty}$ be an FDD for $F$. We say that $\{F_n\}_{n=1}^{\infty}$ is alternately disjointly supported with respect to $\{Z_n\}_{n=1}^{\infty}$ if there exist integers $1 = k(1) < k(2) < \ldots$ such that for each $n \geq 1$, $F_n \subset Z_{k(n)} + Z_{k(n)+1} + \ldots + Z_{k(n+2)-1}$.

An important property of an alternately disjointly supported FDD is that if $(k(n))_{n=1}^{\infty}$ is any increasing sequence of integers and if we drop $\{F_{n(k)}\}_{k=1}^{\infty}$, then the remaining $F_n$'s can be grouped into blocks

$$\tilde{F}_j = \sum_{i=0}^{n(j)-1} F_i$$

which form an FDD that is disjointly supported on the $\{Z_n\}_{n=1}^{\infty}$; more precisely, with the above notation,

$$\tilde{F}_j \subset \sum_{m=k(n(j)+1)}^{k(n(j+1)+1)-1} Z_m$$

for all $j \geq 1$.

We will show that for certain subspaces of a dual space with an FDD, a given FDD can be replaced by one which is alternately disjointly supported.

We first need the following main tool:

**Proposition 2.1.** Let $\{X_n\}_{n=1}^{\infty}$ be a shrinking FDD for $X$, let $Q$ be a quotient mapping of $X$ onto $Y$ and suppose that $\{\tilde{E}_n\}_{n=1}^{\infty}$ is an FDD for $Y$. Then there are a blocking $\{E^\prime_n\}_{n=1}^{\infty}$ of $\{E_n\}_{n=1}^{\infty}$, an FDD $\{W_n\}_{n=1}^{\infty}$ of $X$ which is equivalent to $\{X_n\}_{n=1}^{\infty}$, and $1 = k(1) < k(2) < \ldots$ so that for each
n and each $k(n) ≤ j < k(n+1)$, $QW_j \subset E'_n + E_n$. Moreover, given $ε > 0$, $\{E'_n\}_{n=1}^{∞}$ and $\{W_n\}_{n=1}^{∞}$ can be chosen so that there is an automorphism $T$ on $X$ with $||I - T|| < ε$ and $TX_n = W_n$ for all $n$.

Proof. In order to avoid complicated notation we shall prove the statement for the case where, for every $n ≥ 1$, $X_n$ (and hence also $W_n$) is one-dimensional. The same arguments, with only obvious modifications, yield the FDD case. (Actually, in the proof of the Theorem, only the basis case of Proposition 2.1. is needed. Indeed, in Step 3 of the proof of the Theorem, one can replace $E$ by $E_1 = \{E_n\} (\bigcup \{G_n\})$ and $X$ by $X_1 = \{E_n\} (\bigcup \{G_n\})$, where $\{G_n\}_{n=1}^{∞}$ is a sequence which is dense in the sense of the Banach–Mazur distance in the set of all finite-dimensional spaces, and use the fact [JZ2, [Fe]] that $E_1$ has a basis. In fact, this trick is used in a different way for the proof of the “covering” statement in the Theorem.)

So assume that $X$ has a normalized shrinking basis $\{x_n\}_{n=1}^{∞}$ with biorthogonal functionals $\{x_n^*\}_{n=1}^{∞}$, we are looking for an equivalent basis $\{u_n\}_{n=1}^{∞}$ of $X$ for which the statement holds. First we perturb the basis for $X$ to get another basis whose images under $Q$ are supported on finitely many of the $E_n$'s. This step whose hypothesis that $\{x_n^*\}_{n=1}^{∞}$ be shrinking.

For each $n ≥ 1$ let $Q_n$ be the FDD's natural projection from $Y$ onto $E_1 + \ldots + E_n$. Let $1 > ε > 0$ and set $C = \sup_n ||f_n||$. Choose $p_1 < p_2 < \ldots$ so that for each $n$, $||Qx_n - \sum_{p_1}^{p_2} Qx_n|| < εC^{-1}2^{-n}$. Since $Q$ is a quotient mapping, there is for each $n$ a vector $z_n$ in $X$ with $||z_n|| < εC^{-1}2^{-n}$ and $Qx_n = z_n - \sum_{p_1}^{p_2} Qx_n$. Let $y_n = z_n - x_n$, so that $Qy_n = y_n - \sum_{p_1}^{p_2} y_n$. It is standard to check that $\{y_n\}_{n=1}^{∞}$ is equivalent to $\{x_n\}_{n=1}^{∞}$. Indeed, define an operator $S$ on $X$ by $Sx = \sum_{p_1}^{∞} f_n(x_n)$. Then $||S|| < ε$ and $Sx = x$, so $S$ is an isomorphism from $X$ onto $X$ which maps $x_n$ to $y_n$.

Define a block $\{E_n\}_{n=1}^{∞}$ of $\{E_n\}_{n=1}^{∞}$ by $E_n = E_n + E_{p_1-1} + \ldots + E_{p_2}$ (where $p_0 = 0$). Then for each $n$, $Qy_n$ is in $E_1 + \ldots + E_n$.

Let $Q_0$ be the basis projection from $Y$ onto $E_1 + \ldots + E_n$, $P_0$ the basis projection from $X$ onto $\{y_1, \ldots, y_n\}$, and set $C_1 = \sup_n ||P_n||$. Since $\{y_n\}_{n=1}^{∞}$ is shrinking, $\lim_{n→∞} ||Q_0Q(I - P_0)|| = 0$ for each $n$. Since $Q$ is a quotient mapping, for each $n$ there exists a mapping $T_n$ from $E_1 + \ldots + E_n$ into $X$ so that $QT_n$ is the identity on $E_1 + \ldots + E_n$. Set $M_n = T_n[I - E_1 + \ldots + E_n]$, let $1 > ε > 0$, and recursively choose $0 = k(0) < k(1) < k(2) < \ldots$ so that for each $n$, $||Q(k(n))Q(I - P(k(n+1)-1))|| < (2C_1 M(k(n))/2^{-n}$. Setting $w_j = y_j - T(k(n))Q(k(n+1) Qy_j$ for $k(n+1) ≤ j < k(n+2)$, we see that $Qw_j$ is in $E_1 + \ldots + E_n$ when $k(n+1) ≤ j < k(n+2)$.

The desired block of $\{E_n\}_{n=1}^{∞}$ is defined by $E'_n = E'_n + E_{k(n)} + 1 + \ldots + E_{k(n)}$ but it remains to be seen that $\{u_n\}_{n=1}^{∞}$ is a suitably small perturbation of $\{y_n\}_{n=1}^{∞}$. The inequality $||Q(k(n))Q(I - P(k(n)-1))|| < (2C_1 M(k(n))/2^{-n})$ implies, by composing on the right with $P(k(n+1)-1)$, that $||Q(k(n))Q(P(k(n+1)-1) - P(k(n) - 1))|| < (2M(k(n)))/2^{-n}$. Thus if we define an operator $V$ on $X$ by $V = \sum_{n=1}^{∞} T(k(n))Q(k(n)Q(P(k(n)-1) - P(k(n) - 1))$, we see that $||V|| < ε$ and hence $T = I - V$ is invertible. But for $k(n+1) ≤ j < k(n+2)$, $V_y_j = T(k(n))Q(k(n) Qz_j$; that is, $TV_y_j = w_j$.

Using a duality argument we get from Proposition 2.1 the following.

**Corollary 2.1.** Let $\{Z_n\}_{n=1}^{∞}$ be an $ε_2$-FDD for a space $Z$. Regard $Z$ as the dual of the space $Z_∞ = (\bigcup Z_n)_{∞}$ and let $F$ be a weak*-closed subspace of $Z$ with an FDD. Then $Z$ and $F$ have $ε_2$-FDD's $\{V_n\}_{n=1}^{∞}$ and $\{V_n\}_{n=1}^{∞}$, respectively, so that $\{V_n\}_{n=1}^{∞}$ is alternately disjointly supported with respect to $\{V_n\}_{n=1}^{∞}$. Moreover, given $ε > 0$, $\{V_n\}_{n=1}^{∞}$ can be chosen so that for some block $\{Z_n\}_{n=1}^{∞}$ of $\{Z_n\}_{n=1}^{∞}$, there is an automorphism $T$ of $Z_n$ with $||I - T|| < ε$ and $TV_n = W_n$ for all $n ≥ 1$.

**Proof.** Being weak*-closed, $F$ has a predual $F_*$ that is a quotient space of $Z_n$. By [JZ1, F_2] has a shrinking FDD and consequently, by Theorem 1 of [JZ1], $F_2$ has a shrinking c0-FDD $\{E_n\}_{n=1}^{∞}$. Let $Q: Z_n → F_*$ be the quotient mapping. By Proposition 2.1 there are a block $\{E'_n\}_{n=1}^{∞}$ of $\{E_n\}_{n=1}^{∞}$, an FDD $\{W_n\}_{n=1}^{∞}$ of $Z_n$ which is equivalent to $\{Z_n\}_{n=1}^{∞}$, even the image of $\{W_n\}_{n=1}^{∞}$ under some automorphism on $Z_n$ is arbitrarily close to $I_{Z_n}$ and $ε_2 = 1 < k(2) < \ldots$ so that for each $n$ and $k(n) ≤ j ≤ k(n) + 1$, $W_j ⊂ E'_n + E_{n+1}$. The equivalence implies that $\{W_n\}_{n=1}^{∞}$ is a c0-FDD and being a block of a c0-FDD, $\{E'_n\}_{n=1}^{∞}$ is a c0-FDD. Let $\{V_n\}_{n=1}^{∞}$ (resp. $\{U_n\}_{n=1}^{∞}$) be the dual FDD of $\{W_n\}_{n=1}^{∞}$ (resp. $\{F_n\}_{n=1}^{∞}$) for $Z$ (resp. $F$). Then $\{V_n\}_{n=1}^{∞}$ is an $ε_2$-FDD for $Z$ and $\{U_n\}_{n=1}^{∞}$ is an $ε_2$-FDD for $F$. Moreover, suppose that $u$ is in $U_n$ and $w$ is in $W_j$, where either $j < k(n)$ or $j ≥ k(n) + 1$. Then either $m < n$ or $m > n + 1$ hence $n ≠ m$ and $n ≠ m + 1$. Then $Qw_j ∈ E'_n + E_{n+1}$, hence $w(j) = 0$. This proves that $U_n$ is supported on $\sum_{j=k(n)+1}^{∞} V_j$.

3. **Proof of the Theorem.** The proof consists of four parts, the first three of which are essentially simple special cases of the Theorem.

**Step 1.** $E$ has an FDD $\{E_n\}_{n=1}^{∞}$ with $E_n ⊂ X_n$ for all $n$.

**Proof.** Let $Y = C(K)$ and let $S: E → Y$ be any operator. Using the $L_1(K)$-property of $Y$ (see Theorem 6.1 of [Lin]), one sees that the finite rank operator $S|_{E_n}$ has an extension $S_{n|}_{X_n}: X_n → Y$ with $||S_n|| ≤ (1 + ε)||S||$. Define the extension of $S$ by $S(\sum_{n=1}^{∞} x_n) = \sum_{n=1}^{∞} S_n x_n$. Since $\{X_n\}_{n=1}^{∞}$ is an exact $ε_2$-decomposition, it follows that $||S|| ≤ (1 + ε)||S||$.

**Step 2.** $E$ has an $ε_2$-FDD $\{E_n\}_{n=1}^{∞}$ which is alternately disjointly supported with respect to $\{X_n\}_{n=1}^{∞}$.
Proof. Given \(\delta > 0\), let \(1 \leq (1 + \varepsilon)(1 - \varepsilon)^{-1} < 1 + \delta\) and choose an integer \(N > (2 + \varepsilon)M\varepsilon^{-1}\) where \(M\) is the constant of the \(\ell_2\)-FDD \(\{E_n\}_{n=1}^\infty\), that is, the constant of equivalence of \(\{E_n\}_{n=1}^\infty\) to the natural \(\ell_2\)-FDD for \(\bigoplus E_n\). Let \(Y = C(K)\) and let \(S : E \to Y\) be an operator with \(\|S\| = 1\). For each \(1 \leq j \leq N\) let

\[
Z_j = \text{span}\{E_i : i \neq kN + j, k = 0, 1, 2, \ldots\}.
\]

Each subspace \(Z_j\) has a natural \(\ell_2\)-FDD which is disjointly supported with respect to \(\{X_n\}_{n=1}^\infty\) because \(\{E_n\}_{n=1}^\infty\) is alternately disjointly supported with respect to \(\{X_n\}_{n=1}^\infty\). By Step 1, \(S|Z_j\) has an extension \(T_j : X \to Y\) with

\[
\|T_j\| \leq (1 + \varepsilon)\|S_j\| \leq (1 + \varepsilon)\|S\| = 1 + \varepsilon.
\]

Define \(T : Z \to Y\) by \(T = N^{-1} \sum_{j=1}^N T_j\). Then \(\|T\| \leq (1 + \varepsilon)\|S\| = 1 + \varepsilon\). Moreover, if \(e \in E_i\) and \(i = kN + h\) for some \(1 \leq h \leq N\), then \(T_je = S_je = Se\) for all \(j \neq h\), hence \(T\) is “almost” an extension of \(S\). Indeed, \(\|T_je - Se\| = N^{-1}\|z_i_0 - S_je\| \leq (2 + \varepsilon)N^{-1}\|e\|\) whenever \(e \in E_i\) for some \(i\).

Recalling that the \(\ell_1\)-FDD \(\{E_n\}_{n=1}^\infty\) has constant \(M\), we have

\[
\|T\| \leq M\sup_n \|T|E_n\| - S|E_n\| \leq M(2 + \varepsilon)N^{-1} < \varepsilon.
\]

This proves that \(T\) is a \((1 + \varepsilon, \varepsilon)\)-approximate extension of \(S\) and therefore, by Lemma 2.1, \((E, Z)\) has the \((1 + \varepsilon)(1 - \varepsilon)^{-1}\)-E.P.

Step 3. \(E\) has an FDD.

Proof. By Corollary 2.1 we see that \(X\) and \(E\) have \(\ell_1\)-FDD’s \(\{Z_n\}_{n=1}^\infty\) and \(\{E_n\}_{n=1}^\infty\), respectively, where \(\{E_n\}_{n=1}^\infty\) is alternately disjointly supported with respect to \(\{Z_n\}_{n=1}^\infty\), and, by Remark 2.1, \(\{Z_n\}_{n=1}^\infty\) has constant of equivalence to \(\{\sum Z_n\}\) arbitrarily close to one. Hence, by Step 2, \((E, X)\) has the \((1 + \delta)\)-E.P. for every \(\delta > 0\).

This gives the “moreover” statement when \(E\) has an FDD. When \(E\) just has the approximation property, we enlarge \(X\) to \(X_1 = X \oplus C_1\), where \(C_1 = (\sum G_n)_{n=1}^\infty\) and \(\{G_n\}_{n=1}^\infty\) is a sequence of finite-dimensional spaces which is dense (in the sense of the Banach-Mazur distance) in the set of all finite-dimensional spaces; and we enlarge \(E\) to \(E_1 = E \oplus C_1\). Again, \(X_1\) is an exact \(\ell_1\)-sum of finite-dimensional spaces and \(E_1\) is weak* closed in \(X_1\). Moreover, since \(E\) is a separable dual space which has the approximation property, \(E\) has the metric approximation property [LT1], and hence by [Jo], \(E_1\) is a \(\pi\)-space, whence, since \(E_1\) is a dual space, \(E_1\) has an FDD by [JRZ]. Thus by Step 3, \((E_1, X_1)\) has the \((1 + \delta)\)-E.P. for each \(\delta > 0\), and, therefore, so does \((E, X)\).

Step 4. The general case.

We start with a lemma.

\textbf{Lemma 3.1.} Let \(Z\) be a Banach space and let \(E\) be a subspace of \(Z\). Suppose that \(E\) has a subspace \(F\) such that \((E, Z)\) has the \(\lambda\)-E.P. and \((E/F, Z/F)\) has the \(\mu\)-E.P. Then \((E, Z)\) has the \((\lambda + \mu(1 + \lambda))\)-E.P.

Proof. Let \(Y = C(K)\) and let \(S : E \to Y\) be any operator. Let \(S_1 : Z \to Y\) be an extension of \(S|F\) with \(\|S_1\| \leq \lambda\|S\|\). The operator \(W = S - S_1|E\) from \(E\) into \(Y\) vanishes on \(F\) and so induces an operator \(\tilde{W} : E/F \to Y\) in the usual way, and \(\|\tilde{W}\| = \|W\| \leq \|S\| + \|S_1\| \leq (1 + \lambda)\|S\|\). By our assumptions, \(\tilde{W}\) extends to an operator \(W_1 : Z/F \to Y\) with \(\|W_1\| \leq \mu\|\tilde{W}\| \leq \mu(1 + \lambda)\|S\|\). Let \(Q : Z \to Z/F\) denote the quotient map. Then \(\tilde{T} = S_1 + W_1Q\) is the desired extension of \(S\). Indeed, for every \(e \in E\),

\[
Te = S_1e + W_1Qe = S_1e + W_1e = S_1e + (S - S_1)e = Se
\]

and \(\|Te\| \leq \|S_1\| + \|W_1\| \leq (\lambda + \mu(1 + \lambda))\|S\|\).

Let us now return to the proof of the general case. Being a weak*-closed subspace of \(\ell_1\), \(E\) is the dual of the quotient space \(E_0 = (\sum X_n)_{01}/E_1\). Our main tool in this part of the proof is Theorem IV.4 of [JR] and its proof. This theorem states that \(E_0\) has a subspace \(V\) so that both \(V\) and \(E_0/V\) have shrinking FDD’s. Under these circumstances, Theorem 1 of [Z1] implies that both \(V\) and \(E_0/V\) have \(c_0\)-FDD’s. In order to prove our theorem it suffices, in view of Lemma 3.1, to show that both pairs \((V^*+, X)\) and \((E_0/V^*, X/V^*)\) have the E.P. Now, \((V^*+, X)\) has the \((1 + \delta)\)-E.P. for all \(\delta > 0\) by Step 3, so it remains to discuss the pair \((E_0/V^*, X/V^*)\). This discussion requires some preparation and some minor modification in the proof of Theorem IV.4 of [JR]. We first need a known perturbation lemma:

\textbf{Lemma 3.2.} Suppose \(E, F\) are subspaces of \(X^*\) with \(F\) norm dense in \(X^*\) and \(X^*\) is separable. Then for each \(\varepsilon > 0\) there is an automorphism \(T\) on \(X\) so that \(\|T - I\| < \varepsilon\) and \(T^*E \cap F\) is norm dense in \(T^*E\).

Proof. Let \((x_n, x_n^*)\) be a biorthogonal sequence in \(X \times E\) with \(\text{span} x_n^* = E\) (see, e.g., [Mac]) and take \(y_n^* \in F\) so that \(\sum \|x_n^* - y_n^*\|/\|x_n\| < \varepsilon\). Define \(T : X \to X\) by

\[
Tx = x - \sum_{n=1}^\infty (x_n^* - y_n^*)x_n.
\]

Returning to the proof of the Theorem, we may assume, in view of Lemma 3.2, that \(E \cap \text{span} \{x_n\}_{n=1}^\infty X_n\) is norm dense in \(E\). The standard back-and-forth technique [Mac] for producing biorthogonal sequences yields a biorthogonal sequence \(\{x_n, x_n^*\}_{n=1}^\infty \subset X \times E\) with \(\text{span} \{Qx_n\}_{n=1}^\infty = \text{span} \{x_n\}_{n=1}^\infty QX^*\), \(\text{span} \{x_n^*\}_{n=1}^\infty = E \cap \text{span} \{x_n\}_{n=1}^\infty X_n\), and where \(Q\) is the quotient mapping from the predual \(X_0 = (\sum X_n)_{01}\) of \(X\) onto the predual \(E_0\) of \(E\). This means that for any \(N, x_j^*\) in \(\text{span} \{x_n\}_{n=1}^\infty X_n\) if \(j\) is sufficiently large.
We now refer to the construction in Theorem IV.4 of [JR] and the finite sets \( \Delta_1 \subseteq \Delta_2 \subseteq \ldots \) of natural numbers defined there. From this construction, it is clear that, having defined \( \Delta_n \), the smallest element, \( k(n) \), in \( \Delta_{n+1} \setminus \Delta_n \) can be as large as we desire. In particular, if \( \{x^*_j\}_{j=1}^{\infty} \) is a subset of \( \text{span} \left( \bigcup_{n=1}^{m(n)} X_i \right) \), then we choose \( k(n) \) large enough so that for \( j \geq k(n) \), \( x^*_j \) is in \( \text{span} \left( \bigcup_{n=1}^{m(n)+1} X_i \right) \). Thus setting

\[
Z_n = \text{span} \{ x^*_j : j \in \Delta_n \setminus \Delta_{n+1} \}
\]

(where \( \Delta_0 = 0 \)), we infer that \( \{Z_n\}_{n=1}^{\infty} \) is disjointly supported relative to \( \{X_n\}_{n=1}^{\infty} \). In the notation above and setting \( m(0) = 0 \), we have, for each \( n \),

\[
Z_n \subseteq \text{span} \{ x^*_j : j \in \bigcup_{m(n-1)+1}^{m(n)} \}.
\]

The subspace \( V \) of \( E_\ast \) is defined to be the annihilator of \( \{x^*_j : j \in \bigcup_{n=1}^{\infty} \Delta_n \} \) and, as mentioned earlier, it follows from [JR] and [JZ1] that \( V \) has a \( \epsilon \)-FDD and thus \( V^\perp = E/V^\perp \) has an \( \epsilon \)-FDD. It is also proved in [JR], but is obvious from the “extra” we have added here, that \( \text{span} \{Z_j\}_{j=1}^{\infty} \) is weak*-closed and hence equals \( V^\perp \). It is also obvious from (5) that \( X/V^\perp \) has an \( \epsilon \)-FDD. Therefore, by Step 3, \( (E_\ast/V^\perp, X/V^\perp) \) has the E.P. \( \ast \)

The Extension Property is concerned with extension of operators into \( C(K) \)-spaces. However, in the proof of the theorem, the only place where the fact was used that the range of the mapping is a \( C(K) \)-space was in Step 1, where we needed to extend an operator from a finite-dimensional subspace. This uses only the \( L_\infty \)-property of \( C(K) \)-spaces, so we can state a formally stronger version of the Theorem:

**Corollary 4.1.** Let \( \{X_n\}_{n=1}^{\infty} \) be finite-dimensional and let \( E \) be a weak*-closed subspace of \( X = \bigcup_{n=0}^{\infty} X_n \), regarded as the dual of \( X_n = \bigcup_{m=0}^{n} X_n m \). Let \( T \) be an operator from \( E \) into a space \( Y \). Then there is an extension of \( T \) to an operator \( T \) from \( X \) into \( Y \). Moreover, if \( E \) has the approximation property, then for any \( \epsilon > 0 \), \( T \) can be chosen so that \( \|T\| \leq (\lambda + \epsilon)\|T\| \).

4. Concluding remarks and problems. Very little is known about the Extension Property, so there is no shortage of problems.

**Problem 4.1.** If \( E \) is a subspace of \( X \), and \( X \) is reflexive, does \( (E, X) \) have the E.P.? What if \( X \) is superreflexive? What if \( X \) is \( L_p \), \( 1 < p \neq 2 < \infty \)?

**Problem 4.2.** If \( E \) is a reflexive subspace of the separable space \( X \), does \( (E, X) \) have the E.P.? What if \( E \) is just isomorphic to a conjugate space? In the latter case, what if, in addition, \( X \) is \( \ell_2 \)?

In Problem 4.2 it is necessary to restrict attention to separable \( X \) to avoid known counterexamples. (If \( E \) is an infinite-dimensional reflexive subspace of \( E_\infty \), then no isomorphism from \( E \) into \( C[0, 1] \) can extend to an operator from \( E_\infty \) into \( C[0, 1] \).)

If \( E \) is a subspace of \( X \), then \( (E, X) \) has the \( (1+\varepsilon) \)-E.P. for every \( \varepsilon > 0 \) [LP] but need not have the \( 1 \)-E.P. [JZ1]. We do not know if this phenomenon can occur in the setting of “nice” spaces:

**Problem 4.3.** If \( X \) is a reflexive smooth space and \( (E, X) \) has the \( (1+\varepsilon) \)-E.P. for every \( \varepsilon > 0 \), does \( (E, X) \) have the \( 1 \)-E.P.?

The following observation gives an affirmative answer to Problem 4.3 in a special case.

**Proposition 4.1.** If \( X \) is uniformly smooth and \( (E, X) \) has the \( (1+\varepsilon) \)-E.P. for every \( \varepsilon > 0 \), then \( (E, X) \) has the \( 1 \)-E.P.

**Proof.** In preparation for the proof, we recall Proposition 2 of [Z], which says:

\[ (E, X) \] has the \( \lambda \)-E.P., if and only if there exists a weak*-continuous extension mapping from \( B(E^*) \) to \( \lambda B(X^*) \), that is, a continuous mapping \( \phi : (B(E^*), \text{weak}^*) \to (\lambda B(X^*), \text{weak}^*) \) for which \( (\phi x) \in E^* \) for every \( x^* \in B(E^*) \).

Since \( X \) is uniformly smooth, given \( \varepsilon > 0 \) there exists \( \delta > 0 \) so that if \( y^* \), \( y^* \) in \( X^* \) and \( x \) in \( Y \) satisfy \( \|y^* - y^*\| = 1 = \|x^* - x^*\| = \|y^* - y^*\| \), then \( \|y^* - y^*\| < \varepsilon \). Letting \( \phi \), \( B(E^*), \to (1 + \varepsilon)^{-1} B(X^*) \) be a weakly continuous extension mapping and letting \( f : (B(E^*), |f|) \) be the (uniquely defined, by smoothness) Hahn-Banach extension mapping, we conclude that

\[ \lim_{n \to \infty} \text{sup} \{ \|\phi_n(x^*) - f(x^*)\| : x^* \in B(E^*) \} = 0. \]

That is, \( \{\phi_n\}_{n=1}^{\infty} \) is uniformly convergent to \( f \in B(E^*) \). Since each \( \phi_n \) is weakly continuous, so is \( f \).

If \( E \) is finite-dimensional, then clearly the positively homogeneous extension of \( f \) to a mapping from \( B(E^*) \) to \( B(X^*) \) is a weakly continuous extension mapping. So assume that \( E \) has infinite dimension. But then \( B(E^*) \) is weakly dense in \( B(E^*) \), so by the weak continuity of the \( \phi_n \)’s and the weak lower semicontinuity of the norm, we have

\[ \sup \{ \|\phi_n(x^*) - \phi_m(x^*)\| : x^* \in B(E^*) \} = \sup \{ \|\phi_n(x^*) - \phi_m(x^*)\| : x^* \in B(E^*) \}, \]

which we saw tends to zero as \( n, \) \( m \) tend to infinity. That is, \( \{\phi_n\}_{n=1}^{\infty} \) is a uniformly Cauchy sequence of weakly continuous functions and hence its limit is also weakly continuous. \( \ast \)
It is apparent from the proof of Proposition 4.1 that the 1-E.P. is fairly easy to study in a smooth reflexive space $X$ because every extension mapping from Ball $E^*$ to Ball $X^*$ is, on the unit sphere of $E^*$, the unique Hahn-Banach extension mapping. Let us examine this situation a bit more in the general case. Suppose $E$ is a subspace of $X$ and let $A(E)$ be the collection of all norm one functionals in $E^*$ which attain their norm at a point of Ball $E$. The Bishop-Phelps theorem [BP], [Die] says that $A(E)$ is norm dense in Sphere $E^*$, hence, if $E$ has infinite dimension, $A(E)$ is weak*-dense in Ball $E^*$. Therefore $(E, X)$ has the 1-E.P. if and only if there is a weak*-continuous Hahn-Banach selection mapping $\phi : A(E) \to \text{Ball } X^*$ which has a weak*-continuous extension to a mapping $\phi$ from $A(E)^w = \text{Ball } E^*$ to $\text{Ball } X^*$, since clearly $\phi$ will then be an extension mapping. The existence of $\phi$ is equivalent to saying that whenever $\{x^*_n\}$ is a net in $A(E)$ which weak* converges in $E^*$, then $\{\phi x^*_n\}$ weak* converges in $X^*$ (see, for example, [Bou, I.8.5]). Now, when $X$ is smooth, there is only one mapping $\phi$ to consider, and in this case the above discussion yields the next proposition when dim $E = \infty$ (when dim $E < \infty$ one extends from Sphere $E^* = A(E)^w$ to Ball $E^*$ by homogeneity).

**Proposition 4.2.** Let $E$ be a subspace of the smooth space $X$. The pair $(E, X)$ fails the 1-E.P. if and only if there are nets $\{x^*_n\}$, $\{y^*_n\}$ of functionals in Sphere $X^*$ which attain their norm at points of Sphere $E$ and which weak* converge to distinct points $x^*$ and $y^*$, respectively, which satisfy $x^*|_{B(0, 1)} = y^*|_{B(0, 1)}$.

An immediate, but surprising to us, corollary to Proposition 4.2 is:

**Corollary 4.1.** Let $E$ be a subspace of the smooth space $X$. If the pair $(E, X)$ fails the 1-E.P., then there is a subspace $F$ of $X$ of codimension one which contains $E$ so that $(F, X)$ fails the 1-E.P.

**Proof.** Get $x^*, y^*$ from Proposition 4.2 and set $F = \text{span } E \cup (\ker x^* \cap \ker y^*).$ ■

**Problem 4.4.** Is Corollary 4.1 true for a general space $X$?

**Corollary 4.2.** For $1 < p \neq 2 < \infty$, $L_p$ has a subspace $E$ for which $(E, L_p)$ fails the 1-E.P.

**Proof.** We regard $L_p$ as $L_p(0, 2)$ and make the identifications $L_p^* = L_q(0, 2)$, where $q = p/(p - 1)$ is the conjugate index to $p$. Let

$$f = 1_{[0, 1/2]} - 1_{[1/2, 1]}, \quad g = -2 \cdot 1_{[1/2, 1]} - 1_{[1, 2]},$$

regarded as elements of $L_q$, and define

$$E = \{ f - g \} \subset \{ x \in L_p(0, 2) : \int_0^2 x = 0 \}.$$

Notice that $|f|^{q-1} \text{sign } f$ is in $E$, which implies that $1 = \|f\|_q = \|f\|_{L_p^*} = \|f\|_{L_q}$. So $f$ and $g$ induce the same linear functional on $E$ (we write $f|_E = g|_E$), and $f$ is the unique Hahn-Banach extension of this functional to a functional in $L_p^* = L_q$.

**Claim.** There exists $h$ in $L_q$ supported on $[0, 1/2]$ so that $\int_0^2 h = 0 = \int_0^2 |g + h|^{q-1} \text{sign } (g + h)$.

Assume the claim. Set $\lambda = \|g + h\|_q$ and let $\{h_n\}_{n=1}^{\infty}$ be a sequence of functions which have the same distribution as $h$, are supported on $[0, 1/2]$, and are probabilistically independent as random variables on $[0, 1/2]$ with normalized Lebesgue measure. Then $g_n \equiv \lambda^{-1}(g + h_n)$ defines a sequence on the unit sphere of $L_q(0, 2)$ which converges weakly to $\lambda^{-1} g$. Moreover, $|g_n|^{q-1} \text{sign } g_n$ is in $E$, which means that as a linear functional on $L_p$, $g_n$ attains its norm at a point on the unit sphere of $E$. In view of Proposition 4.2, to complete the proof it suffices to find a sequence $\{f_n\}_{n=1}^{\infty}$ on the unit sphere of $L_q$ which converges weakly in $L_q$ to $\lambda^{-1} f$ so that $|f_n|^{q-1} \text{sign } f_n$ is in $E$. This is easy: take $w$ supported on $[1/2]$ so that

$$\int_0^2 w = 0 = \int_0^2 |w|^{q-1} \text{sign } w = \left( \int_0^2 |f + w|^{q-1} \text{sign } (f + w) \right).$$

and $\|f + w\|_q = 1 + 1 = \|w\|_q = \lambda^q$ (so $w$ can be a multiple of $1_{[1/2, 1] - 1_{[1/3, 2/3]}}$). Let $\{w_n\}_{n=1}^{\infty}$ be a sequence of functions which have the same distribution as $w$, are supported on $[1, 2]$, and are probabilistically independent as random variables on $[1, 2]$. Now set $f_n = \lambda^{-1}(f + w_n)$.

We turn to the proof of the claim. Fix any $0 < \varepsilon < 1/4$. For appropriate $d$, the choice

$$h = d(4\varepsilon^2 L(0, 1/4) - 1_{(1/2 - \varepsilon, 1/2)})$$

works. Indeed, $\int_0^2 h = 0$ no matter what $d$ is, and $g h = 0$, so we need choose $d$ to satisfy

$$\int_0^1 |g|^{q-1} \text{sign } g = \int_0^1 |h|^{q-1} \text{sign } h.$$

The left side of $(\ast)$ is $2^{q-1} + 1 > 0$, while the right side is $|d|^{q-1} \text{sign } d 2^{q-1} \times \left[(1/4)^{q-1} - (e^{2\varepsilon} - 2^{q-1}) \right]$, so such a choice of $d$ is possible for $p \neq 2$. ■

**Problem 4.5.** If $E$ is a weak*-closed subspace of $\ell_1$, does $(E, \ell_1)$ have the $(1 + \varepsilon)$-E.P. for every $\varepsilon > 0$?

A negative answer to Problem 4.5 would be particularly interesting, because it would justify the weird approach we used to prove the Theorem.
Samet [Sam1], [Sam2] proved that if \( E \) is a finite-dimensional subspace of \( \ell_1 \) then \((E, \ell_1)\) has the 1-E.P. Our final proposition shows that for most weak*-closed hyperplanes \( E \) in \( \ell_1 \), \((E, \ell_1)\) does not have the 1-E.P.

**Proposition 4.3.** Let \( x = \{a_n\}_{n=1}^\infty \) be a norm one vector in \( c_0 \) with \( a_n \neq 0 \) for infinitely many \( n \), and let \( E = x^\perp \) in \( \ell_1 = c_0^\perp \). Then \((E, \ell_1)\) does not have the 1-E.P.

**Proof.** By using an onto isometry of \( c_0 \), we can assume, without loss of generality, that \( a_1 = 1 \) and \( a_{2n-1} \) is positive for each \( n \). Assuming for contradiction that \((E, \ell_1)\) has the 1-E.P., we get from Proposition 2 of [Zip] a weak*-continuous extension mapping \( \phi \) from Ball \( E^* \) to Ball \( \ell_\infty = \ell_1^\perp \). For \( n = 1, 2, \ldots \), define a vector \( x(n) \) in \( c_0 \) by having the first \( 2n \) coordinates agree with those of \( x \), the \((2n+1)\)th coordinate be minus one, and other coordinates be zero. Regarding the \( x(n) \)'s as linear functionals on \( \ell_1 \), we have \( x(n)|E| = 1 \) and \( x(n)|E^*| \to 0 \) weak* in \( E^* \). We can write \( \phi(x(n)|E|) = x(n) + b_n x \) and \( \phi(-x(n)|E|) = -x(n) + c_n x \); by weak* continuity of \( \phi \), these two sequences must converge weak* in \( \ell_1^\perp \) to the same functional, namely, to \( \phi(0) \). Since \( \phi \) maps into the unit ball, \( |1 + b_n|, |1 + c_n|, \) and \( |b_n|, |c_n| \) are all at most one. Hence since \( a_{2n+1} > 0, b_n = c_n = 0 \). So \( \phi(x(n)|E|) = x(n) \to x \) and \( \phi(-x(n)|E|) = -x(n) \to -x \) weak* in \( \ell_\infty \), which is a contradiction. \( \blacksquare \)

References


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[JZ2] Extension of operators from subspaces of \( c_0(\gamma) \) into \( C(K) \) spaces, Proc. Amer. Math. Soc. 107 (1989), 751-754.