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Vector-valued Choquet–Deny theorem, renewal equation and self-similar measures

by

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Abstract. The Choquet–Deny theorem and Deny's theorem are extended to the vector-valued case. They are applied to give a simple nonprobabilistic proof of the vector-valued renewal theorem, which is used to study the L^p-dimension, the L^p-density and the Fourier transformation of vector-valued self-similar measures. The results answer some questions raised by Strichartz.

1. Introduction. A *self-similar measure* ν is an invariant measure defined by the equation

$$\nu = \sum_{i=1}^{m} w_i \nu \circ S_i^{-1},$$

where the w_i's are probability weights and the S_i's are contractive similitudes (see [H]). In deriving the L^2-dimension and estimating the L^2-density of such measures, two of the present authors reduced the above invariance to the well known *renewal equation* on $[0,\infty)$ (see [Fe]):

$$f(x) = f * \mu(x) + h(x), \quad x \ge 0, \tag{1.1}$$

where f is a bounded continuous function, μ is a probability measure and h is an "error term" (see [L1], [LW], where the L^2-density and the L^2-dimension are called the *mean quadratic variation* (m.q.v.) and the m.q.v. index instead). Moreover, by combining a Tauberian theorem and the solution of the renewal equation, they showed that certain quadratic averages (which depend on the L^2-dimension) of the Fourier transformation of the above self-similar measures ν are asymptotically multiplicatively periodic. This property was first discovered by Strichartz [Str1, 2]. The renewal equation

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was also used by Lalley [La] to calculate the packing dimension of some self-similar fractals in a different setting.

More recently, Strichartz [Str3] made use of the directed multigraph ([CM], [EM]) set up and introduced *vector-valued* self-similar and self-conformal measures. In the calculation of the L^p-dimension and L^p-density he left open the question of asymptotic behavior of such measures and of their Fourier transformation because the notions of solution of the vector-valued renewal equation and of its asymptotic behavior have yet to be formulated.

In [Fe] the scalar-valued renewal equation (1.1) is studied through the homogeneous convolution equation

$$f(x) = f * \mu(x), \quad x \in \mathbb{R}.$$

The equation has been investigated for a long time and different methods have been used. Schwartz ([Sch], [K]) first characterized the continuous (not necessarily bounded) solutions f, where μ is assumed to have compact support (such f are called *mean-periodic functions* on $\mathbb{R}$); Choquet and Deny [CD] replaced $\mathbb{R}$ by a locally compact abelian group and assumed f to be bounded and μ to be a probability measure; Deny [D] further extended this to the case where both f and μ are positive (including the unbounded case); Fürstenberg [Fu] and Benyamini and Weit [BW] considered the Choquet–Deny type of restriction on Lie groups. Also in the 80's there was a series of papers that extended Deny's theorem to semigroups (e.g. $\mathbb{R}^+$ and sub-semigroups in $\mathbb{R}^{n+}$) and the results were used to study various characterization problems in statistics (DS], [LR], [LZ], [RS]; cf. [RL] for complete references).

The scalar-valued Choquet–Deny theorem and Deny's theorem can be stated as follows:

THEOREM 1.1 (Choquet–Deny). *Let $(G,+)$ be a metrizable separable locally compact abelian group, and let μ be a probability measure on G. Suppose f is a bounded continuous solution of*

$$f = f * \mu.$$

Then $f(\cdot) = f(\cdot + a)$ for all $a \in \operatorname{supp} \mu$. In particular, if $\operatorname{supp} \mu$ generates the group G, then f is a constant function.

We will call a positive regular Borel measure a *Radon measure*.

THEOREM 1.2 (Deny). *Let G be as above and let μ be a Radon measure on G such that $\operatorname{supp} \mu = G$. Then any positive continuous solution of $f = f * \mu$ is given by*

$$f = \int_{\mathcal{E}_\mu} g \, dP(g),$$

where P is a Radon measure on

$$\mathcal{E}_\mu = \Big\{ g : g \text{ continuous}, \ g(x+y) = g(x)g(y), \ g(0) = 1, \ \int_G g(-y)\, dy = 1 \Big\}.$$

Moreover, the above integral representation for f is unique.

Our first purpose in this paper is to extend the above two theorems to the case where f is an $\mathbb{R}^n$-valued function and μ is an $n \times n$ matrix-valued measure (Theorems 3.1 and 3.7). By applying such a Choquet–Deny theorem and modifying the technique in [Fe], we obtain the solution of the vector-valued renewal equation on $\mathbb{R}^+$ together with its asymptotic behavior (Theorems 4.2 and 4.3). Our second aim is to use this result to answer the question raised in [Str3] on the asymptotic behavior of the L^p-density and Fourier transformation of self-similar measures (Theorem 5.3).

The basic idea of the proof of the extension of Theorems 1.1 and 1.2 is to use iterated substitution together with some linear algebraic technique, to reduce the vector-valued case to the scalar-valued case. This way we obtain the general solutions for each coordinate and then match them up to form the vector-valued solution. The vector-valued renewal equation considered here has overlap with the one in the Markov renewal theory [Ç, Chapter 10]. However, instead of being probabilistic, our approach is completely analytic and self-contained. Also, without using the Markov matrices, the solutions and results come out to be more symmetrical. In the development, we mainly consider the case corresponding to irreducible matrices. The case of reducible matrices turns out to be quite interesting in connection with the asymptotic behavior of the solution of the renewal equation and the L^p-density of a self-similar measure (Theorems 4.5 and 5.3; see also [MW, Theorem 4]).

In this paper we have not considered the case where the functions and measures take values in an infinite-dimensional space. This would be more complicated as shown in the theory of nonnegative infinite matrices in [S]. The Deny theorem extended here can also be modified to be considered on semigroups as in [DS] and [LZ] without much difficulty; it can be used to extend some applications in [LR] and [RL] to the vector-valued case. Another possible extension of Deny's theorem is to set up the convolution equation with functions and measures taking values in the cone of positive self-adjoint operators in Hilbert space or in the cone of positive elements of a C^*-algebra. This direction is considered in [CL].

2. Linear algebra preliminaries. For the set of states $\{1, \ldots, n\}$, we use $\gamma = (i_1, \ldots, i_k)$ to denote the *path* which starts from state i_1 and visits states $i_2, \ldots, i_k$ successively. Such a γ is called a *cycle* if $i_1 = i_k$, and a *simple cycle* if it is a cycle and all $i_1, \ldots, i_{k-1}$ are distinct.

Let $\mathbb{P}_n$ be the class of all $n \times n$ matrices with nonnegative entries. For $\mathbf{P} \in \mathbb{P}_n$, let $\mathbf{P} = [p_{ij}]_{1 \le i,j \le n}$. The *weight* of a path $\gamma = (i_1, \ldots, i_k)$ with respect to $\mathbf{P}$ is defined to be the product $p_{i_1 i_2} \cdots p_{i_{k-1} i_k}$. Let

$$q_k(i,j) = \begin{cases} p_{ij} & \text{if } k = 0, \\ \sum_{l \ne j} p_{il} q_{k-1}(l,j) & \text{if } k > 0, \end{cases} \qquad q(i,j) = \sum_{k=0}^{\infty} q_k(i,j).$$

If the sum of each row is 1, then $\mathbf{P}$ is called a *Markov matrix*; it then corresponds to a Markov chain. In that case $q_k(i,j)$ can be interpreted as the probability of the first visit of the jth state starting from the ith state in $k+1$ steps, and $q(i,j)$ is the probability of the Markov chain starting from i ever visiting j. A matrix $\mathbf{P} \in \mathbb{P}_n$ is called *irreducible* if for each pair $i,j \in \{1, \ldots, n\}$, there exists k such that $q_k(i,j) > 0$; this amounts to the condition that any two states are connected by a path with positive weight. It follows from the Perron–Frobenius theorem that if $\mathbf{P}$ is irreducible, then the spectral radius of $\mathbf{P}$ equals the maximal eigenvalue; the eigenvalue is simple and the corresponding eigenvector is (coordinatewise) positive ([M], [S]). Also, a matrix $\mathbf{P}$ is irreducible if and only if every principle submatrix (i.e., the square submatrix obtained by crossing out any j rows and the corresponding j columns, $1 \le j < n$) has maximal eigenvalue strictly less than that of $\mathbf{P}$ (see [M]).

Let $\mathbf{P}_{ij}$ be the submatrix of $\mathbf{P}$ obtained by deleting the ith row and the jth column, $\mathbf{p}_{i\hat{j}}$ the $(n-1)$-row vector obtained by deleting the jth entry of $[p_{i1}, \ldots, p_{in}]$, and $\mathbf{p}^t_{\hat{i}j}$ the $(n-1)$-column vector obtained by deleting the ith entry of $[p_{1j}, \ldots, p_{nj}]^t$.

LEMMA 2.1. *Let $\mathbf{P} \in \mathbb{P}_n$. Then for any i, j,*

$$q(i,j) = p_{ij} + \mathbf{p}_{i\hat{j}} \sum_{k=0}^{\infty} (\mathbf{P}_{ij})^k \mathbf{p}^t_{\hat{i}j}. \tag{2.1}$$

Proof. By induction and a direct calculation, we have

$$q_{k+1}(i,j) = \sum_{l \ne j} p_{il} q_k(l,j) = \sum_{l \ne j} p_{il} (\mathbf{p}_{l\hat{j}} (\mathbf{P}_{lj})^{k-1} \mathbf{p}^t_{\hat{l}j}) = \mathbf{p}_{i\hat{j}} (\mathbf{P}_{ij})^k \mathbf{p}^t_{\hat{i}j}$$

and (2.1) follows.

PROPOSITION 2.2. *Let $\mathbf{P} \in \mathbb{P}_n$ be irreducible. Then 1 is the maximal eigenvalue of $\mathbf{P}$ if and only if $q(i,i) = 1$ for some (and hence all) $1 \le i \le n$.*

Proof. We first proof the sufficiency. Without loss of generality we can assume that $i = 1$. We claim that the matrix $\mathbf{P}_{11}$ has maximal eigenvalue less than 1. Indeed, the irreducibility of $\mathbf{P}$ implies that any two states can be connected by a path of length less than n^2 and of weight greater than some $\eta > 0$. If $\mathbf{P}_{11}$ has maximal eigenvalue greater than or equal to 1, then there exists an entry i_0, j_0 of $\sum_{k=0}^{\infty} (\mathbf{P}_{11})^k$ whose weight is greater than η^{-2}. Consider the matrix $\mathbf{P}$; there exist two paths connecting 1 to i_0 and j_0 to 1 with length $r+1$ and $s+1$ respectively, and with positive weights greater than η. Furthermore, we can take the first (second) path to pass through states contained in $\{2, \ldots, n\}$ except the starting state (finishing state respectively) at 1. Hence

$$1 = q(1,1) > (\mathbf{p}_{1\hat{1}} (\mathbf{P}_{11})^r) \sum_{k=0}^{\infty} (\mathbf{P}_{11})^k ((\mathbf{P}_{11})^s \mathbf{p}^t_{\hat{1}1}) > \eta \eta^{-2} \eta = 1.$$

This is a contradiction and the claim is proved. Now $\sum_{k=0}^{\infty} (\mathbf{P}_{11})^k$ will converge and the limit is $(\mathbf{I} - \mathbf{P}_{11})^{-1}$. It follows that

$$\begin{aligned} \det(\mathbf{I} - \mathbf{P}) &= \det(\mathbf{I} - \mathbf{P})_{11}(p_{11} - 1) + \sum_{k=2}^{n} (-1)^k p_{1k} \det(\mathbf{I} - \mathbf{P})_{1k} \\ &= \det(\mathbf{I} - \mathbf{P}_{11})(p_{11} - 1) \\ &\quad + \sum_{k=2}^{n} \sum_{l=2}^{n} (-1)^{k+l} p_{1k} (p_{l1} \det(\mathbf{I} - \mathbf{P})_{1k,l1}) \\ &= \det(\mathbf{I} - \mathbf{P}_{11})(p_{11} - 1) + \mathbf{p}_{1\hat{1}} \operatorname{Adj}(\mathbf{I} - \mathbf{P}_{11}) \mathbf{p}^t_{\hat{1}1} \\ &= \det(\mathbf{I} - \mathbf{P}_{11})((p_{11} - 1) + \mathbf{p}_{1\hat{1}} (\mathbf{I} - \mathbf{P}_{11})^{-1} \mathbf{p}^t_{\hat{1}1}) \\ &= \det(\mathbf{I} - \mathbf{P}_{11})(q(1,1) - 1) = 0. \end{aligned} \tag{2.2}$$

Here $(\mathbf{I} - \mathbf{P})_{1k,l1}$ is the $(n-2) \times (n-2)$ matrix obtained by deleting the first and lth rows, and the first and kth columns of $\mathbf{I} - \mathbf{P}$. This shows that 1 is an eigenvalue of $\mathbf{P}$. For $\lambda > 1$, the above calculation implies that

$$\det(\lambda \mathbf{I} - \mathbf{P}) = \det(\lambda \mathbf{I} - \mathbf{P}_{11})((p_{11} - \lambda) + \mathbf{p}_{1\hat{1}} (\lambda \mathbf{I} - \mathbf{P}_{11})^{-1} \mathbf{p}^t_{\hat{1}1}).$$

Note that $\det(\lambda \mathbf{I} - \mathbf{P}_{11})$ is nonzero since the maximal eigenvalue of $\mathbf{P}_{11}$ is less than 1. For the second factor, using the nonnegativity of p_{ij} and $(\mathbf{I} - \mathbf{P}_{11})^{-1} = \sum_{k=0}^{\infty} (\mathbf{P}_{11})^k$, we have

$$\begin{aligned} (p_{11} - \lambda) + \mathbf{p}_{1\hat{1}} (\lambda \mathbf{I} - \mathbf{P}_{11})^{-1} \mathbf{p}^t_{\hat{1}1} &< (p_{11} - 1) + \mathbf{p}_{1\hat{1}} (\mathbf{I} - \mathbf{P}_{11})^{-1} \mathbf{p}^t_{\hat{1}1} \\ &= q(1,1) - 1 = 0. \end{aligned}$$

Hence $\det(\lambda \mathbf{I} - \mathbf{P}) \ne 0$ and $\lambda > 1$ cannot be an eigenvalue of $\mathbf{P}$. The maximal eigenvalue of $\mathbf{P}$ is hence 1.

To prove the last statement we observe that if $\mathbf{P}$ is irreducible and has maximal eigenvalue 1, then $\mathbf{P}_{ii}$ has maximal eigenvalue strictly less than 1 [M, Theorem 5.3] and $\mathbf{I} - \mathbf{P}_{ii}$ is invertible. In view of identity (2.2) proved above, we conclude that $q(i,i) = 1$ for each i.

Throughout we assume that G is a separable metrizable locally compact abelian group. Let

$$\mathbf{M} = \begin{bmatrix} \mu_{11} & \dots & \mu_{1n} \\ \vdots & & \vdots \\ \mu_{n1} & \dots & \mu_{nn} \end{bmatrix}$$

be a matrix-valued Radon measure (i.e., each μ_{ij}, $1 \le i,j \le n$, is a Radon measure on G). If $\|\mu_{ij}\| < \infty$ for all i, j, we use

$$\mathbf{M}_{\text{var}} = \begin{bmatrix} \|\mu_{11}\| & \dots & \|\mu_{1n}\| \\ \vdots & & \vdots \\ \|\mu_{n1}\| & \dots & \|\mu_{nn}\| \end{bmatrix}$$

to denote the matrix of variations of μ_{ij}. For the measures μ_{ij} in $\mathbf{M}$, and for any path $\gamma = (i_1, \dots, i_k)$, we use the notation

$$\mu_\gamma = \mu_{i_1 i_2} * \dots * \mu_{i_{k-1} i_k}.$$

It is easy to show that for Radon measures μ_1, μ_2 on G,

$$\text{(2.3)} \qquad \begin{aligned} \operatorname{supp}(\mu_1 * \mu_2) &= \overline{\operatorname{supp}\mu_1 + \operatorname{supp}\mu_2}, \\ \operatorname{supp}(\mu_1 + \mu_2) &= \operatorname{supp}\mu_1 \cup \operatorname{supp}\mu_2. \end{aligned}$$

Let $\mathbf{M}^0 = I$, $\mathbf{M}*\mathbf{M} = [\sum_{l=1}^n \mu_{il} * \mu_{lj}]$ and $\mathbf{M}^{*k}$ be the k-fold convolution of $\mathbf{M}$. Let $\langle A \rangle$ denote the closed subgroup generated by A.

LEMMA 2.3. *Let* $\mathbf{M}$ *be a matrix-valued Radon measure defined on* G, *and let*

$$\tau_i = \mu_{ii} + \boldsymbol{\mu}_{i\hat{i}} * \sum_{k=0}^{\infty} (\mathbf{M}_{ii})^{*k} * \boldsymbol{\mu}^t_{\hat{i}i}, \qquad 1 \le i \le n.$$

Then for each $1 \le i \le n$, $\langle \operatorname{supp}\tau_i \rangle$ *equals the closed subgroup* $G_{\mathbf{M}}$ *generated by*

$$\bigcup \{\operatorname{supp}\mu_\gamma : \gamma \textit{ a simple cycle on } \{1, \dots, n\}\}.$$

Proof. Consider $i = 1$. Then τ_1 is the sum of μ_{11} and the measures of the form

$$\mu_\gamma = \mu_{1j_1} * \dots * \mu_{j_l 1}, \quad j_1, \dots, j_k \in \{2, \dots, n\},\ l = 1, 2, \dots$$

By changing the order of convolution of μ_γ, it is easy to see that τ_1 is the sum of convolutions of the $\mu_{\gamma'}$ where γ' are simple cycles on $\{1, \dots, n\}$. Now using (2.3), we conclude that $\langle \operatorname{supp}\tau_1 \rangle$ equals the closed subgroup generated by $\bigcup\{\operatorname{supp}\mu_\gamma : \gamma$ a simple cycle on $\{1, \dots, n\}\}$.

As a simple example, consider the case where $\mathbf{M}$ is a 2×2 matrix-valued measure. Then $G_{\mathbf{M}}$ is the closed subgroup generated by

$$\operatorname{supp}\mu_{11}, \quad \operatorname{supp}\mu_{22} \quad \text{and} \quad \operatorname{supp}\mu_{12} * \mu_{21}.$$

Using (2.3) we can show that $\operatorname{supp}\mu_{12} - \operatorname{supp}\mu_{12}$ and $\operatorname{supp}\mu_{21} - \operatorname{supp}\mu_{21}$ will also be in $G_{\mathbf{M}}$. However, $\operatorname{supp}\mu_{12}$ and $\operatorname{supp}\mu_{21}$ may not be in $G_{\mathbf{M}}$ (see Remark 3.4 for a concrete example).

3. Vector-valued Choquet–Deny theorem.

We will use the term λ-*eigenvector* for an eigenvector of a matrix corresponding to the eigenvalue λ; by a *normalized vector* we mean a vector of length 1.

THEOREM 3.1. *Let* $\mathbf{M}$ *be a Radon measure on* G. *Suppose* $\mathbf{M}_{\text{var}}$ *is irreducible and has maximal eigenvalue* 1. *Then any bounded continuous solution of*

$$\text{(3.1)} \qquad \mathbf{f} = \mathbf{f} * \mathbf{M}$$

satisfies:

(i) *If* $G = G_{\mathbf{M}}$, *then* $\mathbf{f} = c\mathbf{v}$ *for some constant* c, *where* $\mathbf{v}$ *is the unique normalized left* 1-*eigenvector of* $\mathbf{M}_{\text{var}}$.

(ii) *If* $G_{\mathbf{M}}$ *is a proper subgroup of* G, *then there exists a bounded continuous function* p *on* G *such that* $p(\cdot + a) = p(\cdot)$ *for all* $a \in G_{\mathbf{M}}$, *and* $\mathbf{f} = [f_1, \dots, f_n]$ *satisfies*

$$\text{(3.2)} \qquad [f_1(\cdot + a_{11}), f_2(\cdot + a_{12}), \dots, f_n(\cdot + a_{1n})] = p(\cdot)\mathbf{v},$$

where $a_{1j} \in \operatorname{supp}\mu_{\gamma(1,j)}$, $j = 1, \dots, n$ *and* $\gamma(1, j)$ *is any path from* 1 *to* j *such that* $\mu_{\gamma(1,j)} \neq 0$.

Conversely, the $\mathbf{f}$ *described in* (i), (ii) *satisfies* (3.1).

Remark 3.2. The continuity in the theorem can be replaced by Borel measurability; equations are then understood in the sense of almost everywhere.

Remark 3.3. In (ii) the left side of (3.2) is independent of the choice of the paths from 1 to j with nonzero weight. In particular, if each μ_{1j} is nonzero, we can just take $a_{1j} \in \operatorname{supp}\mu_{1j}$. Note also that each f_j has period a for any $a \in G_{\mathbf{M}}$. Hence for $a_{j1} \in \operatorname{supp}\mu_{\gamma(j,1)}$,

$$f_j(x + a_{1j}) = f_j(x + (a_{1j} + a_{j1}) - a_{j1}) = f_j(x - a_{j1}).$$

Furthermore, the choice of $a_{1j} \in \operatorname{supp}\mu_{\gamma(1,j)}$ can be replaced by $a_{ij} \in \operatorname{supp}\mu_{\gamma(i,j)}$ for any fixed $i \in \{1, \dots, n\}$. This can be seen from the proof of the theorem.

Remark 3.4. Since $\operatorname{supp}\mu_{\gamma(1,1)} \subseteq G_{\mathbf{M}}$, we have $f_1(x + a_{11}) = f_1(x)$. If in addition $\operatorname{supp}\mu_{\gamma(1,j)} \subseteq G_{\mathbf{M}}$ for each j, then (3.2) reduces to

$$[f_1(x), f_2(x), \dots, f_n(x)] = p(x)\mathbf{v}.$$

In general this is not true, and (3.2) amounts to saying that each f_j is a *shift* of a periodic function p times a constant multiple determined by the coordinates of $\mathbf{v}$. For example, consider $G=\mathbb{R}$ and let

$$\mathbf{M}=\begin{bmatrix}\frac{1}{2}\delta_{1+\sqrt{2}} & \frac{1}{2}\delta_1\\ \frac{1}{3}\delta_{\sqrt{2}} & \frac{2}{3}\delta_{1+\sqrt{2}}\end{bmatrix}.$$

Then $G_{\mathbf{M}}=\langle 1+\sqrt{2}\rangle$, and $[2,3]$ is a left 1-eigenvector. Hence

$$f_1(x)=2p(x)\quad\text{and}\quad f_2(x)=3p(x+1),$$

where p is a periodic function of period $1+\sqrt{2}$.

Proof of Theorem 3.1. We use the technique of repeated substitution to reduce the equation to the real-valued case, and then apply the Choquet–Deny theorem. First let us consider the case $n=2$:

$$\begin{cases} f_1=f_1*\mu_{11}+f_2*\mu_{21},\\ f_2=f_1*\mu_{12}+f_2*\mu_{22}.\end{cases}\tag{3.3}$$

It follows that

$$\begin{aligned} f_1&=f_1*\mu_{11}+(f_1*\mu_{12}+f_2*\mu_{22})*\mu_{21}\\ &=f_1*\mu_{11}+f_1*\mu_{12}*\mu_{21}+(f_1*\mu_{12}+f_2*\mu_{22})*\mu_{22}*\mu_{21}=\ldots\\ &=f_1*\Big(\mu_{11}+\mu_{12}*\sum_{k=0}^{l}\mu_{22}^{*k}*\mu_{21}\Big)+f_2*\mu_{22}^{*l+1}*\mu_{21}.\end{aligned}\tag{3.4}$$

Since $\mathbf{M}_{\mathrm{var}}$ is irreducible, $\|\mu_{22}\|<1$, the remainder term converges to zero and hence

$$f_1=f_1*\Big(\mu_{11}+\mu_{12}*\sum_{k=0}^{\infty}\mu_{22}^{*k}*\mu_{21}\Big).\tag{3.5}$$

Denote the measure on the right by τ_1. By assumption $\mathbf{M}_{\mathrm{var}}$ has maximal eigenvalue 1, and Proposition 2.2 implies that

$$\|\tau_1\|=\|\mu_{11}\|+\|\mu_{12}\|\sum_{k=0}^{\infty}\|\mu_{22}^{*k}\|\cdot\|\mu_{21}\|=1.$$

By the Choquet–Deny theorem, f_1 satisfies $f_1(x+a)=f_1(x)$ for $a\in\langle\operatorname{supp}\tau_1\rangle$ $=G_{\mathbf{M}}$ (Lemma 2.3). Similarly f_2 is of the same form.

In case (i), both f_1 and f_2 are constant functions. A direct substitution into (3.3) yields $\mathbf{f}=c\mathbf{v}$, where $\mathbf{v}$ is the unique normalized left 1-eigenvector of $\mathbf{M}_{\mathrm{var}}$.

In case (ii), since

$$\operatorname{supp}\mu_{\gamma(1,1)},\operatorname{supp}\mu_{\gamma(2,2)},\operatorname{supp}\mu_{\gamma(1,2)}+\operatorname{supp}\mu_{\gamma(2,1)}\subseteq G_{\mathbf{M}},$$

and f_1 and f_2 have periods a for each $a\in G_{\mathbf{M}}$, we have

$$\begin{aligned} f_1(x+a_{11})&=f(x)=f_1(x)\|\mu_{11}\|+f_2*\mu_{21}(x)\\ &=f_1(x)\|\mu_{11}\|+\int_G f_2(x+a_{12}-a_{12}-y)\,d\mu_{21}(y)\end{aligned}$$

$$\begin{aligned} &=f_1(x)\|\mu_{11}\|+f_2(x+a_{12})\int_G d\mu_{21}(y)\\ &=f_1(x+a_{11})\|\mu_{11}\|+f_2(x+a_{12})\|\mu_{21}\|\end{aligned}$$

for any $a_{1j}\in\operatorname{supp}\mu_{\gamma(1,j)}$, $j=1,2$, and similarly

$$\begin{aligned} f_2(x+a_{12})&=\int_G f_1(x+(a_{12}+a_{21})-(a_{21}+y))\,d\mu_{12}(y)+f_2(x+a_{12})\|\mu_{22}\|\\ &=f_1(x)\int_G d\mu_{12}(y)+f_2(x+a_{12})\|\mu_{22}\|\\ &=f_1(x+a_{11})\|\mu_{12}\|+f_2(x+a_{12})\|\mu_{22}\|.\end{aligned}$$

This implies that for each $x\in G$, $[f_1(x+a_{11}),f_2(x+a_{12})]$ is a left 1-eigenvector of $\mathbf{M}$. For the fixed normalized left 1-eigenvector $\mathbf{v}$ of $\mathbf{M}$, we can write the above vector as $p(x)\mathbf{v}$. It is easy to show that $p(x+a)=p(x)$ for $a\in G_{\mathbf{M}}$, and assertion (ii) follows from this.

For the general case we make use of

$$[f_2,\ldots,f_n]=f_1*\boldsymbol{\mu}_{1\hat{1}}+[f_2,\ldots,f_n]*\mathbf{M}_{11},$$

where $\boldsymbol{\mu}_{1\hat{1}}=[\mu_{12},\ldots,\mu_{1n}]$, and $\mathbf{M}_{11}$ is the $(n-1)\times(n-1)$ matrix obtained by deleting the first row and first column of $\mathbf{M}$. By a substitution similar to that in (3.4) we have

$$\begin{aligned} f_1&=f_1*\mu_{11}+[f_2,\ldots,f_n]*\boldsymbol{\mu}_{\hat{1}1}\\ &=f_1*(\mu_{11}+\boldsymbol{\mu}_{1\hat{1}}*\boldsymbol{\mu}_{\hat{1}1}^t)+[f_2,\ldots,f_n]*\mathbf{M}_{11}*\boldsymbol{\mu}_{\hat{1}1}^t\\ &=f_1*\Big(\mu_{11}+\boldsymbol{\mu}_{1\hat{1}}*\sum_{k=0}^{l}(\mathbf{M}_{11})^{*k}*\boldsymbol{\mu}_{\hat{1}1}^t\Big)+[f_2,\ldots,f_n]*(\mathbf{M}_{11})^{*l+1}*\boldsymbol{\mu}_{\hat{1}1}^t.\end{aligned}$$

Since $\mathbf{M}_{\mathrm{var}}$ has maximal eigenvalue 1 and is irreducible, the submatrix $(\mathbf{M}_{\mathrm{var}})_{11}$ has maximal eigenvalue less than 1 (see [M]) so that the last term converges to zero, and we have

$$f_1=f_1*\Big(\mu_{11}+\boldsymbol{\mu}_{1\hat{1}}*\sum_{k=1}^{\infty}(\mathbf{M}_{11})^{*k}*\boldsymbol{\mu}_{\hat{1}1}^t\Big)=f_1*\tau_1.\tag{3.7}$$

Similarly we can set up the convolution equation for $f_2,\ldots,f_n$ with $\tau_2,\ldots$ $\ldots,\tau_n$ respectively. The rest of the argument is the same as above.

A function $g:G\to\mathbb{R}^+$ is called an *exponential function* if $g(x+y)=g(x)g(y)$. Let $\mathcal{E}$ denote the class of continuous exponential functions with $g(0)=1$. For a Radon measure τ, we let

$$\mathcal{E}_\tau=\Big\{g\in\mathcal{E}:\int_G g(-y)\,d\tau(y)=1\Big\}.$$

The topology on $\mathcal{E}_\tau$ is the weak topology generated by $C_c(G,\mathbb{R})$, the space of continuous functions on $\mathbb{R}$ with compact support (see [D]).

We will use the notation $\mu(h)$ to denote $\int_G h(y)\,d\mu(y)$. For $\mathbf{M} = [\mu_{ij}]$, we let $\mathbf{M}(h) = [\mu_{ij}(h)]$.

LEMMA 3.5. *Let μ be a Radon measure on G. Then for $g \in \mathcal{E}$,*

$$g * \mu^{*k}(x) = g(x)\Big(\int_G g(-y)\,d\mu(y)\Big)^k = g(x)(\mu(\widetilde{g}))^k,$$

where $\widetilde{g}(y) = g(-y)$.

PROPOSITION 3.6. *Let $\mathbf{M}$ be a matrix-valued Radon measure on G such that $\mathbf{M}_{\mathrm{var}}$ is irreducible. Let*

$$\mathcal{E}_{\mathbf{M}} = \{g \in \mathcal{E} : \mathbf{M}(\widetilde{g}) \text{ has maximal eigenvalue } 1\}$$

and let

$$\tau_i = \mu_{ii} + \boldsymbol{\mu}_{ii} * \sum_{k=0}^{\infty} (\mathbf{M}_{ii})^{*k} * \boldsymbol{\mu}_{ii}^t, \quad i = 1,\ldots,n.$$

Then for each i, $\mathcal{E}_{\tau_i} = \{g \in \mathcal{E} : \tau_i(\widetilde{g}) = 1\} = \mathcal{E}_{\mathbf{M}}$.

Proof. Let $g \in \mathcal{E}$ be such that $\tau_i(\widetilde{g}) = 1$. By Lemma 3.5 above, we have

$$1 = \tau_i(\widetilde{g}) = \mu_{ii}(\widetilde{g}) + \boldsymbol{\mu}_{ii}(\widetilde{g}) \sum_{k=0}^{\infty} (\mathbf{M}_{ii})^k(\widetilde{g})\boldsymbol{\mu}_{ii}^t(\widetilde{g}).$$

By Proposition 2.2, $\mathbf{M}(\widetilde{g})$ has maximal eigenvalue 1. Conversely, assume $\mathbf{M}(\widetilde{g})$ has maximal eigenvalue 1. Since $\mathbf{M}_{\mathrm{var}}$ is irreducible, so is $\mathbf{M}(\widetilde{g})$. Proposition 2.2 applies again and $\tau_i(\widetilde{g}) = 1$ for each i.

THEOREM 3.7. *Let $\mathbf{M}$ be a matrix-valued Radon measure on G such that $\mathbf{M}_{\mathrm{var}}$ is irreducible and $G_{\mathbf{M}} = G$. For $g \in \mathcal{E}_{\mathbf{M}}$, let $\mathbf{v}(g)$ denote the unique normalized left 1-eigenvector of $\mathbf{M}(\widetilde{g})$ ($\mathbf{M}(\widetilde{g})$ is also irreducible since it has the same nonzero entries as $\mathbf{M}_{\mathrm{var}}$). Then $g\mathbf{v}(g)$ is a solution of*

$$\mathbf{f} = \mathbf{f} * \mathbf{M}. \tag{3.8}$$

The general nonnegative continuous solution of (3.8) is given by

$$\mathbf{f} = \int_{\mathcal{E}_{\mathbf{M}}} g\mathbf{v}(g)\,dP(g), \tag{3.9}$$

where P is a Radon measure on $\mathcal{E}_{\mathbf{M}}$.

Proof. The first statement is just a direct check. We will prove the main part by considering the 2-dimensional case. The general case will follow by a suitable modification. By (3.4) we see that

$$\{f_2 * \mu_{22}^{*k} * \mu_{21}\}$$

is a decreasing sequence. We claim that its pointwise limit h is zero. Note that h satisfies $h = h * \mu_{22}$. Hence

$$\begin{aligned} h * \tau_1 &= h * \Big(\mu_{11} + \mu_{12} * \sum_{k=0}^{\infty} \mu_{22}^{*k} * \mu_{21}\Big) \\ &= h * \mu_{11} + \mu_{12} * \Big(\sum_{k=0}^{\infty} h * \mu_{22}^{*k}\Big) * \mu_{21} \\ &= h * \mu_{11} + \mu_{12} * \Big(\sum_{k=0}^{\infty} h\Big) * \mu_{21}. \end{aligned}$$

On the other hand, by substituting h into (3.4) again we have

$$f_1 = f_1 * \tau_1 + h = (f_1 * \tau_1 + h) * \tau_1 + h.$$

Now $f_1(x) < \infty$ implies that $(h * \tau_1)(x) < \infty$, and hence $h(x) = 0$. It follows that $f_1 = f_1 * \tau_1$. Applying the same argument to f_2, we also have $f_2 = f_2 * \tau_2$. By Proposition 3.6 we get $\mathcal{E}_{\tau_1} = \mathcal{E}_{\tau_2} = \mathcal{E}_{\mathbf{M}}$, and by Deny's theorem,

$$f_i = \int_{\mathcal{E}_{\mathbf{M}}} g\,dP_i(g), \quad i = 1,2.$$

Putting this back into (3.8), we have

$$\begin{aligned} \int_{\mathcal{E}_{\mathbf{M}}} g\,dP_1(g) &= \int_{\mathcal{E}_{\mathbf{M}}} g(\mu_{11}(\widetilde{g})\,dP_1(g) + \mu_{21}(\widetilde{g})\,dP_2(g)), \\ \int_{\mathcal{E}_{\mathbf{M}}} g dP_2(g) &= \int_{\mathcal{E}_{\mathbf{M}}} g(\mu_{12}(\widetilde{g})\,dP_1(g) + \mu_{22}(\widetilde{g})\,dP_2(g)). \end{aligned}$$

The uniqueness of the representing measure implies that the measures on both sides are equal, i.e.

$$[dP_1(g), dP_2(g)] = [dP_1(g), dP_2(g)] \begin{bmatrix} \mu_{11}(\widetilde{g}) & \mu_{12}(\widetilde{g}) \\ \mu_{21}(\widetilde{g}) & \mu_{22}(\widetilde{g}) \end{bmatrix}. \tag{3.10}$$

Let $\mathbf{v}(g)$ be the normalized left 1-eigenvector of $[\mu_{ij}(\widetilde{g})]$. Then (3.10) can be expressed as

$$[dP_1(g), dP_2(g)] = \mathbf{v}(g)dP(g),$$

where P is a Radon measure on $\mathcal{E}_{\mathbf{M}}$ and (3.9) follows.

For $n > 2$, we will use induction. A repeated substitution of

$$f_1 = f_1 * \mu_{11} + [f_2,\ldots,f_n] * \boldsymbol{\mu}_{11}^t \tag{3.11}$$

into $[f_2,\ldots,f_n]$ yields

(3.12) $$\begin{aligned}[f_2,\ldots,f_n] &= f_1 * \mu_{1\hat{1}} + [f_2,\ldots,f_n] * \mathbf{M}_{11}\\ &= f_1 * \mu_{11} * \boldsymbol{\mu}_{1\hat{1}} + [f_2,\ldots,f_n] * (\mathbf{M}_{11} + \boldsymbol{\mu}^t_{\hat{1}1}\boldsymbol{\mu}_{1\hat{1}})\\ &\ \ \vdots\\ &= f_1 * \mu_{11}^{*l+1} * \boldsymbol{\mu}_{1\hat{1}} + [f_2,\ldots,f_n] * \Big(\mathbf{M}_{11} + \boldsymbol{\mu}^t_{\hat{1}1} * \sum_{k=0}^{l} \mu_{11}^{*k} * \boldsymbol{\mu}_{1\hat{1}}\Big),\end{aligned}$$

By the same argument as at the beginning of the proof, we can show that $f_1 * \mu_{11}^{*l+1} * \boldsymbol{\mu}_{1\hat{1}}$ converges to zero, hence

$$\begin{aligned}[f_2,\ldots,f_n] &= [f_2,\ldots,f_n] * \Big(\mathbf{M}_{11} + \boldsymbol{\mu}^t_{\hat{1}1} * \boldsymbol{\mu}_{1\hat{1}} * \sum_{k=0}^{\infty} \mu_{11}^{*k}\Big)\\ &= [f_2,\ldots,f_n] * (\mathbf{M}_{11} + \mathbf{Q}_{11}).\end{aligned}$$

The induction hypothesis implies that $[f_2,\ldots,f_n]$ is a mixture of vectors of the form $g\mathbf{u}(g)$, where $g \in \mathcal{E}_1$, the set of exponential functions defined by the matrix-valued Radon measure $\mathbf{M}_{11} + \mathbf{Q}_{11}$ as in Proposition 3.6; $\mathbf{u}(g)$ is the normalized left 1-eigenvector of $(\mathbf{M}_{11} + \mathbf{Q}_{11})(\widetilde{g})$. By making use of $g\mathbf{u}(g)$ and a repeated substitution of (3.11) into itself and observing that $(\mu_{11}(\widetilde{g}))^k \to 0$ as $k \to \infty$, we have

$$f_1 = g\Big(\sum_{k=0}^{\infty} (\mu_{11}(\widetilde{g}))^k\Big)\mathbf{u}(g)\boldsymbol{\mu}^t_{\hat{1}1}(\widetilde{g}).$$

It follows that f_1 is also a scalar multiple of $g \in \mathcal{E}_1$. We conclude that if $[f_1,\ldots,f_n]$ is a solution of (3.8), then each f_i is a mixture of $g \in \mathcal{E}_1$. Such g must satisfy $\tau_1(\widetilde{g}) = 1$ (Deny's theorem), hence $\mathcal{E}_1 \subseteq \mathcal{E}_{\mathbf{M}}$. Conversely, it can be checked directly that each $g \in \mathcal{E}_{\mathbf{M}}$ is in $\mathcal{E}_1$, hence $\mathcal{E}_1 = \mathcal{E}_{\mathbf{M}}$. Now an application of the argument in (3.10) yields the theorem.

Remark 3.8. In Proposition 3.6 the set $\mathcal{E}_{\mathbf{M}}$ may be empty even in the scalar-valued case; the only solution f is then the zero function. If $\mathbf{M}$ is defined on $\mathbb{R}$ and is supported by $[0,\infty)$, then $\mathcal{E}_{\mathbf{M}}$ contains at most one element. Also, if $\mathbf{M}$ is defined on G and if $\mathbf{M}_{\text{var}}$ has maximal eigenvalue 1, then $\mathcal{E}_{\mathbf{M}}$ only contains the constant function; Theorem 3.7 is just case (i) of Theorem 3.1.

4. Vector-valued renewal equation. In this section we will consider the inhomogeneous convolution equation of the form

(4.1) $$\mathbf{f}(x) = \mathbf{z}(x) + \mathbf{f} * \mathbf{M}(x),$$

where $\mathbf{M}$ is a matrix-valued Radon measure on $\mathbb{R}$ that vanishes on $(-\infty,0)$, and $\mathbf{z}$ also vanishes on $(-\infty,0)$. In the scalar-valued case (4.1) is called the *renewal equation* because of its closed connection with renewal theory ([Fe], [Ç]).

Formally, the solution $\mathbf{f}$ is of the form

$$\mathbf{f}(x) = \mathbf{z} * \sum_{k=0}^{\infty} \mathbf{M}^{*k}(x).$$

The class of functions $\mathbf{z}$ in (4.1) under consideration are the directly Riemann integrable functions. A function $h : \mathbb{R} \to \mathbb{R}$ is called *directly Riemann integrable* if it is Riemann integrable on any finite interval and $\sum_k \|h\chi_{(k,k+1]}\|_\infty < \infty$; this implies that (as in the definition given in [Fe]) for any $\varepsilon > 0$, and for η sufficiently small,

$$\sum_k \overline{m}_k \text{ and } \sum_k \underline{m}_k \text{ converge absolutely, and } \sum_k (\overline{m}_k - \underline{m}_k)\eta < \varepsilon,$$

where $\overline{m}_k$ and $\underline{m}_k$ are the supremum and infimum of h on the interval $(k\eta,(k+1)\eta]$ respectively. It is clear that a continuous function with compact support is directly Riemann integrable; also, a function which is decreasing on $[0,\infty)$ and vanishes on $(-\infty,0]$ belongs to this class if and only if it is Riemann (or Lebesgue) integrable. The main purpose of introducing this class of functions is the following lemma.

LEMMA 4.1. *Suppose $\{\mu_k\}$ is a sequence of measures defined on $\mathbb{R}$ such that $\{\mu_k(I)\}$ is uniformly bounded for all intervals of fixed length $|I|$, and $\{\mu_k\}$ converges vaguely to the Lebesgue measure. Then*

$$\int_{\mathbb{R}} h(x)\,d\mu_k(x) \to \int_{\mathbb{R}} h(x)\,dx$$

for any h directly Riemann integrable.

Proof. The proof is contained in [Fe, p. 349]. By definition $\{\mu_k\}$ converges vaguely if $\int_{\mathbb{R}} \varphi\,d\mu_k$ converges for any continuous φ with compact support. If the limiting measure is the Lebesgue measure, then the vague convergence is equivalent to

$$\int_{\mathbb{R}} \varphi\,d\mu_k \to \int_{\mathbb{R}} \varphi\,dx$$

for every step function φ (see [Ch]). The uniform boundedness of $\{\mu_k\}$ implies that the step functions in the above convergence can be replaced by $\psi = \sum_k a_k\chi_{(k\eta,(k+1)\eta]}$, where $\sum_k |a_k| < \infty$. The lemma follows by approximating the directly Riemann integrable function h with such ψ.

In the following we will first consider the asymptotic behavior of $\mathbf{U} = \sum_{k=0}^{\infty} \mathbf{M}^{*k}$, where $\mathbf{M}$ is a matrix-valued Radon measure defined on $\mathbb{R}$ and vanishes on $(-\infty, 0)$. It is easy to see that the closed subgroup $G_{\mathbf{U}}$ generated by $\mathbf{U}$ equals the subgroup $G_{\mathbf{M}}$ defined in Lemma 2.3. For convenience we will write the matrix-valued measure $\mathbf{M}$ by means of

$$\mathbf{F} = \begin{bmatrix} F_{11} & \dots & F_{1n} \\ \vdots & & \vdots \\ F_{n1} & \dots & F_{nn} \end{bmatrix},$$

where $\mathbf{F}_{ij}(x) = \mu_{ij}(-\infty, x]$, and write $\mathbf{F}(\infty) = [F_{ij}(\infty)]$; also by a slight abuse of notation we use $F_{ij}(x, x+h]$ to denote $F_{ij}(x+h) - F_{ij}(x)$. Let $\mathbf{m} = [m_{ij}] = [\int_0^\infty x\,dF_{ij}(x)]$ denote the moment matrix.

THEOREM 4.2. *Suppose* $\mathbf{F}$ *is a matrix-valued Radon measure defined on* $\mathbb{R}$ *such that each entry is nondegenerate at* 0 *and vanishes on* $(-\infty, 0)$. *Also, suppose* $\mathbf{F}(\infty)$ *is irreducible and has maximal eigenvalue* 1. *Let* $\mathbf{U} = \sum_{k=0}^{\infty} \mathbf{F}^{*k}$. *Then*:

(i) *If* $G_{\mathbf{M}} = \mathbb{R}$, *then*

$$\lim_{x\to\infty} \mathbf{U}(x, x+h] = \mathbf{A}h, \tag{4.3}$$

where

$$\mathbf{A} = \frac{1}{\alpha}\begin{bmatrix} u_1v_1 & \dots & u_1v_n \\ \vdots & & \vdots \\ u_nv_1 & \dots & u_nv_n \end{bmatrix}, \quad \alpha = [v_1, \dots, v_n]\begin{bmatrix} m_{11} & \dots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \dots & m_{nn} \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

($\mathbf{A} = \mathbf{0}$ *if one of the* m_{ij} *is* ∞) *and* $\mathbf{u}, \mathbf{v}$ *are the unique normalized right and left* 1-*eigenvectors of* $\mathbf{F}(\infty)$ *respectively.*

(ii) *If* $G_{\mathbf{M}} = \langle \varrho \rangle$ *for some* $\varrho > 0$, *then* (4.3) *can be adjusted to*: *for any* $a_{ij} \in \operatorname{supp} \mu_{\gamma(i,j)}$ (see Theorem 3.1),

$$\lim_{x\to\infty} [U_{ij}(x + a_{ij}, x + a_{ij} + \varrho]] = \mathbf{A}\varrho. \tag{4.3'}$$

Proof. We first show that for each fixed $l > 0$, $\mathbf{U}(x, x+l]$ is uniformly bounded. Let $\mathbf{v} = [v_1, \dots, v_n]$ be the unique normalized left 1-eigenvector of $\mathbf{F}(\infty)$ which is irreducible. Then by the Perron–Frobenius theorem, $v_i > 0$ for each i. Let $v_0 = \min_i v_i$. Note that

$$(\mathbf{I} - \mathbf{F}) * \mathbf{U}(x) = \mathbf{I}, \quad x \ge 0. \tag{4.4}$$

By using the coordinatewise ordering, we have

$$\begin{aligned} \mathbf{v} &= \mathbf{v}(\mathbf{I} - \mathbf{F}) * \mathbf{U}(x) = \mathbf{v}(\mathbf{F}(\infty) - \mathbf{F}) * \mathbf{U}(x) \\ &\ge v_0 \int_0^x \Big[\sum_{i=1}^n (F_{i1}(\infty) - F_{i1}(x-y)), \dots, \sum_{i=1}^n (F_{in}(\infty) - F_{in}(x-y))\Big] d\mathbf{U}(y) \\ &\ge v_0 \int_{x-\delta}^x \dots \\ &\ge v_0 \Big[\sum_{i=1}^n (F_{i1}(\infty) - F_{i1}(\delta)), \dots, \sum_{i=1}^n (F_{in}(\infty) - F_{in}(\delta))\Big] \mathbf{U}(x-\delta, x]. \end{aligned}$$

Since the F_{ij}'s are nondegenerate at 0, we can find $\delta > 0$ such that the vector $[\sum_{i=1}^n (F_{i1}(\infty) - F_{i1}(\delta)), \dots]$ above is positive. Hence for each $x \in \mathbb{R}$, $\mathbf{U}(x - \delta, x]$ is uniformly bounded. It follows that $\mathbf{U}(x, x+l]$ is uniformly bounded for fixed l.

From this we see that for the family of measures $\{\mathbf{U}(\cdot + t)\}_{t>0}$ there is a subsequence $\{t_k\}$ with $t_k \to \infty$ as $k \to \infty$ such that for each $x \in \mathbb{R}$,

$$\mathbf{U}(x + t_k) \to \mathbf{L}(x) \quad \text{as } k \to \infty.$$

(Note that $\mathbf{U}$ is supported by $[0, \infty)$, but $\mathbf{L}$ is supported by $\mathbb{R}$.)

Consider the convolution equation

$$\mathbf{f}(x) = \mathbf{z}(x) + \mathbf{f} * \mathbf{F}(x), \quad x \ge 0, \tag{4.5}$$

where $\mathbf{z}$ is a continuous function with compact support and vanishes on $(-\infty, 0)$. Then $\mathbf{f} = \mathbf{z} * \mathbf{U}$ is well defined and a direct substitution implies that it is a solution of (4.5). Note that

$$\mathbf{f}(t_k + x) = \mathbf{z} * \mathbf{U}(t_k + x) \to \mathbf{z} * \mathbf{L}(x) = \boldsymbol{\xi}(x), \quad x \in \mathbb{R},$$

and by (4.5),

$$\boldsymbol{\xi}(x) = \boldsymbol{\xi} * \mathbf{F}(x), \quad x \in \mathbb{R}. \tag{4.6}$$

If $G_{\mathbf{M}} = \mathbb{R}$, by Theorem 3.1(i) and Remark 3.2, we have $\boldsymbol{\xi}(x) = a\mathbf{v}$ for some constant a, where $\mathbf{v}$ is the normalized left 1-eigenvector of $\mathbf{F}(\infty)$. It follows that

$$\mathbf{z} * \mathbf{U}(t_k + x) \to \mathbf{z} * \mathbf{L}(x) = a\mathbf{v} \tag{4.7}$$

for all $\mathbf{z}$ as described above. In particular, by taking $\mathbf{z} = [0, \dots, z_i, \dots, 0]$ separately, we conclude that each row of $\mathbf{L}$ is independent of x, hence proportional to the Lebesgue measure and

$$\mathbf{U}(t_k, t_k + h] \to \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \mathbf{v}h = \begin{bmatrix} a_1v_1 & \dots & a_1v_n \\ \vdots & & \vdots \\ a_nv_1 & \dots & a_nv_n \end{bmatrix} h = \mathbf{A}h \tag{4.8}$$

as $k \to \infty$ (using the vague convergence).

We will now determine $a_1,\ldots,a_n$. Let $\mathbf{u}=[u_1,\ldots,u_n]^t$ be the normalized right 1-eigenvector of $\mathbf{F}(\infty)$. Let $\mathbf{z}$ be such that

$$\mathbf{z}(x)=\begin{cases}(\mathbf{F}(\infty)-\mathbf{F}(x))\mathbf{u}\ (\text{also } =(\mathbf{I}-\mathbf{F}(x))\mathbf{u}) & \text{if } x\ge 0,\\ 0 & \text{if } x<0,\end{cases}\tag{4.9}$$

and let $\mathbf{f}=\mathbf{U}*\mathbf{z}$. Note that

$$\int_0^\infty (\mathbf{F}(\infty)-\mathbf{F}(x))\,dx=\Big[\int_0^\infty (F_{ij}(\infty)-F_{ij}(x))\,dx\Big]=\Big[\int_0^\infty x\,dF_{ij}(x)\Big]=[m_{ij}]=\mathbf{m}.$$

Assume that all the entries of $\mathbf{m}$ are finite measures. Then each coordinate of $\mathbf{z}$ is decreasing and integrable, so directly Riemann integrable. By using $\mathbf{I}=\mathbf{U}*(\mathbf{I}-\mathbf{F})$, (4.8), (4.9) and Lemma 4.1 we have

$$\mathbf{u}=\mathbf{f}(t_k)=\int_0^{t_k} d\mathbf{U}(t_k-x)\mathbf{z}(x)\to\int_0^\infty \mathbf{A}\mathbf{z}(x)\,dx=\mathbf{A}\Big(\int_0^\infty(\mathbf{F}(\infty)-\mathbf{F}(x))\,dx\Big)\mathbf{u}.\tag{4.10}$$

This implies that

$$\mathbf{u}=\mathbf{A}\mathbf{m}\mathbf{u}=\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix}(\mathbf{v}\mathbf{m}\mathbf{u})\tag{4.11}$$

so that

$$\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix}=(\mathbf{v}\mathbf{m}\mathbf{u})^{-1}\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}=\frac{1}{\alpha}\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}.$$

Hence (4.8) yields

$$\mathbf{z}*\mathbf{U}(t_k+x)\to\frac{1}{\alpha}\begin{bmatrix}u_1v_1 & \ldots & u_1v_n\\ \vdots & & \vdots\\ u_nv_1 & \ldots & u_nv_n\end{bmatrix}.$$

If $m_{ij}=\infty$ for some i,j, we can make use of the following fact: if $\mu_n\to\mu$ vaguely, and if h is lower semicontinuous, then

$$\liminf_{n\to\infty}\int_{\mathbb{R}} h\,d\mu_n\ge\int_{\mathbb{R}} h\,d\mu.$$

Note that each coordinate of $\mathbf{z}(x)$ in (4.9) is lower semicontinuous, we can hence replace (4.10) by $\ge$, so that (4.11) can be replaced by

$$\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}\ge\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix}(\mathbf{v}\mathbf{m}\mathbf{u}).$$

This implies that $a_1=\ldots=a_n=0$.

Finally, we observe that in (4.8) the limit

$$\mathbf{U}(t_k,t_k+h]\to\mathbf{A}h\quad\text{as } k\to\infty$$

is independent of the choice of $\{t_k\}$ tending to infinity; hence we conclude that

$$\mathbf{U}(t,t+h]\to\mathbf{A}h\quad\text{as } t\to\infty.$$

This completes the proof for the case $G_{\mathbf{M}}=\mathbb{R}$.

If $G_{\mathbf{M}}=\langle\varrho\rangle$ for some $\varrho>0$, we use (4.6), Theorem 3.1(ii) and Remark 3.3 to conclude that $z_i*U_{ij}(t_k+x)\to z_i*L_{ij}(x)$ which is a periodic function of period ϱ, and for $a_{ij}\in\operatorname{supp}\mu_{\gamma(i,j)}$,

$$\mathbf{z}*[U_{ij}(t_k+x+a_{ij})]\to\mathbf{z}*\mathbf{L}(x+a_{ij})=p(x)\mathbf{v}\tag{4.12}$$

for some continuous periodic function p of period ϱ. It follows that

$$\mathbf{U}(t_k+x+a_{ij},t_k+x+a_{ij}+\varrho]\to\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix}\mathbf{v}\varrho=\begin{bmatrix}a_1v_1 & \ldots & a_1v_n\\ \vdots & & \vdots\\ a_nv_1 & \ldots & a_nv_n\end{bmatrix}\varrho=\mathbf{A}\varrho$$

as $k\to\infty$. To determine $a_1,\ldots,a_n$, we proceed similarly to case (i). First we let $\mathbf{z}$ be defined as in (4.9) and let $\mathbf{f}'=\mathbf{U}'*\mathbf{z}$, where

$$\mathbf{U}'(x)=[U_{ij}(x+a_{ij})],\qquad a_{ij}\in\operatorname{supp}\mu_{\gamma(i,j)}.$$

Note that $\mathbf{U}'*(\mathbf{I}-\mathbf{F})=\mathbf{I}$ (as in (4.4)); by the same argument as in (4.10), (4.11) and so on, we can conclude the proof of (ii).

THEOREM 4.3. *Under the same hypotheses on* $\mathbf{F}$ *as in Theorem* 4.2, *let* $\mathbf{z}$ *be a directly Riemann integrable function with* $\mathbf{z}(x)=0$ *for* $x<0$. *Then* $\mathbf{f}(x)=\mathbf{z}*\mathbf{U}(x)$ *is a bounded continuous solution of*

$$\mathbf{f}(x)=\mathbf{z}(x)+\mathbf{f}*\mathbf{M}(x),\qquad x\ge 0,\tag{4.13}$$

and it is unique in the class of continuous solutions that vanish on $(-\infty,0)$. *Furthermore, if* $G_{\mathbf{M}}=\mathbb{R}$, *then*

$$\lim_{x\to\infty}\mathbf{f}(x)=\Big(\int_0^\infty \mathbf{z}(t)\,dt\Big)\mathbf{A},$$

where $\mathbf{A}$ *is defined as in Theorem* 4.2. *If* $G_{\mathbf{M}}=\langle\varrho\rangle$ *for some* $\varrho>0$, *then* *for each* $x > 0$, *and* $a_{1j} \in \operatorname{supp} \mu_{\gamma(1,j)}$,

$$\lim_{n\to\infty} [f_1(x + a_{11} + n\varrho), \ldots, f_n(x + a_{1n} + n\varrho)] = \Big(\sum_k \mathbf{z}(x + k\varrho)\Big)\mathbf{A}.$$

Proof. Lemma 4.1 implies that the convolution $\mathbf{z} * \mathbf{U}(x)$ is well defined. It is direct to check that $\mathbf{f}(x) = \mathbf{z}*\mathbf{U}(x)$ is a solution of the renewal equation. To prove the uniqueness, we let $\mathbf{f}_1$ be another solution. Let $\mathbf{g} = \mathbf{f} - \mathbf{f}_1$. Since $\mathbf{g}$ and $\mathbf{F}$ both vanish on $(-\infty, 0)$, $\mathbf{g}$ satisfies the convolution equation

$$\mathbf{g}(x) = \int_0^x \mathbf{g}(x - y)\, d\mathbf{F}(y), \quad x > 0.$$

By iteration,

$$\mathbf{g}(x) = \int_0^x \mathbf{g}(x - y)\, d\mathbf{F}^n(y), \quad x > 0.$$

Since $\mathbf{F}^n(0, x] \to 0$ as $n \to \infty$, it follows that $\mathbf{g}(x) = \mathbf{0}$ for $x > 0$.

To prove the asymptotic property of $\mathbf{f}$ for the nonarithmetic case, we need only use (4.3) and Lemma 4.1:

$$\mathbf{f}(x) = \int_0^x \mathbf{z}(y)\, d\mathbf{U}(x - y) \to \Big(\int_0^\infty \mathbf{z}(t)\, dt\Big)\mathbf{A} \quad \text{as } x \to \infty.$$

For the arithmetic case, we can use (4.3′) instead to draw the conclusion.

Remark 4.4. If the matrix $\mathbf{F}(\infty)$ in Theorem 4.3 has maximal eigenvalue less than 1, then as in the previous case,

$$\mathbf{f}(x) = \mathbf{z} * \Big(\sum_{k=0}^\infty \mathbf{F}^{*k}\Big)(x) = \mathbf{z} * \mathbf{U}(x)$$

is a solution of (4.13). Moreover, $\mathbf{f}$ is also directly Riemann integrable and hence $\mathbf{f}(x) \to 0$ as $x \to \infty$. For simplicity we will demonstrate this in the scalar case: Let μ be a bounded positive measure supported by $[0, \infty)$, and let z be directly Riemann integrable. For any $x \in [k, k + 1]$ and any $k > 0$,

$$z * \mu(x) = \int_0^x z(x - y)\, d\mu(y) \le \sum_{i=0}^k \|z\chi_{[k-i-1,k-i+1]}\|_\infty \mu[i, i + 1),$$

so that

$$\sum_{k=0}^\infty \|(z * \mu)\chi_{[k,k+1)}\|_\infty \le \sum_{k=0}^\infty \sum_{i=0}^k \|z\chi_{[k-i-1,k-i+1]}\|_\infty \mu[i, i + 1)$$
$$\le 2\sum_{k=0}^\infty \|z\chi_{[k,k+1)}\|_\infty \mu[0, \infty) < \infty$$

and $z * \mu$ is directly Riemann integrable.

In the following we will discuss the asymptotic behavior of the solution in Theorem 4.3 when the matrix $\mathbf{F}(\infty)$ is *reducible*. For simplicity we will first look at the case where $\mathbf{F}(\infty)$ can be decomposed into two irreducible components and the moment $\mathbf{m}$ of $\mathbf{F}$ is finite. Consider the following four cases:

$$\begin{bmatrix} \mathbf{L}_1 & \mathbf{0} \\ \mathbf{K} & \mathbf{L}_2 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{E}_1 & \mathbf{0} \\ \mathbf{K} & \mathbf{L}_2 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{L}_1 & \mathbf{0} \\ \mathbf{K} & \mathbf{E}_2 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{E}_1 & \mathbf{0} \\ \mathbf{K} & \mathbf{E}_2 \end{bmatrix},$$

where $\mathbf{L}_1(\infty)$ and $\mathbf{L}_2(\infty)$ have maximal eigenvalue less than 1 and $\mathbf{E}_1(\infty)$, $\mathbf{E}_2(\infty)$ have maximal eigenvalue equal to 1. Let us write the renewal equation in the form

$$[\mathbf{f}_1, \mathbf{f}_2] = [\mathbf{z}_1, \mathbf{z}_2] + [\mathbf{f}_1, \mathbf{f}_2] * \mathbf{F},$$

where the number of entries of $\mathbf{f}_1$ and $\mathbf{f}_2$ corresponds to the decomposition, and $\mathbf{z}_2 \ne \mathbf{0}$. If $\mathbf{K} = \mathbf{0}$, then the equation reduces to two separate ones and the solutions for $\mathbf{f}_1$ and $\mathbf{f}_2$ are given by Theorem 4.3 and Remark 4.4.

Assume that $\mathbf{K} \ne \mathbf{0}$. In the first case,

$$\mathbf{f}_2(x) = \mathbf{z}_2(x) + \mathbf{f}_2 * \mathbf{L}_2(x) \to 0$$

as $x \to \infty$ by Remark 4.4, and

$$\mathbf{f}_1 = (\mathbf{z}_1 + \mathbf{f}_2 * \mathbf{K}) + \mathbf{f}_1 * \mathbf{L}_1 = \mathbf{z}' + \mathbf{f}_1 * \mathbf{L}_1,$$

where $\mathbf{z}' := \mathbf{z}_1 + \mathbf{f}_2 * \mathbf{K}$ is directly Riemann integrable by Remark 4.4. Hence $\mathbf{f}_1$ is asymptotically equal to 0 as $x \to 0$.

The second case follows from the same argument: $\mathbf{f}_2(x) \to 0$ as $x \to \infty$ and $\mathbf{f}_1(x)$ is asymptotically periodic by Theorem 4.3.

In the last two cases, $\mathbf{f}_2 \ge 0$ is a bounded, asymptotically periodic function by Theorem 4.3. If

$$\mathbf{f}_1 = (\mathbf{z} + \mathbf{f}_2 * \mathbf{K}) + \mathbf{f}_1 * \mathbf{L}_1$$

(the third case), then

$$\mathbf{f}_1 = (\mathbf{z} + \mathbf{f}_2 * \mathbf{K}) * \sum_{k=0}^l \mathbf{L}_1^{*k} + \mathbf{f}_1 * \mathbf{L}_1^{*l+1}.$$

The right side converges to $(\mathbf{z} + \mathbf{f}_2 * \mathbf{K}) * \sum_{k=0}^\infty \mathbf{L}_1^{*k}$, which is also asymptotically periodic.

If

$$\mathbf{f}_1 = (\mathbf{z} + \mathbf{f}_2 * \mathbf{K}) + \mathbf{f}_1 * \mathbf{E}_1$$

(the fourth case), let $\mathbf{u}$ be a 1-eigenvector of $\mathbf{E}_1(\infty)$ with all entries positive. There is a constant $c > 0$ such that

$$\mathbf{z}(x) + \mathbf{f}_2 * \mathbf{K}(x) \ge c\mathbf{u}$$

(coordinatewise) for sufficiently large x. Hence

$$\begin{aligned}\mathbf{f}_1 &\geq c\mathbf{u} + \mathbf{f}_1 * \mathbf{E}_1 \geq c\mathbf{u} + c\mathbf{u}\cdot\mathbf{E}_1(\infty) + \mathbf{f}_1 * \mathbf{E}_1^{*2}\\ &= 2c\mathbf{u} + \mathbf{f}_1 * \mathbf{E}_1^{*2} \geq \ldots\\ &\geq Nc\mathbf{u}\end{aligned}$$

for any N, which means $\lim_{x\to\infty}\mathbf{f}_1(x) = \infty$.

For the general reducible case we divide the set of states $\{1,\ldots,n\}$ in $\mathbf{F}(\infty)$ as follows: let S be the set of states such that the corresponding irreducible components of $\mathbf{F}(\infty)$ have spectral radius 1; $S_0 \subseteq S$ consists of those states j that cannot be reached from any state of S corresponding to the other components. For example, let

$$\mathbf{F} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{0} & & \cdots & & \mathbf{0}\\ \times & \mathbf{E}_1 & \mathbf{0} & & & \\ \times & \times & \mathbf{L}_2 & & & \vdots\\ \times & \mathbf{K} & \times & \mathbf{E}_2 & & \\ \times & \times & \times & \mathbf{0} & \mathbf{E}_3 & \mathbf{0}\\ \times & \times & \times & \times & \times & \mathbf{L}_3 \end{bmatrix},$$

where $\mathbf{K} \neq \mathbf{0}$. Then S consists of states corresponding to $\mathbf{E}_i$ for $i = 1, 2, 3$ and S_0 consists of states corresponding to $\mathbf{E}_i$ for $i = 2, 3$.

By using the above argument we have the following theorem for the general reducible case:

THEOREM 4.5. *Suppose* $\mathbf{F}$ *is a matrix-valued Radon measure defined on* $\mathbb{R}^+$ *such that each entry is nondegenerate at* 0 *and has finite moment. Also, suppose that* $\mathbf{F}(\infty)$ *has maximal eigenvalue* 1. *Let* S *and* S_0 *be defined as above and let* $\mathbf{z}$ *be a directly Riemann integrable function on* $\mathbb{R}^+$ *such that* $z_i \neq 0$ *for* $i \in S_0$. *If* $\mathbf{f}$ *is continuous, bounded on finite intervals, vanishes on* $(-\infty, 0]$, *and satisfies the renewal equation in Theorem* 4.3, *then* $\mathbf{f} = \mathbf{z} * \sum_{k=0}^{\infty}\mathbf{F}^{*k}$, *and the components* f_i *satisfy*:

(i) *If* $i \in S_0$, *then*

$$\lim_{x\to\infty}(f_i(x) - p_i(x)) = 0,$$

where p_i *is either a periodic or a constant function* (see Theorem 4.3).

(ii) *If* $i \in S\setminus S_0$, *then* $\lim_{x\to\infty} f_i(x) = \infty$.

(iii) *If* $i \notin S$ *and there is no path from* S *to* i, *then* $\lim_{x\to\infty} f_i(x) = 0$.

(iv) *If* $i \notin S$ *and there is a path from* S_0 *to* i, *but no path from* $S\setminus S_0$ *to* i, *then*

$$\lim_{x\to\infty}(f_i(x) - p_i(x)) = 0$$

for some p_i *as in* (i).

(v) *If* $i \notin S$ *and there is a path from* $S\setminus S_0$ *to* i, *then* $\lim_{x\to\infty} f_i(x) = \infty$.

The proof is simple in view of the four cases above: (i) corresponds to $\mathbf{E}_1$ in the second case, or $\mathbf{E}_2$ in the third case; (ii) corresponds to $\mathbf{E}_2$ in the fourth case; (iii) coresponds to $\mathbf{L}_1$, $\mathbf{L}_2$ in the first case or $\mathbf{L}_2$ in the second case; (iv) corresponds to $\mathbf{L}_1$ in the third case; (v) uses (ii) and the same argument as for $\mathbf{L}_1$ in the third case.

5. Self-similar family of measures. In this section, we will apply the renewal theory developed in Section 4 to study a self-similar family of measures. Recall that for a finite family $\{S_j\}_{j=1}^m$ of contractive maps on $\mathbb{R}^d$, there exists a unique compact subset K in $\mathbb{R}^d$ satisfying $K = \bigcup_j S_jK$. The set K can be obtained by iterating the maps using the *cascade algorithm*, starting from any fixed bounded set or point. For this reason we call $\{S_j\}_{j=1}^m$ an *iterated function system* (IFS) and K the *attractor* of the system. If we associate a probability weight w_j to each S_j, then the iteration will produce a unique probability measure μ satisfying

$$\mu = \sum_{j=1}^{m} w_j\mu\circ S_j^{-1}.$$

In particular, when the S_j's are contractive similitudes, i.e., $S_jx = \varrho_jR_jx + b_j$, $0 < \varrho_j < 1$, R_j a rotation and $b_j \in \mathbb{R}^d$, we call K a *self-similar set* and μ a *self-similar measure*. If in addition the S_jK's are disjoint, then each S_jK is an identical copy of K, and the same holds for μ restricted to S_jK. This basic concept of self-similarity was introduced by Mandelbrot in his momentous monograph [Ma], the above mathematical set up was given by Hutchinson in [H] and the iterated function system notion was invented by Barnsley [B]. The reader can refer to [F] for more details.

The condition that the S_jK's are disjoint is too restrictive (e.g., the standard Koch curve and the Sierpiński gasket do not satisfy it). In [H], Hutchinson defined another very useful separation condition called the *open set condition*: there exists a bounded open set U such that

$$S_jU \subset U \quad\text{and}\quad S_iU\cap S_jU = \emptyset, \quad i\neq j.$$

Then U is called a *basic open set* of the IFS. It was proved in [LW] that under this condition, the self-similar measure μ defined by the S_j's satisfies either $\mu(\partial U) = 1$ or 0 (∂U is the boundary of U); using a theorem of Schief [Sc], we can find a basic open set U such that $\mu(\partial U) = 0$ (see [L2]).

To define a self-similar family of measures, we will follow the notations of [MW] and [Str3]. Let (V, E) be a directed multigraph, i.e. V is a finite set of vertices and E a finite set of directed edges. We use $E_{u,v}$ to denote the set of edges joining u to v, which may be empty or may have more than one element. As defined in Section 2, a *path* from u to v in the multigraph is a sequence of edges $(e_1,\ldots,e_n)$ where $e_j \in E_{u_j,u_{j+1}}$ for some $u_j \in V$ and

$u_1 = u$, $u_{n+1} = v$. A *cycle* is a path $(e_1, \ldots, e_n)$ such that $u_1 = u_{n+1}$. We say that the multigraph is *connected* if for any $u, v \in V$ there is a path from u to v or from v to u. If both such paths exist then we say that the graph (V, E) is *strongly connected*. Without loss of generality we may assume that the graph under consideration is connected.

Let $\{S(e) : e \in V\}$ be a family of similitudes such that the ratios $r(e)$ satisfy the contractive condition

$$r(e_1)\ldots r(e_n) < 1$$

for any cycle $(e_1, \ldots, e_n)$. Assume that $w(e) > 0$ for each edge $e \in E$, and

$$\sum_{u\in V}\sum_{e\in E_{u,v}} w(e) = 1 \quad \text{for all } v \in V.$$

Then there exists a unique family $\{\mu_v\}$ of measures such that

$$\mu_v = \sum_{u\in V}\sum_{e\in E_{u,v}} w(e)\mu_u \circ S(e)^{-1} \tag{5.1}$$

for all $v \in V$ (see [MW]). The family $\{\mu_v\}$ is called a *self-similar family of measures*. For the special case of self-similar measures defined by the $\{S_j\}_{j=1}^m$ at the beginning of the section, we can take $V = \{1, \ldots, m\}$, $E = \{e : e = (i,j),\ i,j = 1, \ldots, m\}$, $S(e) = S_i$ and $p(e) = p_i$ for $e = (i,j)$, and $\mu_j = \mu$. If further $\{S_j\}_{j=1}^m$ satisfies the open set condition with basic open set U, then μ_j is supported by the closure of U.

By extending the definition in [H], we say that $\{S(e)\}$ satisfies the *open set condition* if there exist open sets U_u for $u \in V$ such that

$$S(e)U_u \subseteq U_v$$

for all $e \in E_{u,v}$, and for each $v \in V$, the sets $\{S(e)U_u\}$ are disjoint, for u running through V and $e \in E_{u,v}$. In this case the measure μ_u is supported by the closure $\overline{U}_u$. Similarly to the scalar case, we can find basic open sets U_u for $u \in V$ such that

$$\mu_u(\partial U_u) = 0, \tag{5.2}$$

provided that the graph is strongly connected (see [W]).

For $0 < p < \infty$ and $\beta \in \mathbb{R}$, we define the (L^p, β)-*density* of a measure μ on $\mathbb{R}^n$ by

$$D^p_\beta(\mu) := \limsup_{t\to 0^+} \frac{1}{t^{(n+\beta(p-1))/p}} \|\mu(B_t(\cdot))\|_{L^p(dx)},$$

where $B_t(x)$ is the ball centered at x with radius t, and the L^p-*dimension* of μ by

$$\sup\{\beta : D^p_\beta(\mu) < \infty\}.$$

For a self-similar family $\{\mu_v\}$ of measures, let α_p be the positive number such that the matrix

$$[a_{u,v}] := \Big[\sum_{e\in E_{u,v}} w(e)^p r(e)^{-\alpha_p}\Big], \quad u, v \in V, \tag{5.3}$$

has maximal eigenvalue 1 (see [MW]). Under a separation condition stronger than the open set condition, Strichartz [Str3] showed that if (V, E) is strongly connected then

$$\phi_v(t) := \frac{1}{t^{n+\alpha_p}} \int \mu_v(B_t(x))^p\,dx \tag{5.4}$$

is positive and bounded. It follows that the L^p-dimension of each μ_v is $\alpha_p/(p-1)$. He then conjectured that, as in the scalar case in [LW], $\{\mu_v\}$ is asymptotically multiplicatively periodic, i.e. there is a multiplicatively periodic function $q(t)$ such that

$$\lim_{t\to 0^+} (\phi_v(t) - q(t)\varrho_v) = 0,$$

where $\boldsymbol{\varrho} = [\varrho_v]$, $v \in V$, is the unique eigenvector such that

$$\sum_{u\in V} \varrho_u a_{u,v} = \varrho_v$$

for all $v \in V$ and $\sum \varrho_u = 1$. In the following we will show that the conjecture is essentially true.

THEOREM 5.1. *Let (V, E) be a strongly connected multigraph and let $\{\mu_v\}$ be a self-similar family of measures satisfying the open set condition. Let*

$$\phi_v(t) = \frac{1}{t^{n+\alpha_p}} \int \mu_v(B_t(x))^p\,dx.$$

(i) *In the arithmetic case, i.e., if $\langle -(\ln r(e_1) + \ldots + \ln r(e_n)) : (e_1, \ldots, e_n)$ is a simple cycle$\rangle$ can be generated by $-\ln r$ for some $r > 0$, then there exists a multiplicatively periodic function q with period r such that*

$$\lim_{t\to 0^+} (\phi_v(t) - q(r_{u,v}^{-1}t)\varrho_v) = 0 \quad \textit{for } v \in V,$$

where $r_{u,v} = r(e_1)\ldots r(e_n)$ for any path $(e_1, \ldots, e_n)$ from $u \in V$ to v.

(ii) *In the nonarithmetic case, there is a constant c such that*

$$\lim_{t\to 0^+} \phi_v(t) = c\varrho_v \quad \textit{for } v \in V.$$

Proof. By (5.1) we have

$$\phi_v(t) = \frac{1}{t^{n+\alpha_p}} \int \Big(\sum_{u\in V}\sum_{e\in E_{u,v}} p(e)\mu_u \circ S(e)^{-1}(B_t(x))\Big)^p\,dx.$$

Note that $\mu_u \circ S(e)^{-1}$ is supported by the closure of $S(e)U_v$, and for any $e \neq e'$ the intersection of $S(e)U_v$ and $S(e')U_v$ has empty interior. Hence

$$\phi_v(t) = \frac{1}{t^{n+\alpha_p}} \sum_{u\in V} \sum_{e\in E_{u,v}} p(e)^p \int_{d(x,\partial(S(e)U_u))\geq t} (\mu_u \circ S(e)^{-1}(B_t(x)))^p\, dx$$
$$+ \frac{1}{t^{n+\alpha_p}} \int_I \Big(\sum_{u\in V} \sum_{e\in E_{u,v}} p(e)\mu_u \circ S(e)^{-1}(B_t(x))\Big)^p dx,$$

where the set I consists of points x such that $d(x, \partial(S(e)U_u)) < t$ for some $e \in E_{u,v}$ and some $u \in V$. The inequality

$$\Big(\sum_{j=1}^m x_j\Big)^p \leq c \sum_{j=1}^m x_j^p, \quad x_i \geq 0,$$

for some $c > 0$ (depending only on m) implies that the second term of $\phi_v(t)$ is less than

$$\frac{2c}{t^{n+\alpha_p}} \sum_{u\in V} \sum_{e\in E_{u,v}} p(e)^p \int_{d(x,\partial(S(e)U_u))<t} (\mu_u \circ S(e)^{-1}(B_t(x)))^p\, dx.$$

We can write

$$\phi_v(t) = \frac{1}{t^{n+\alpha_p}} \sum_{u\in V} \sum_{e\in E_{u,v}} p(e)^p \int (\mu_u \circ S(e)^{-1}(B_t(x)))^p\, dx + R_v(t), \tag{5.5}$$

where

$$R_v(t) \leq \frac{2c+1}{t^{n+\alpha_p}} \sum_{u\in V} \sum_{e\in E_{u,v}} p(e)^p \int_{d(x,\partial(S(e)U_u))<t} (\mu_u \circ S(e)^{-1}(B_t(x)))^p\, dx. \tag{5.6}$$

Note that

$$\frac{1}{t^{n+\alpha_p}} \int_{d(x,\partial(S(e)U_u))<t} (\mu_u \circ S(e)^{-1}(B_t(x)))^p\, dx$$
$$= \frac{r(e)}{t^{n+\alpha_p}} \int_{d(x,\partial U_u)<r(e)^{-1}t} \mu_u(B_{r(e)^{-1}t}(x))^p\, dx.$$

By an argument similar to that in Lemma 3.6 of [LW] and making use of (5.2), we can show that

$$\frac{1}{t^{n+\alpha_p}} \int_{d(x,\partial U_u)<t} \mu_u(B_t(x))^p\, dx$$

is of order $o(t^\varepsilon)$. Hence so is $R_v(t)$.

Now let $t = e^{-s}$, $f_v(s) = \phi_v(t)$ and $z_v(s) = R_v(t)$. Define the matrix $M = [\lambda_{u,v}]$ of measures as

$$\lambda_{u,v} = \sum_{e\in E_{u,v}} p(e)^p r(e)^{-\alpha_p} \delta_{-\ln r(e)}.$$

It follows that the matrix of variations

$$[\|\lambda_{u,v}\|] = [a_{u,v}]$$

is irreducible and has maximal eigenvalue 1 with eigenvector ϱ. Now (5.5) becomes

$$f_v(s) = \sum_{u\in V} \int_0^\infty f_u(s-s')\, d\lambda_{u,v}(s') + z_v(s)$$
$$= \sum_{u\in V} \int_0^s f_u(s-s')\, d\lambda_{u,v}(s') + z_v'(s) \quad \text{for } s \geq 0,$$

where $z_v'(s) := \sum_u \int_s^\infty f_u(s-s')\, d\lambda_{u,v}(s') + z_v(s)$. The fact that the $\lambda_{u,v}$'s have compact support implies that $\sum_u \int_s^\infty f_u(s-s')\, d\lambda_{u,v}(s') = 0$ for s large. Also $z_v(s) = o(e^{-\varepsilon s})$ as $s \to \infty$ for some positive number ε, hence each z_v' is directly Riemann integrable and Theorem 4.3 applies. By converting f_v back to ϕ_v, the theorem follows.

One of the main reasons for studying the L^p-density, in particular the L^2-density, of self-similar measures is to consider the asymptotic behavior of their Fourier transformation ([L1], [LW], [Str1,2,3]). Under the open set condition and by using a new form of Tauberian theorem, it was proved in [LW] that the average $(1/T^{n-\beta}) \int_{|\xi|\leq T} |\widehat{\mu}(\xi)|^2\, d\xi$ of the Fourier transformation of μ is asymptotically multiplicatively periodic as $T \to \infty$. Using the same argument, we have

THEOREM 5.2. *Under the condition of Theorem* 5.1, *there is a multiplicatively periodic function Q such that*

$$\lim_{T\to\infty} \Big(\frac{1}{T^{n-\beta}} \int_{|\xi|\leq T} |\widehat{\mu}_v(\xi)|^2\, d\xi - Q(r_{u,v}T)\varrho_v \Big) = 0,$$

and in case (i) *the above Q is a constant function.*

To conclude this section we discuss the case when the graph (V, E) is not strongly connected, which means that the matrix $[a_{u,v}]$ defined in (5.3) is reducible. A subgraph H is said to be a *strongly connected component* of (V, E) if it is a maximal strongly connected subgraph. The strongly connected components are pairwise disjoint.

Let α_p be the number such that the matrix $[a_{u,v}]$ in (5.3) has maximal eigenvalue 1. Define ϕ_v as in (5.4). For any strongly connected component

H of (V, E) the maximal eigenvalue of $[a_{u,v}]$, $u, v \in H$, is less than or equal to 1. There is at least one such component so that the maximal eigenvalue of the corresponding submatrix of $[a_{u,v}]$ is equal to 1. Let S be the set of vertices in all such components. Let $S_0 \subseteq S$ consist of vertices v that cannot be reached from any vertex of S in the other components. By Theorem 4.5, we have

THEOREM 5.3. *Let (V, E) be a connected multigraph and let $\{\mu_v\}$ be a self-similar family of measures satisfying the open set condition. Then*:

(i) *If $v \in S_0$, then*

$$\lim_{t\to 0^+} (\phi_v(t) - q_v(t)) = 0$$

for some multiplicatively periodic function $q_v > 0$.

(ii) *If $v \in S \setminus S_0$, then $\lim_{t\to 0^+} \phi_v(t) = \infty$.*

(iii) *If $v \notin S$ and there is no path from S to v, then $\lim_{t\to 0^+} \phi_v(t) = 0$.*

(iv) *If $v \notin S$ and there is a path from S_0 to v, but no path from $S \setminus S_0$ to v, then*

$$\lim_{t\to 0^+} (\phi_v(t) - q_v(t)) = 0$$

for some bounded multiplicatively periodic function $q_v > 0$.

(v) *If $v \notin S$ and there is a path from $S \setminus S_0$ to v, then $\lim_{t\to 0^+} \phi_v(t) = \infty$.*

As a direct corollary of the theorem, we have an analogue of Theorem 4 of [MW].

COROLLARY 5.4. *Let $\beta_p = \alpha_p/(p-1)$, where α_p is defined in (5.3). Then for $v \in S$, μ_v has L^p-dimension β_p. Moreover, each μ_v has finite (L^p, β_p)-density if and only if $S = S_0$, that is, if there is no path between two vertices of S which are in different components.*

Proof. For $v \in S_0$, it follows from Theorem 5.3(i) that μ_v has dimension β_p. For $v \in S \setminus S_0$, it follows from Theorem 5.3(ii) that the L^p-dimension of μ_v is not greater than β_p. On the other hand, for $\beta < \beta_p$, the matrix

$$[a'_{u,v}] := \Big[\sum_{e\in E_{u,v}} p(e)^p r(e)^{-\beta(p-1)}\Big], \quad u, v \in V,$$

has maximal eigenvalue less than 1. By Remark 4.4 and the proof of Theorem 5.3,

$$\limsup_{t\to 0^+} \frac{1}{t^{n+\beta(p-1)}} \int \mu_v(B_t(x))^p\, dx = 0.$$

This implies that the L^p-dimension of μ_v, $v \in S \setminus S_0$, is also β_p.

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Extension Gevrey et rigidité dans un secteur

par

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Abstract. We study a rigidity property, at the vertex of some plane sector, for Gevrey classes of holomorphic functions in the sector. For this purpose, we prove a linear continuous version of Borel–Ritt's theorem with Gevrey conditions.

0. Introduction. On sait qu'une fonction f holomorphe dans un voisinage V de 0 dans $\mathbb{C}$ est complètement déterminée par la donnée des $f^{(n)}(0)$, $n \in \mathbb{N}$. En fait, cette donnée peut être remplacée par celle des $f^{(n)}(z_n)$, $n \in \mathbb{N}$, où $(z_n)_{n\ge 0}$ est une suite convenable de points de V. Divers auteurs, dont J. A. Marti [Ma] dresse la liste, ont ainsi donné des conditions suffisantes très précises sur la suite $(z_n)_{n\ge0}$ pour que soit vérifiée cette propriété, dite de "rigidité". A titre d'exemple, lorsque V contient le disque unité fermé, le corollaire 2.8 de [Ma] stipule que la condition

$$(*)\qquad \sum_{p=1}^{\infty}\frac{(n+p)!}{n!p!}|z_n|^p \le C < 2 \quad \text{pour tout } n \text{ de } \mathbb{N}$$

est suffisante pour que la nullité des $f^{(n)}(z_n)$, $n\in\mathbb{N}$, implique celle de f. La clef du problème dans [Ma] consiste à montrer que, moyennant une condition comme $(*)$, les formes linéaires $L_n : f \to \frac{1}{n!}f^{(n)}(0)$ et $K_n : f \to \frac{1}{n!}f^{(n)}(z_n)$ sont "suffisamment proches" en un sens topologiquement adéquat et, comme conséquence, que l'orthogonalité par rapport aux K_n implique l'orthogonalité par rapport aux L_n.

Il est alors naturel de chercher à savoir ce qui se passe dans la situation suivante : 0 est maintenant un point du bord de l'ouvert V et on considère des fonctions f holomorphes dans V, de classe C^∞ sur $\overline{V}$, présentant encore la propriété d'être uniquement déterminées par la donnée des $f^{(n)}(0)$, $n\in\mathbb{N}$. Existe-t-il alors encore une notion de rigidité comme celle évoquée auparavant?

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