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Some algebras without submultiplicative norms or positive functionals

by

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Abstract. We prove a conjecture of Yood regarding the nonexistence of submultiplicative norms on the algebra $C(T)$ of all continuous functions on a topological space T which admits an unbounded continuous function. We also exhibit a quotient of $C(T)$ which does not admit a nonzero positive linear functional. Finally, it is shown that the algebra $L(X)$ of all linear operators on an infinite-dimensional vector space X admits no nonzero submultiplicative seminorm.

Introduction and results. Let $\mathcal{A}$ be an algebra over the field of complex numbers. A seminorm $\|\ \|$ on $\mathcal{A}$ will be called *submultiplicative* if it satisfies $\|ab\| \le \|a\|\,\|b\|$ for all elements $a, b \in \mathcal{A}$.

If T is a topological space, let $C(T)$ denote the algebra of all continuous complex-valued functions on T. If T is compact, and hence all functions $f \in C(T)$ are bounded, then the algebra $C(T)$ carries a submultiplicative norm, namely the usual uniform norm $\|f\| = \sup_{t\in T}|f(t)|$.

In [3] B. Yood gives a condition on T which ensures that the algebra $C(T)$ does not admit a submultiplicative norm, and conjectures that this is the case whenever $C(T)$ contains an unbounded function. This conjecture will be proven below.

THEOREM 1. *If the algebra $C(T)$ contains an unbounded function, then it does not admit a submultiplicative norm.*

The algebra $C(T)$ carries an involution, namely complex conjugation ($f^* = \bar{f}$). Let $I = C_{00}(T)$ be the ideal of all functions $f \in C(T)$ which have compact support in T. Then the ideal I is invariant under the involution of $C(T)$ and hence the quotient $\mathcal{A} = C(T)/I$ carries a unique involution for which the quotient map $Q : f \in C(T) \to f + I \in \mathcal{A}$ is a $*$-homomorphism.

THEOREM 2. *Assume that the Hausdorff space T satisfies the following condition: There exists a function $g_0 \in C(T)$ such that the set $K_n = \{t \in T :$*

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$|g_0(t)| \leq n\}$ *is compact for each* $n \geq 1$. *Then the quotient* $\mathcal{A} = C(T)/I$ *does not admit a nonzero positive linear functional.*

Remarks. The space $T = \mathbb{R}^n$ satisfies the assumption of Theorem 2. On the other hand, if the space T is countably compact, then every continuous function on T is bounded. Thus a noncompact, countably compact space T does not admit a function g_0 as above [2, 17.1 and 17.J].

A very general construction of involutive Banach algebras $\mathcal{A}$ such that all positive linear functionals on $\mathcal{A}$ vanish on $\mathcal{A}^2$ can be found in [1, page 202, Example 16].

If X is an infinite-dimensional vector space (over the complex numbers), let $L(X)$ denote the algebra of all linear maps from X to X.

THEOREM 3. *If* X *is an infinite-dimensional complex vector space, then the algebra* $L(X)$ *does not admit a nonzero submultiplicative seminorm.*

Proofs. Let $\mathcal{A}$ be an algebra over the complex numbers. An element $a \in \mathcal{A}$ will be called *weakly regular* if the two-sided ideal generated by a in $\mathcal{A}$ is the entire algebra $\mathcal{A}$, or equivalently, if a is not contained in any proper two-sided ideal in $\mathcal{A}$.

If $\mathcal{A}$ has an identity and if there exist elements $u, v \in \mathcal{A}$ such that $uav = 1$, then the element a is weakly regular in $\mathcal{A}$.

LEMMA 1. *If the complex algebra* $\mathcal{A}$ *contains elements* a, b_n, $n \geq 1$, *such that*

$$(1) \qquad b_n \neq 0 \quad \text{and} \quad ab_n = nb_n \quad \text{for all } n \geq 1,$$

then $\mathcal{A}$ *does not admit a submultiplicative norm. If in addition the* b_n *can be chosen to be weakly regular, then* $\mathcal{A}$ *does not admit a nonzero submultiplicative seminorm.*

Proof. Assume that $\|\ \|$ is a submultiplicative norm on $\mathcal{A}$. Then the relations (1) imply that $n\|b_n\| = \|nb_n\| = \|ab_n\| \leq \|a\|\,\|b_n\|$, and consequently $\|a\| \geq n$ for all $n \geq 1$, contradicting the finiteness of $\|a\|$.

Assume now that the b_n are in addition weakly regular and let α be any submultiplicative seminorm on $\mathcal{A}$. Let $J = \ker(\alpha)$. Then J is a two-sided ideal in $\mathcal{A}$. We must show that $J = \mathcal{A}$.

Assume on the contrary that J is a proper ideal in $\mathcal{A}$ and let $Q : \mathcal{A} \to \mathcal{A}/J$ denote the quotient map. The seminorm α induces a submultiplicative norm on the quotient $\mathcal{A}/J$. However, we also have $Q(a)Q(b_n) = nQ(b_n)$, $n \geq 1$, where the elements $Q(b_n) \in \mathcal{A}/J$ are nonzero, since the weakly regular elements b_n are not in the proper two-sided ideal $J \subseteq \mathcal{A}$. According to the first part of the lemma, the quotient $\mathcal{A}/J$ cannot carry a norm. This is the desired contradiction. ∎

Proof of Theorem 1. Let $\mathcal{A} = C(T)$. Passing to $|f|$ if necessary, we may assume that $\mathcal{A}$ contains an unbounded nonnegative function f. Inductively choose points $t_n \in T$ such that

$$f(t_{n+1}) > f(t_n) + 3 \quad \text{for all } n \geq 1.$$

Then the closed intervals $I_n = [f(t_n) - 1, f(t_n) + 1] \subseteq \mathbb{R}$, $n \geq 1$, are pairwise disjoint and consequently there exists a continuous function $h : \mathbb{R} \to \mathbb{R}$ such that $h(x) = n$ for all $x \in I_n$ and all $n \geq 1$. Moreover, for each $n \geq 1$, we can choose a continuous function $g_n : \mathbb{R} \to \mathbb{R}$ such that $g_n(f(t_n)) = 1$ and $\operatorname{supp}(g_n) \subseteq I_n$.

Now set $a = h \circ f \in \mathcal{A}$ and $b_n = g_n \circ f \in \mathcal{A}$ and note that $b_n \neq 0$, since $b_n(t_n) = 1$. Also, if $t \in T$ and $b_n(t) = g_n(f(t)) \neq 0$, then $f(t) \in \operatorname{supp}(g_n) \subseteq I_n$ and so $a(t) = h(f(t)) = n$. This implies that $ab_n = nb_n$ and Lemma 1 can now be applied. ∎

Proof of Theorem 3. Let X be an infinite-dimensional complex vector space and $L(X)$ the algebra of all linear operators on X. Note first that each surjective element $S \in L(X)$ is right-invertible in $L(X)$. Indeed, if $Z \subseteq X$ is any algebraic complement of the kernel of S in X, then the restriction $S|_Z : Z \to X$ is bijective and consequently its inverse $U = (S|_Z)^{-1}$ is an element of $L(X)$ which satisfies $SU = I$.

By splitting a Hamel basis B of X into countably many disjoint sets of the same cardinality as B, we obtain a decomposition

$$X = \bigoplus_{n \geq 1} X_n \quad \text{(algebraic direct sum)},$$

where each subspace X_n is isomorphic to X. Let $P_n : X \to X_n$ denote the projection onto X_n, according to this decomposition, and define the linear operator $A : X \to X$ by the condition $A = nP_n$ on the subspace X_n for all $n \geq 1$. Then $A, P_n \in L(X)$, $P_n \neq 0$ and $AP_n = nP_n$ for all $n \geq 1$.

By Lemma 1 it will now suffice to show that the projections P_n are weakly regular in $L(X)$. Since $X_n \cong X$, we can choose a linear surjection $Q_n : X_n \to X$. Then $Q_nP_n \in L(X)$ is a surjection and hence is right-invertible in $L(X)$. Thus there exists $U_n \in L(X)$ such that $Q_nP_nU_n$ is the identity of $L(X)$, and so the element $P_n \in L(X)$ is weakly regular. ∎

Proof of Theorem 2. For a function $f \in C(T)$ and a compact subset $K \subseteq T$ we shall write $\|f\|_K = \sup_{t \in K} |f(t)|$, as usual.

Recall that a linear functional μ on an algebra $\mathcal{A}$ with involution is called *positive* if it satisfies $\mu(a^*a) \geq 0$ for all $a \in \mathcal{A}$.

Let $C(T)_+$ denote the family of functions $f \in C(T)$ which satisfy $f(t) \geq 0$ for all $t \in T$. Such f satisfies $f = h^*h$, where $h = \sqrt{f} \in \mathcal{A}$, and consequently $\mu(f) \geq 0$ for each positive linear functional μ on $C(T)$.

Note that $C(T)$ is the linear span of $C(T)_+$. Consequently, each nonzero linear functional μ on $C(T)$ satisfies $\mu(f) \neq 0$ for some $f \in C(T)_+$. If in addition μ is positive, then we have $\mu(f) > 0$ for some $f \in C(T)_+$.

Let now $\mathcal{A} = C(T)/I$, where $I = C_{00}(T)$ is the ideal of functions $f \in C(T)$ with compact support. Assume that ω is a nonzero positive linear functional on $\mathcal{A}$. Since the quotient map $Q : C(T) \to \mathcal{A}$ is a $*$-homomorphism, the composition $\mu = \omega \circ Q$ is a nonzero positive linear functional on $C(T)$. Consequently, there exists an element $f \in C(T)_+$ such that $\mu(f) > 0$.

Let $g_0 \in C(T)$ and $K_n \subseteq T$ be as in the assumption of Theorem 2 and set $g = \max\{f, |g_0|\} \in C(T)$. Then $0 \le f(t) \le g(t)$ for all $t \in T$, and consequently $0 < \mu(f) \le \mu(g)$. Moreover, all the sets

$$C_n = \{t \in T : g(t) \le n\} \subseteq T, \quad n = 1, 2, \ldots,$$

are compact (C_n is a closed subset of K_n). Let now $n \ge 1$ be arbitrary. Clearly $x \ge n \Rightarrow x^2 \ge nx$ for each real number x. Thus the function $g^2 \in C(T)$ satisfies $g^2 \ge ng$, except possibly on the compact set $C_n \subseteq K_n$.

Now set $h_n = n\|g\|_{K_n} \max\{0, n+1-|g_0|\}$ and note that $h_n \in C(T)_+$, h_n has compact support (contained in the set K_{n+1}) and $h_n(t) \ge n\|g\|_{K_n}$ for all $t \in K_n$. Consequently, $Q(h_n) = 0$ and $g^2 + h_n \ge ng$ at each point of T. Thus

$$\omega(Q(g^2)) = \mu(g^2 + h_n) \ge \mu(ng) = n\mu(g).$$

Recall that $\mu(g) > 0$ and let $n \uparrow \infty$ to obtain the contradiction $\omega(Q(g^2)) = +\infty$. ∎

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Editorial note: See also the paper by Alexander R. Pruss in this issue.

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Sur la caractérisation topologique des compacts à l'aide des demi-treillis des pseudométriques continues

par

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Abstract. For a Tikhonov space X we denote by $\mathrm{Pc}(X)$ the semilattice of all continuous pseudometrics on X. It is proved that compact Hausdorff spaces X and Y are homeomorphic if and only if there is a positive-homogeneous (or an additive) semi-lattice isomorphism $T : \mathrm{Pc}(X) \to \mathrm{Pc}(Y)$.

A topology on $\mathrm{Pc}(X)$ is called admissible if it is intermediate between the compact-open and pointwise topologies on $\mathrm{Pc}(X)$. Another result states that Tikhonov spaces X and Y are homeomorphic if and only if there exists a positive-homogeneous (or an additive) semi-lattice homeomorphism $T : (\mathrm{Pc}(X), \tau_X) \to (\mathrm{Pc}(Y), \tau_Y)$, where τ_X, τ_Y are admissible topologies on $\mathrm{Pc}(X)$ and $\mathrm{Pc}(Y)$.

Des résultats caractérisant un espace compact X à l'aide de l'espace $C(X)$ des fonctions continues sont bien connus et classiques. Rappelons ici le théorème de I. M. Gelfand et A. N. Kolmogorov [GK], affirmant qu'un espace compact X est déterminé complètement par l'anneau $C(X)$ des fonctions continues, ou le théorème de Banach–Stone [Ba, XI, §4] caractérisant un espace compact X au moyen de l'espace de Banach $C(X)$. Il s'avère (voir [Se, 7.8.2]) que la caractérisation de Gelfand–Kolmogorov résulte du théorème de I. Kaplansky, qui a démontré dans [Ka] que le treillis $C(X)$ des fonctions continues détermine complètement un espace compact X. Dans [Sh] T. Shirota a généralisé ce théorème de I. Kaplansky en montrant qu'il reste vrai pour les espaces Hewitt-complets; entre autres, il a démontré dans [Sh] que le treillis $C(X)$ muni d'une topologie intermédiaire entre la topologie de la convergence simple et la topologie compacte-ouverte determinait complètement un espace de Tikhonov X.

Le but de cet article est d'obtenir des résultats analogues caractérisant un espace compact (ou de Tikhonov) X à l'aide de l'espace $\mathrm{Pc}(X)$ formé de toutes les pseudométriques continues sur X. L'espace $\mathrm{Pc}(X)$ avec l'ordre

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