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The one-sided minimal operator and the one-sided reverse Hölder inequality

by

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Abstract. We introduce the one-sided minimal operator, $\mathrm{m}^+ f$, which is analogous to the one-sided maximal operator. We determine the weight classes which govern its two-weight, strong and weak-type norm inequalities, and show that these two classes are the same. Then in the one-weight case we use this class to introduce a new one-sided reverse Hölder inequality which has several applications to one-sided (A_p^+) weights.

1. Introduction. In our papers [1] and [2] we introduced a new operator, the minimal operator, so named since it is analogous to the Hardy–Littlewood maximal operator. Given a measurable function f, define the minimal function of f, $\mathrm{m}f$, by

$$\mathrm{m}f(x) = \inf_I \frac{1}{|I|} \int_I |f|\,dy,$$

where the infimum is taken over all cubes I with sides parallel to the coordinate axes which contain x. In [1] we used the minimal operator to study the structure of functions which satisfy the reverse Hölder inequality; in [2] we considered the weighted norm inequalities which hold for the minimal operator, and applied this to the problem of differentiability of the integral.

The maximal operator, as originally defined by Hardy and Littlewood, was a one-sided maximal operator on $\mathbb{R}$ (see [4]). The weighted norm inequalities for the one-sided maximal operator were first considered by Sawyer [10] and then by Martín-Reyes and others [5]–[8]. In light of this we define a one-sided minimal operator.

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DEFINITION 1.1. If f is a measurable function on $\mathbb{R}$, then define the *(forward) one-sided minimal function*, m^+f, by

$$\mathrm{m}^+f(x) = \inf_{t>0} \frac{1}{t} \int_x^{x+t} |f|\,dy.$$

We define the *backward one-sided minimal function* m^-f similarly.

In this paper we examine the weighted norm inequalities for the one-sided minimal operator. All of our results are similar to those which hold for the minimal operator, but the proofs are generally more difficult. The material is organized as follows:

Section 2 contains two preliminary results which are used repeatedly in later sections.

In Section 3 we show that for $p > 0$, the weak-type norm inequality

$$u(\{x : \mathrm{m}^+f(x) < 1/t\}) \le \frac{C}{t^p} \int_{\mathbb{R}} \frac{v}{|f|^p}\,dx$$

holds if and only if the pair of weights (u, v) satisfies the (W_p^+) condition: there exists a constant C such that, given any interval $I = [a, b]$,

$$\frac{1}{|I^-|} \int_{I^-} u\,dx \le C\bigg(\frac{1}{|I|} \int_I v^{1/(p+1)}\,dx\bigg)^{p+1},$$

where $I^- = [a, c]$ is such that $2|I^-| = |I|$.

In Section 4 we show that for $p > 0$, the strong-type norm inequality

$$\int_{\mathbb{R}} \frac{u}{(\mathrm{m}^+f)^p}\,dx \le C \int_{\mathbb{R}} \frac{v}{|f|^p}\,dx$$

holds if and only if the pair of weights (u, v) satisfies the $(W_p^+)^*$ condition: there exists a constant C such that, given any interval I,

$$\int_I \frac{u}{\mathrm{m}^+(\sigma/\chi_I)^p}\,dx \le C \int_I \sigma\,dx,$$

where $\sigma = v^{1/(p+1)}$.

In Section 5 we show that the two classes (W_p^+) and $(W_p^+)^*$ are the same. This is a striking difference between the one-sided minimal operator and the one-sided maximal operator, since for the maximal operator the corresponding classes are not equal [8].

In Section 6 we consider the special case when $u = v$. We begin by defining a one-sided reverse Hölder inequality: a weight w is in (RH_s^+), $s > 1$, if there exists a constant C such that, for every interval $I = [a, b]$,

$$\bigg(\frac{1}{|I^-|} \int_{I^-} w^s\,dx\bigg)^{1/s} \le \frac{C}{|I|} \int_I w\,dx,$$

where again $I^- = [a, c]$ is such that $2|I^-| = |I|$. We show that this condition is equivalent to a "weak" reverse Hölder condition defined by Martín-Reyes, Pick and de la Torre [7]. They showed that their inequality characterized the class (A_∞^+), which is the union of the (A_p^+) classes. The latter classes of weights control the one-weight, weighted norm inequalities for the one-sided maximal operator. Using this characterization we show that in the one-weight case, each (W_p^+) class, $p > 0$, is equal to (A_∞^+). Further, by using the one-sided reverse Hölder condition we give new proofs of several properties of (A_p^+) weights, and thereby simplify the proof of the one-weight strong-type norm inequality for the one-sided maximal operator. Doing so answers a question posed by Martín-Reyes [5].

2. Notation and preliminary results. Throughout this paper all notation is standard or will be defined as necessary. All weight functions are assumed to be locally integrable. Given an interval I and a non-negative function v, let $|I|$ denote the length of I, and define $v(I) = \int_I v\,dx$ and $I(v) = v(I)/|I|$. Finally, given $p > 1$, $p' = p/(p-1)$ denotes the conjugate exponent of p, and the letter C denotes a positive constant whose value may be different at each appearance.

The next result is a technical lemma due to Muckenhoupt [9]. For convenience we repeat its short proof.

LEMMA 2.1. *Given a function σ and an interval I, let $\{I_\alpha\}$ be a collection of intervals contained in I such that, for each α, $\int_{I_\alpha} \sigma\,dx \le N|I_\alpha|$. If $J = \bigcup_\alpha I_\alpha$, then $\int_J \sigma\,dx \le 2N|J|$. A similar result holds if we reverse the inequalities and replace $2N$ by $N/2$ in the conclusion.*

Proof. We prove only the first half of the result; the second is proved in almost exactly the same way. Fix $\varepsilon > 0$ and let $\delta > 0$ be such that $\int_E \sigma\,dx < \varepsilon$ whenever $|E| < \delta$. Then there exists a finite subcollection $I_1, \ldots, I_n$ of the I_α's such that

$$|J_n| \equiv \Big|\bigcup_{k=1}^n I_k\Big| > |J| - \delta$$

and no point is contained in more than two of the I_k's. Then

$$\int_J \sigma\,dx = \int_{J\setminus J_n} \sigma\,dx + \int_{J_n} \sigma\,dx$$
$$\le \varepsilon + \sum_{k=1}^{n} \int_{I_k} \sigma\,dx \le \varepsilon + N\sum_{k=1}^{n}|I_k| \le \varepsilon + 2N|J|.$$

Since ε is arbitrary we are done. ∎

In later sections we are going to need the following decomposition of finite intervals. It is similar to the Whitney decomposition of open sets in $\mathbb{R}$.

DEFINITION 2.2. Given a finite interval $I = [a,b]$ we form the *plus-minus decomposition* of I as follows: let $x_0 = a$, and for $k > 0$ define $x_k = (b + x_{k-1})/2$. Then for $k \ge 1$, define the intervals $J_k = [x_{k-1}, x_{k+1}]$, $J_k^- = [x_{k-1}, x_k]$ and $J_k^+ = [x_k, x_{k+1}]$.

It is immediate from this definition that for all k, $|J_k^-| = 2|J_k^+|$, that I is the union of the J_k^-'s, and that the J_k's have finite overlap.

3. Weak-type norm inequalities. For completeness, we repeat the definition of the (W_p^+) condition.

DEFINITION 3.1. The pair of non-negative weights (u, v) satisfies the (W_p^+) *condition*, $p > 0$, if there exists a constant C such that, given any interval $I = [a,b]$,

$$\frac{1}{|I^-|}\int_{I^-} u\,dx \le C\bigg(\frac{1}{|I|}\int_I v^{1/(p+1)}\,dx\bigg)^{p+1},$$

where $I^- = [a,c]$ is such that $2|I^-| = |I|$.

In order to prove the weak-type inequality, we first need to show that the (W_p^+) condition is actually equivalent to a seemingly stronger condition.

LEMMA 3.2. *Given $p > 0$, a pair of weights (u,v) is in (W_p^+) if and only if there exists a constant C such that, given λ, $0 < \lambda < 1$, and given any interval $I = [a,b]$ and the subinterval $I^- = [a,c]$ with $|I^-|/|I| = \lambda$,*

$$(1)\qquad \frac{1}{|I^-|}\int_{I^-} u\,dx \le \frac{C}{\lambda(1-\lambda)^p}\bigg(\frac{1}{|I|}\int_I \sigma\,dx\bigg)^{p+1},$$

where $\sigma = v^{1/(p+1)}$.

Proof. If $\lambda = 1/2$ this is the (W_p^+) condition. Therefore it will suffice to prove that the (W_p^+) condition implies this condition. If $\lambda < 1/2$ this is immediate. Therefore we may assume that $1/2 < \lambda < 1$. Fix λ and let $n > 1$ be the least integer such that $\lambda \le (2^n - 1)/2^n$. Now fix the intervals I and I^- as above. Form the plus-minus decomposition of I described in Definition 2.2. Then I^- is contained in the union of J_k^-'s, $1 \le k \le n$. By the (W_p^+) condition,

$$\int_{J_k^-} u\,dx \le C|J_k|\bigg(\frac{1}{2|J_k|}\int_I \sigma\,dx\bigg)^{p+1}.$$

Therefore,

$$\int_J u\,dx \le \sum_{k=1}^{n}\int_{J_k^-} u\,dx \le C|I|\sum_{k=1}^{n} 2^{p(k-1)}\bigg(\frac{1}{|I|}\int_I \sigma\,dx\bigg)^{p+1}$$
$$\le C|I^-|2^{(n-1)p}\bigg(\frac{1}{|I|}\int_I \sigma\,dx\bigg)^{p+1}.$$

Since by our choice of n, $2^{1-n} \ge 1 - \lambda$, we are done. ∎

In passing, note that the same proof shows that if the pair (u,v) satisfies inequality (1) for some ratio λ then (u,v) is in (W_p^+).

THEOREM 3.3. *For $p > 0$, the pair of weights (u,v) satisfies the (W_p^+) condition if and only if there exists a constant C such that the weak-type inequality*

$$u(\{x : \mathrm{m}^+ f(x) < 1/t\}) \le \frac{C}{t^p}\int_{\mathbb{R}} \frac{v}{|f|^p}\,dx$$

holds for all f such that $1/f$ is in $L^p(v)$.

Proof. Suppose first that the weak-type inequality holds. Fix an interval I and let $f = \sigma/\chi_I$, where $\sigma = v^{1/(p+1)}$. Partition I into adjacent intervals I^- and I^+ of equal length, I^- to the left of I^+. Then for any $x \in I^-$,

$$\mathrm{m}^+ f(x) \le \frac{1}{|I^+|}\int_I \sigma\,dx.$$

Let $1/t$ equal the right-hand side. Then

$$u(I^-) \le u(\{x : \mathrm{m}^+ f(x) < 1/t\})$$
$$\le \frac{C}{|I^+|^p}\bigg(\int_I \sigma\,dx\bigg)^p\bigg(\int_I \frac{v}{\sigma^p}\,dx\bigg) = \frac{C}{|I|^p}\bigg(\int_I \sigma\,dx\bigg)^{p+1}.$$

Since C is independent of I, (u,v) is in (W_p^+).

Conversely, suppose that (u,v) is in (W_p^+). We will first consider the special case where f is such that $1/f$ has compact support. Fix $t > 0$ and let $E_t = \{x : \mathrm{m}^+ f(x) < 1/t\}$. Since $\mathrm{m}^+ f$ is upper semicontinuous, E_t is open and so is the union of disjoint open intervals I_j, $j \ge 1$. Since $1/f$ has compact support, each of the I_j's is bounded. Fix j and form the plus-minus decomposition of I_j described in Definition 2.2. We now claim that for each k,

$$\frac{1}{|J_k^+|}\int_{J_k}|f|\,dx\le 8/t. \tag{2}$$

If this is true then by Lemma 3.2,

$$u(I_j)=\sum_k u(J_k^-)\le\frac{C}{t^p}\sum_k\frac{u(J_k^-)}{J_k(|f|)^p}\le\frac{C}{t^p}\sum_k\sigma(J_k)^{p+1}\Big(\int_{J_k}|f|\,dx\Big)^{-p}.$$

By Hölder's inequality,

$$\int_{J_k}\sigma\,dx=\int_{J_k}\frac{\sigma}{|f|^{p/(p+1)}}\cdot|f|^{p/(p+1)}\,dx \le\Big(\int_{J_k}\frac{v}{|f|^p}\,dx\Big)^{1/(p+1)}\Big(\int_{J_k}|f|\,dx\Big)^{p/(p+1)}.$$

Therefore,

$$u(I_j)\le\frac{C}{t^p}\sum_k\int_{J_k}\frac{v}{|f|^p}\,dx\le\frac{C}{t^p}\int_{I_j}\frac{v}{|f|^p}\,dx,$$

the last inequality holding since the J_k's have finite overlap. If we take the sum over all j we get the desired inequality.

It therefore remains to show that (2) holds. To do this we will first show that

$$\frac{1}{b-x_{k-1}}\int_{x_{k-1}}^{b}|f|\,dx\le 2/t. \tag{3}$$

To see this, note that if x is in (x_{k-1},b) then x is in E_t, so there exists a point y such that

$$\frac{1}{y-x}\int_x^y|f|\,dx\le 1/t.$$

If $y>b$ then, since b is not in E_t, it is easy to see that the same inequality holds if we replace y by b. Therefore, we may suppose that $y\le b$. But if this is true for every such x, then by Lemma 2.1, we have inequality (3). Inequality (2) now follows at once since $b-x_{k-1}=2|J_k^-|=4|J_k^+|$.

To complete the proof we now consider an arbitrary f. Define the sequence of functions $f_n=f/\chi_{[-n,n]}$. Then the weak-type inequality holds for each f_n. Now the sequence $\{f_n\}$ decreases to f. Further, the sequence $\{\mathfrak{m}^+f_n\}$ is also decreasing and $\mathfrak{m}^+f\le\lim\mathfrak{m}^+f_n$. To see that equality holds, fix x in $\mathbb{R}$ and $\varepsilon>0$. Then there exists an interval $I=[x,y]$ such that for all n sufficiently large,

$$\mathfrak{m}^+f(x)\ge I(|f|)-\varepsilon\ge\mathfrak{m}^+f_n(x)-\varepsilon.$$

Therefore, by the monotone convergence theorem, the weak-type inequality holds for f and we are done. ■

4. Strong-type norm inequalities. We begin by restating the $(W_p^+)^*$ condition.

DEFINITION 4.1. The pair of non-negative weights (u,v) satisfies the $(W_p^+)^*$ *condition*, $p>0$, if there exists a constant C such that, given any interval I,

$$\int_I\frac{u}{\mathfrak{m}^+(\sigma/\chi_I)^p}\,dx\le C\int_I\sigma\,dx,$$

where $\sigma=v^{1/(p+1)}$.

The next proof follows that of the strong-type inequality for the one-sided maximal operator given by Martín-Reyes, Ortega Salvador and de la Torre in [6].

THEOREM 4.2. *For $p>0$, the pair of weights (u,v) satisfies the $(W_p^+)^*$ condition if and only if there exists a constant C such that the strong-type inequality*

$$\int_{\mathbb{R}}\frac{u}{(\mathfrak{m}^+f)^p}\,dx\le C\int_{\mathbb{R}}\frac{v}{|f|^p}\,dx$$

holds for all f such that $1/f$ is in $L^p(v)$.

Proof. The $(W_p^+)^*$ condition follows immediately from the strong-type inequality if we fix an interval I and let $f=\sigma/\chi_I$, where $\sigma=v^{1/(p+1)}$.

To prove the converse, suppose that (u,v) is in $(W_p^+)^*$. It will suffice to prove the strong-type inequality for v everywhere positive; for if (u,v) is in $(W_p^+)^*$ then so is $(u,v+\varepsilon)$, $\varepsilon>0$, and the strong-type inequality would follow for (u,v) by letting ε tend to zero. Further, by an argument identical to that at the end of the proof of Theorem 3.3, we may assume that f is such that $1/f$ has compact support.

Fix a function f and, for each k in $\mathbb{Z}$, define

$$O_k=\{x:\mathfrak{m}^+f(x)<1/2^k\}.$$

Each set O_k is open, so $O_k=\bigcup_j I_{jk}$, where the I_{jk}'s are disjoint open intervals. Since $1/f$ has compact support, the I_{jk}'s are uniformly bounded in length. Then by the proof of inequality (3) in Theorem 3.3, each interval $I_{jk}=(a_{jk},b_{jk})$ has the property that if x is in I_{jk} then

$$\frac{1}{b_{jk}-x}\int_x^{b_{jk}}|f|\,dy\le 2^{-k+1}.$$

For each j and k, define

$$E_{jk} = \{x : \mathrm{m}^+ f(x) \ge 2^{-k-1}\} \cap I_{jk}.$$

The sets E_{jk} are pairwise disjoint and

$$\bigcup_j E_{jk} = \{x : 2^{-k-1} \le \mathrm{m}^+ f(x) < 2^{-k}\} \equiv E_k.$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}} \frac{u}{(\mathrm{m}^+ f)^p}\,dx &= \sum_k \int_{E_k} \frac{u}{(\mathrm{m}^+ f)^p}\,dx \\ &= \sum_k \sum_j \int_{E_{jk}} \frac{u}{(\mathrm{m}^+ f)^p}\,dx \le 2^p \sum_k \sum_j \int_{E_{jk}} 2^{kp} u\,dx \\ &\le 4^p \sum_k \sum_j \int_{E_{jk}} \bigg(\frac{1}{b_{jk}-x}\int_x^{b_{jk}} |f|\,dy\bigg)^{-p} u\,dx.\end{aligned}$$

To estimate the last term, let $X = \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}$ and let ω be the measure defined on X by $\nu \times \nu \times m$, where ν is the counting measure and m is Lesbegue measure. Now define the function ϕ on X by

$$\phi(j,k,x) = \chi_{E_{jk}}(x) u(x) \bigg(\frac{1}{b_{jk}-x}\int_x^{b_{jk}} \sigma\,dy\bigg)^{-p},$$

and define the operators S and T on $L^2(\sigma)$ by

$$Sh(j,k,x) = \frac{\int_x^{b_{jk}} \sigma\,dy}{\int_x^{b_{jk}} h\sigma\,dy} \quad \text{and} \quad Th(j,k,x) = \frac{\int_x^{b_{jk}} h\sigma\,dy}{\int_x^{b_{jk}} \sigma\,dy} \quad \text{if } x \in E_{jk},$$

and 0 otherwise. Note that by Hölder's inequality, we have $Sh(j,k,x) \le T(h^{1-r'})(j,k,x)^{r-1}$ for any $r > 1$. Suppose that T is a bounded operator from $L^2(\sigma)$ to $L^2(X, \phi d\omega)$. Then if we let $r = 1 + 2/p$, the last term of the above inequality is bounded by

$$\begin{aligned}4^p \int_X S(|f|/\sigma)^p \phi\,d\omega &\le 4^p \int_X T(\sigma^{r'-1}/|f|^{r'-1})^2 \phi\,d\omega \\ &\le 4^p C \int \frac{\sigma^p}{|f|^p}\sigma\,dx = 4^p C \int \frac{v}{|f|^p}\,dx,\end{aligned}$$

which is the desired inequality. It therefore remains to show that T is bounded. To do this we will apply the Marcinkiewicz interpolation. But T is clearly bounded on L^∞, so we need only show that T is weak $(1,1)$, that is, T satisfies

$$\int_{\{|Th|>\lambda\}} \phi\,d\omega \le \frac{C}{\lambda} \int h\sigma\,dx.$$

To prove this, let $A_{jk}(\lambda) = \{x : Th(j,k,x) > \lambda\} \cap E_{jk}$ and let $s_{jk}(\lambda) = \inf A_{jk}(\lambda)$. If we define

$$J_{jk} = J_{jk}(\lambda) = [s_{jk}(\lambda), b_{jk}),$$

then J_{jk} and J_{lm} are either disjoint or one is contained in the other. Furthermore, by the definition of $s_{jk}(\lambda)$,

$$\frac{1}{\sigma(J_{jk})} \int_{J_{jk}} h\sigma \ge \lambda. \tag{4}$$

Let $\{J_i\}$ be the maximal elements of the family $\{J_{jk}\}$. These maximal elements exist since $\sup |I_{jk}| < \infty$ and so the intervals J_{jk} have uniformly bounded lengths. Obviously, inequality (4) holds for each J_i. Further, by their maximality, the J_i's are disjoint. Therefore

$$\begin{aligned}\int_{\{|Th|>\lambda\}} \phi\,d\omega &= \sum_k \sum_j \int_{A_{jk}(\lambda)} \bigg(\frac{1}{b_{jk}-x}\int_x^{b_{jk}} \sigma\,dy\bigg)^{-p} u\,dx \\ &= \sum_i \sum_{\{(k,j): J_{jk} \subset J_i\}} \int_{A_{jk}(\lambda)} \bigg(\frac{1}{b_{jk}-x}\int_x^{b_{jk}} \sigma\,dy\bigg)^{-p} u\,dx \\ &\le \sum_i \int_{J_i} \frac{u}{\mathrm{m}^+(\sigma/\chi_{J_i})^p}\,dx \le C \sum_i \sigma(J_i) \\ &\le \frac{C}{\lambda} \sum_i \int_{J_i} h\sigma\,dx \le \frac{C}{\lambda} \int_{\mathbb{R}} h\sigma\,dx.\end{aligned}$$

(The second inequality is the $(W_p^+)^*$ condition.) Thus T is weak $(1,1)$ and the proof is complete. ■

5. The equivalence of the (W_p^+) and $(W_p^+)^*$ conditions. In this section we show that the two classes (W_p^+) and $(W_p^+)^*$ are the same. The proof requires two lemmas.

LEMMA 5.1. *Suppose (u,v) is in (W_p^+). Then there exists a constant C such that, given an interval $I = [a,b]$,*

$$\int_{I^-} \frac{u}{\mathrm{m}^+(\sigma/\chi_I)^p}\,dx \le C\sigma(I^- \cup I^+),$$

where $I^- = [a,c]$ and $I^+ = [c,d]$ are such that $2|I^-| = |I|$ and $2|I^+| = |I^-|$.

Proof. For each $t > 0$ define the set

$$E_t = \{x \in I^- : \mathrm{m}^+(\sigma/\chi_I)(x) < 1/t\}.$$

Then

$$\int_{I^-} \frac{u}{\mathfrak{m}^+(\sigma/\chi_I)^p}\,dx = p\int_0^\infty t^{p-1}u(E_t)\,dt = \int_0^r + \int_r^\infty,$$

where r is a constant to be chosen below. Denote the first integral on the right-hand side by α and the second by β. Then we immediately have the estimate $\alpha \le u(I^-)r^p$. To estimate β, note that the set E_t is open, and so it is the union of disjoint intervals I_j. Fix j and form the plus-minus decomposition of I_j described in Definition 2.2. Then

$$u(I_j) = \sum_{k=1}^\infty u(J_k^-) \le C\sum_{k=1}^\infty \sigma(J_k)\left(\frac{\sigma(J_k)}{|J_k|}\right)^p \le C\sum_{k=1}^\infty \frac{|J_k|}{t^{p+1}} \le \frac{C}{t^{p+1}}|I_j|.$$

The first inequality follows from the (W_p^+) condition and Lemma 3.2, the second from inequality (2) of Theorem 3.3, and the third since the J_k's have finite overlap. Therefore, if we sum over all j we see that $u(E_t) \le C|E_t|/t^{p+1} \le C|I^-|/t^{p+1}$. Hence

$$\beta \le \frac{C|I^-|}{r}.$$

Let

$$r^p = \frac{\sigma(I^- \cup I^+)}{u(I^-)};$$

then by the (W_p^+) condition (and Lemma 3.2),

$$r \ge \frac{C|I^-|}{\sigma(I^- \cup I^+)}.$$

If we combine these with the above estimates we get the desired inequality. ∎

LEMMA 5.2. *Fix $p > 0$ and let (u, v) be in (W_p^+). Then if $\{K_i\}$ is a sequence of nested intervals such that $K_{i+1} \subset K_i$ for all $i \ge 0$, and $|K_i|$ tends to zero as i tends to infinity, then*

$$\lim_{i\to\infty} \int_{K_i} \frac{u}{\mathfrak{m}^+(\sigma/\chi_{K_i})^p}\,dx = 0.$$

Proof. Since the K_i's are nested, $\mathfrak{m}^+(\sigma/\chi_{K_i}) \le \mathfrak{m}^+(\sigma/\chi_{K_{i+1}})$, and since $|K_i|$ tends to zero, $1/\mathfrak{m}^+(\sigma/\chi_{K_i})$ tends to zero almost everywhere in K_0. From Lemma 5.1 we see that $u/\mathfrak{m}^+(\sigma/\chi_{K_0})^p$ is integrable on K_0 (let $I^- = K_0$), so by the dominated convergence theorem,

$$\lim_{i\to\infty} \int_{K_0} \frac{u}{\mathfrak{m}^+(\sigma/\chi_{K_i})^p}\,dx = 0.$$

The conclusion follows at once. ∎

THEOREM 5.3. *The pair (u, v) is in (W_p^+) if and only if it is in $(W_p^+)^*$.*

Proof. Clearly it will suffice to show that if (u, v) is in (W_p^+) then it is in $(W_p^+)^*$. Fix an interval I and form the plus-minus decomposition of I described in Definition 2.2. Additionally, let J_k' be the interval whose left and right endpoints are the right endpoints of J_k^- and I. Then we have the estimate

$$\int_I \frac{u}{\mathfrak{m}^+(\sigma/\chi_I)^p}\,dx \le \int_{J_1^-} + \int_{J_1'}.$$

By Lemma 5.1, the first integral is bounded by $C\sigma(J_1)$, where C is some constant that depends only on (u, v). Since $\mathfrak{m}^+$ is calculated only using forward averages, the second integral is equal to

$$\int_{J_1'} \frac{u}{\mathfrak{m}^+(\sigma/\chi_{J_1'})^p}\,dx.$$

Now $J_1' = J_2^- \cup J_2'$, so we may repeat this argument for this integral. Hence, by induction we get for all $n \ge 0$ the estimate

$$\int_I \frac{u}{\mathfrak{m}^+(\sigma/\chi_I)^p}\,dx \le C\sum_{k=1}^n \sigma(J_k) + \int_{J_n'} \frac{u}{\mathfrak{m}^+(\sigma/\chi_{J_n'})^p}\,dx.$$

By Lemma 5.2 the last term tends to zero as n tends to infinity. Therefore, since the J_k's have finite overlap, we have

$$\int_I \frac{u}{\mathfrak{m}^+(\sigma/\chi_I)^p}\,dx \le C\sum_{k=1}^\infty \sigma(J_k) \le C\sigma(I).$$

Hence (u, v) is in $(W_p^+)^*$. ∎

6. One-weight norm inequalities and the one-sided reverse Hölder inequality.

In this section we consider the special case of weights w such that (w, w) is in (W_p^+). We will show that such weights are related to the (A_p^+) weights, which control the weighted norm inequalities for the one-sided maximal operator.

In [7], Martín-Reyes, Pick and de la Torre defined the class of weights (A_∞^+).

DEFINITION 6.1. A function w is in (A_∞^+) if there exist positive constants C and δ such that, given two adjacent intervals I^- and I^+, I^- to the left of I^+, and given a measurable subset E of I^+, we have

$$\frac{|E|}{|I^- \cup I^+|} \le C\left(\frac{w(E)}{w(I^-)}\right)^\delta.$$

This is not the obvious one-sided analogue of the (A_∞) condition. Using it, however, the authors of [7] proved the following result.

THEOREM 6.2. *Given a weight w, the following are equivalent:*

(1) *w is in (A_∞^+).*

(2) *There exists p, $1 < p < \infty$, such that w is in (A_p^+).*

(3) *w satisfies a weak reverse Hölder inequality: there exist constants C and δ such that for every interval $I = [a,b]$,*

$$\int_I w^{1+\delta}\,dx \le C \int_I w\,dx \cdot M(w\chi_I)(b)^\delta.$$

(4) *There exist constants C and δ such that, given two adjacent intervals I^- and I^+, I^- to the left of I^+, and given a measurable subset E of I^-, we have*

$$\frac{w(E)}{w(I^-\cup I^+)} \le C\left(\frac{|E|}{|I^+|}\right)^\delta.$$

We want to replace the weak reverse Hölder condition of Theorem 6.2 by the one-sided reverse Hölder condition given above. For convenience we repeat the definition.

DEFINITION 6.3. A weight w is in (RH_s^+), $s > 1$, if there exists a constant C such that, for every interval $I = [a,b]$,

$$\left(\frac{1}{|I^-|}\int_{I^-} w^s\,dx\right)^{1/s} \le \frac{C}{|I|}\int_I w\,dx,$$

where $I^- = [a,c]$ is such that $2|I^-| = |I|$.

To prove Theorem 6.5 below we need a lemma which is a special case of Lemma 3.2.

LEMMA 6.4. *Given $s > 1$, a weight w is in (RH_s^+) if and only if there exists a constant C such that, given λ, $0 < \lambda < 1$, and given any interval $I = [a,b]$ and the subinterval $I^- = [a,c]$ with $|I^-|/|I| = \lambda$, we have*

$$\left(\frac{1}{|I^-|}\int_{I^-} w^s\,dx\right)^{1/s} \le \frac{C}{\lambda^{1/s}(1-\lambda)^{1/s'}}\frac{1}{|I|}\int_I w\,dx.$$

THEOREM 6.5. *A function w satisfies the weak reverse Hölder condition (3) of Theorem 6.2 with constant δ if and only if it is in $(RH_{1+\delta}^+)$.*

Proof. First suppose that w is in $(RH_{1+\delta}^+)$ for some δ. Fix an interval $I = [a,b]$ and form the plus-minus decomposition of I described in Definition 2.2. Additionally, let I_k be the interval whose left endpoint is the left endpoint of J_k and whose right endpoint is the right endpoint of I. Then

$|I_k| = 2|J_k|$. Therefore, by the $(RH_{1+\delta}^+)$ condition and Lemma 6.4,

$$\begin{aligned}\int_I w^{1+\delta}\,dx &= \sum_{k=1}^\infty \int_{J_k^-} w^{1+\delta}\,dx \le C\sum_{k=1}^\infty \int_{J_k} w\,dx\left(\frac{1}{|J_k|}\int_{J_k} w\,dx\right)^\delta\\ &\le C\sum_{k=1}^\infty \int_{J_k} w\,dx\left(\frac{1}{|I_k|}\int_{I_k} w\,dx\right)^\delta\\ &\le C\sum_{k=1}^\infty \int_{J_k} w\,dx\cdot M(w\chi_I)(b)^\delta \le C\int_I w\,dx\cdot M(w\chi_I)(b)^\delta.\end{aligned}$$

Hence w satisfies condition (3) with constant δ.

Conversely, suppose that w satisfies this weak reverse Hölder condition. Again fix an interval $I = [a,b]$ and partition it into adjacent intervals I^+ and I^- of equal length, I^- to the left of I^+. For each point y in I^+, let $I_y = [a,y]$ and define

$$M = \inf_{y\in I^+} M(w\chi_{I_y})(y).$$

If $M = 0$ then w must be equal to zero a.e. on I^-, so there is nothing to prove. Therefore we may assume that $M > 0$. For each point y there exists an interval J_y containing y and contained in I_y such that $M \le 2J_y(w)$. But then by Lemma 2.1 we must have $M \le CI(w)$. Now fix a point y such that $M(w\chi_{I_y})(y) \le 2M$. Then by condition (3),

$$\begin{aligned}\int_{I^-} w^{1+\delta}\,dx \le \int_{I_y} w^{1+\delta}\,dx &\le C\int_{I_y} w\,dx\cdot(2M)^\delta\\ &\le C\int_I w\,dx\left(\frac{1}{|I|}\int_I w\,dx\right)^\delta.\end{aligned}$$

Hence w is in $(RH_{1+\delta}^+)$. ■

The relationship between the (W_p^+) classes and (A_∞^+) follows at once from the next lemma, which extends a result of Strömberg and Wheeden [11] to the one-sided case.

LEMMA 6.6. *A weight w is in (RH_s^+) if and only if w^s is in (A_∞^+).*

Proof. Suppose first that w is in (RH_s^+). To prove that w^s is in (A_∞^+) we will use another condition due to Martín-Reyes, Pick and de la Torre [7] which is equivalent to the (A_∞^+) condition. It will suffice to show that there is a constant C such that if $I = [a,d]$ is any interval, and $I^- = [a,b]$ and

$I^+ = [c,d]$ are subintervals such that $b \le c$ and $|I^-| = |I^+| = |I|/4$, then

$$\frac{1}{|I^-|}\int_{I^-} w^s\,dx \cdot \exp\left(\frac{1}{|I^+|}\int_{I^+} -\log w^s\,dx\right) \le C.$$

To see this, partition I into adjacent intervals J^- and J^+ of equal length. Then

$$\begin{aligned}\frac{1}{|I^-|}\int_{I^-} w^s\,dx \cdot \exp\left(\frac{1}{|I^+|}\int_{I^+}\log w^{-s}\,dx\right) &\le C\left(\frac{1}{|J^-|}\int_{J^-} w\,dx\right)^s \cdot \exp\left(\frac{1}{|I^+|}\int_{I^+}\log w^{1-p'}\,dx\right)^{s(p-1)}\\ &\le C\left(\frac{1}{|J^-|}\int_{J^-} w\,dx\right)^s\left(\frac{1}{|I^+|}\int_{I^+} w^{1-p'}\,dx\right)^{s(p-1)}\\ &\le C\left(\frac{1}{|J^-|}\int_{J^-} w\,dx\right)^s\left(\frac{1}{|J^+|}\int_{J^+} w^{1-p'}\,dx\right)^{s(p-1)}.\end{aligned}$$

The first inequality follows from the (RH_s^+) condition, the second from Jensen's inequality. By Theorem 6.5, w is in (A_∞^+), so by Theorem 6.2 it is in (A_p^+) for some $p > 1$. Hence for the appropriate choice of p the last term is uniformly bounded for all intervals I.

Now, conversely, suppose that w^s is in (A_∞^+). Then w^s is in (A_q^+) for some $q > 1$, so by Hölder's inequality, if I is any interval, then

$$\begin{aligned}I^-(w^s) &= \frac{I^-(w^s)I^+(w^{s(1-q')})^{q-1}}{I^+(w^{s(1-q')})^{q-1}}\\ &\le I^-(w^s)I^+(w^{s(1-q')})^{q-1}I^+(w)^s \le CI(w)^s.\end{aligned}$$

Hence w is in (RH_s^+). ■

As an immediate consequence we get the following result.

THEOREM 6.7. *For $0 < p < \infty$, the pair (w,w) is in (W_p^+) if and only if w is in (A_∞^+).*

We want to conclude this section by showing that the one-sided reverse Hölder inequality can be used to simplify the proofs of several theorems about the structure of (A_p^+) weights.

First, note that by Lemma 6.4 and a calculation identical to the one which shows that if a weight w is in (RH_s) it is in (A_∞), we can show that if w is in (RH_s^+) for some $s > 1$ then it satisfies condition (4) of Theorem 6.2. (See García-Cuerva and Rubio de Francia [3, p. 401] for details.)

Second, and more importantly, we can show that if w is in (A_p^+) for some $p > 1$, then there exists $\varepsilon > 0$ such that w is in $(A_{p-\varepsilon}^+)$. This gives a proof of this key result which only uses the structural properties of the (A_p^+) weights and does not use the weighted norm inequalities for the one-sided maximal operator. This answers a question posed by Martín-Reyes [5].

To prove this fact, first note that if w is in (A_p^+) then it is also in (RH_s^+) for all $s > 1$ which are sufficiently close to one. Then by Lemma 6.6, w^s is in (A_∞^+). Now parallel to the "positive" classes (A_p^+), (A_∞^+) and (RH_s^+) are the "negative" classes (A_p^-), (A_∞^-) and (RH_s^-), the definition of each being the mirror image of the associated "positive" class. Identical proofs show that the exact same relationships hold among them, and the positive and negative classes are related by the fact that if w is in (A_p^+) then $w^{1-p'}$ is in $(A_{p'}^-)$. Hence for some $s > 1$ the same argument as above shows that $w^{s(1-p')}$ is in (A_∞^-). But Martín-Reyes, Pick and de la Torre [7] showed, using only the structural properties of (A_p^+) weights, that this implies that w^s is in (A_p^+). Therefore, by Hölder's inequality, w is in $(A_{p-\varepsilon}^+)$ for some $\varepsilon > 0$.

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Weak Cauchy sequences in $L_\infty(\mu, X)$

by

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Abstract. For a finite and positive measure space (Ω, Σ, μ) characterizations of weak Cauchy sequences in $L_\infty(\mu, X)$, the space of μ-essentially bounded vector-valued functions $f : \Omega \to X$, are presented. The fine distinction between Asplund and conditionally weakly compact subsets of $L_\infty(\mu, X)$ is discussed.

1. Introduction and preliminaries. In his celebrated paper [Ta, Th. 1] M. Talagrand gave a parametric Rosenthal ℓ_1-dichotomy. With the help of this result conditionally weakly compact subsets of $L_p(\mu, X)$, $1 \le p < \infty$, the space of Bochner integrable functions, can be characterized. A characterization for $p = \infty$ has not been found yet. The relatively weakly compact subsets of $L_\infty(\mu, X)$ were considered in special cases by K. T. Andrews and J. J. Uhl [AU] and in general by the author [S3]. A basic tool in both papers is the celebrated factorization lemma of Davis, Figiel, Johnson and Pełczyński.

Here, in a modified version, this method will be applied to give a complete (i.e. for all Banach spaces X) characterization of conditionally weakly compact subsets and weak Cauchy sequences of $L_\infty(\mu, X)$. It is mainly based on a result on parametrizing operators $T : X \to L_1(\mu, Y)^*$ (see the definition below). In Section 3 a fine distinction between Asplund sets and conditionally weakly compact sets is sketched for $L_\infty(\mu, X)$. In the survey article of L. H. Riddle and J. J. Uhl [AU], this was given for arbitrary Banach spaces by means of topology, vector measures and geometry. Here, this will be illustrated in the particular situation of $L_\infty(\mu, X)$.

First we fix some notations and definitions which are used in the paper. X and Y denote Banach spaces; $B(X)$ resp. $S(X)$ is the unit ball resp. the unit sphere of the Banach space X. If not indicated otherwise, we consider a positive and finite measure space, which will be denoted by (Ω, Σ, μ). Then $L_p(\mu, X) := L_p(\Omega, \Sigma, \mu, X)$ for $1 \le p \le \infty$ is the usual Bochner space. $L_\infty(\mu, X^*, X)$ is the set of equivalence classes of w^*-measurable and

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