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Geometric characteristics for convergence and asymptotics of successive approximations of equations with smooth operators

by

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Abstract. We discuss the problem of characterizing the possible asymptotic behaviour of the iterates of a sufficiently smooth nonlinear operator acting in a Banach space in small neighbourhoods of a fixed point. It turns out that under natural conditions, for the most part of initial approximations these iterates tend to "lie down" along a finite-dimensional subspace generated by the leading (peripherical) eigensubspaces of the Fréchet derivative at the fixed point and moreover the asymptotic behaviour of "projections" of the iterates on this subspace is determined by the arithmetic properties of the leading eigenvalues.

Let $\mathcal{X}$ be a Banach space and $\mathbf{A}$ a smooth (nonlinear) operator in $\mathcal{X}$ with fixed point x_*. A basic problem in numerical methods is to find conditions under which this fixed point x_* may be obtained as the limit

$$x_* = \lim_{n\to\infty} x_n$$

of the successive approximations

$$x_{n+1} = \mathbf{A}x_n \quad (n = 0, 1, \ldots) \tag{1}$$

for any initial value x_0 sufficiently close to x_*. A well-known sufficient condition (see, e.g., [9]) is that the spectral radius $\varrho(\mathbf{A}'(x_*))$ of the Fréchet derivative of $\mathbf{A}$ at x_* is strictly less than 1; some more precise conditions may be found in [6, 7, 11].

In case $\varrho(\mathbf{A}'(x_*)) < 1$, the successive approximations (1) converge to x_* at least as fast as a geometric progression with ratio $\varrho(\mathbf{A}'(x_*))$; this means that

$$\limsup_{n\to\infty} \sqrt[n]{\|x_n - x_0\|} \le \varrho(\mathbf{A}'(x_*)).$$

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It is of interest to determine the convergence rate more explicitly in terms of the initial value x_0. If $\mathcal{X}$ is a Hilbert space and $\mathbf{A}'(x_*)$ has a simple positive leading eigenvalue, this problem has been investigated by M. A. Krasnosel'skiĭ and Ya. B. Rutitskiĭ [9] who showed that the "portion" of those initial values x_0 in the ball

$$B(x_*, r) = \{x \in \mathcal{X} : \|x - x_*\| \le r\}$$

for which

$$\lim_{n\to\infty} \sqrt[n]{\|x_n - x_0\|} = \varrho(\mathbf{A}'(x_*)) \tag{2}$$

tends to 100% as $r \to 0$. This is closely related to the fact that, starting from an initial value x_0 satisfying (2), the sequence $x_n - x_*$ tends towards the direction of some normalized eigenvector e which corresponds to the eigenvalue $\varrho(\mathbf{A}'(x_*))$, i.e.

$$\lim_{n\to\infty} \|x_n - x_*\|^{-1}(x_n - x_*) = e.$$

Analogous results have been obtained under more general hypotheses in [4, 5].

In this paper we consider the case when $\mathbf{A}$ is just a completely continuous operator, or, more generally, the derivative $\mathbf{A}'(x_*)$ is completely continuous, or, still more generally, the peripherical spectrum of $\mathbf{A}'(x_*)$ (i.e. the part of the spectrum lying on the circumference $|\lambda| = \varrho(\mathbf{A}'(x_*))$, see [10]) is Fredholm. It turns out that analogous statements also hold in this case. Moreover, one can show that the successive approximations (1) "converge", as $n \to \infty$, towards an invariant subspace of eigenvectors corresponding to eigenvalues λ with $|\lambda| = \varrho(\mathbf{A}'(x_*))$. In contrast to the cases described above, however, the nature of this "convergence" is here much more complicated.

1. Equivalent norms and separated spectra. Consider a linear operator $\mathbf{B}$ in a Banach space $\mathcal{X}$ whose spectrum splits into two parts σ_0 and σ^0, where σ_0 is contained in the disc $|\lambda| \le r_0$, and σ^0 is contained in the annulus $r_- \le |\lambda| \le r_+$ $(0 \le r_0 < r_- \le r_+ < \infty)$. In this case [2, 8], the space $\mathcal{X}$ splits in turn into a direct sum $\mathcal{X} = \mathcal{X}_0 \dotplus \mathcal{X}^0$ of two $\mathbf{B}$-invariant subspaces $\mathcal{X}_0$ and $\mathcal{X}^0$ such that σ_0 is the spectrum of the restriction $\mathbf{B}_0$ of $\mathbf{B}$ to $\mathcal{X}_0$, and σ^0 is the spectrum of the restriction $\mathbf{B}^0$ of $\mathbf{B}$ to $\mathcal{X}^0$. Denote by $\mathbf{P}_0$ and $\mathbf{P}^0$ the projections (which commute with $\mathbf{B}$) of $\mathcal{X}$ onto $\mathcal{X}_0$ and $\mathcal{X}^0$, respectively.

Recall [9] that, given a continuous linear operator $\mathbf{C}$ in $\mathcal{X}$ and $\varepsilon > 0$, there is an equivalent norm $\|\cdot\|_\varepsilon$ in $\mathcal{X}$ such that $\|\mathbf{C}\|_\varepsilon \le \varrho(\mathbf{C}) + \varepsilon$. An important consequence is that the spectral radius of a continuous linear operator $\mathbf{C}$ in $\mathcal{X}$ is the infimum of $\|\mathbf{C}\|$ over all equivalent norms in $\mathcal{X}$. The norm $\|\cdot\|_\varepsilon$ for which $\|\mathbf{C}\|_\varepsilon \le \varrho(\mathbf{C}) + \varepsilon$ may be defined, for example, by

$$\|x\|_\varepsilon = (\varrho+\varepsilon)^{n-1}\|x\| + (\varrho+\varepsilon)^{n-2}\|\mathbf{C}x\| + \ldots + \|\mathbf{C}^{n-1}x\|, \tag{3}$$

where n is large enough such that $\|\mathbf{C}^n\|^{1/n} \le \varrho(\mathbf{C}) + \varepsilon$. (The existence of such an n is an immediate consequence of the definition of $\varrho(\mathbf{C})$.)

In the sequel we shall need a refinement of this result. First of all, we state a simple fact:

LEMMA 1. *If $\mathbf{D}$ is a continuous linear operator commuting with $\mathbf{C}$, then* $\|\mathbf{D}\|_\varepsilon \le \|\mathbf{D}\|$.

Proof. This follows from the obvious estimate

$$\|\mathbf{C}^k\mathbf{D}x\| \le \|\mathbf{D}\mathbf{C}^k x\| \le \|\mathbf{D}\| \cdot \|\mathbf{C}^k x\|$$

$(k = 0, 1, \ldots, n-1;\ \mathbf{C}^0 = \mathbf{I})$. ∎

By means of Lemma 1, we can state a more general assertion.

LEMMA 2. *Let $\mathfrak{M}$ be a compact family of mutually commuting continuous linear operators in $\mathcal{X}$. Then for any $\varepsilon > 0$, there is an equivalent norm $\|\cdot\|_\varepsilon$ in $\mathcal{X}$ such that*

$$\|\mathbf{A}\|_\varepsilon \le \varrho(\mathbf{A}) + \varepsilon \tag{4}$$

for each $\mathbf{A} \in \mathfrak{M}$.

Proof. Suppose that the operators $\{\mathbf{A}_1, \ldots, \mathbf{A}_n\}$ form an $(\varepsilon/3)$-net in $\mathfrak{M}$. We first construct an equivalent norm $\|\cdot\|_\varepsilon$ such that $\|\mathbf{A}_i\|_\varepsilon \le \varrho(\mathbf{A}_i) + \varepsilon/3$ for $i = 1, \ldots, n$. To this end, we consider a norm $\|\cdot\|_1$ as in (3), putting $\mathbf{C} = \mathbf{A}_1$ and $n = n_1$, i.e. we require that $\|\mathbf{A}_1^{n_1}\|^{1/n_1} \le \varrho(\mathbf{A}_1) + \varepsilon/3$. Thus we have $\|\mathbf{A}_1\|_1 \le \varrho(\mathbf{A}_1) + \varepsilon/3$, and the norms $\|\mathbf{A}\|_1$ of the other operators $\mathbf{A} \in \mathfrak{M}$ do not increase in this way, by Lemma 1.

Next, we take $\|\cdot\|_1$ as original norm in $\mathcal{X}$ and construct in the same way a norm $\|\cdot\|_2$, putting in (3) $\mathbf{C} = \mathbf{A}_2$ and $n = n_2$, i.e. $\|\mathbf{A}_2^{n_2}\|^{1/n_2} \le \varrho(\mathbf{A}_2) + \varepsilon/3$. Again we have $\|\mathbf{A}_2\|_2 \le \varrho(\mathbf{A}_2) + \varepsilon/3$, and, by Lemma 1, $\|\mathbf{A}_1\|_2 \le \|\mathbf{A}_1\|_1 \le \varrho(\mathbf{A}_1) + \varepsilon/3$. Continuing this way, we arrive at an equivalent norm $\|\cdot\|_\varepsilon = \|\cdot\|_n$ with $\|\mathbf{A}_i\|_\varepsilon \le \varrho(\mathbf{A}_i) + \varepsilon/3$ for $i = 1, \ldots, n$.

We claim that the norm $\|\cdot\|_\varepsilon$ has the required properties. Given $\mathbf{A} \in \mathfrak{M}$, choose $\mathbf{A}_i$ such that $\|\mathbf{A} - \mathbf{A}_i\| \le \varepsilon/3$, hence $|\varrho(\mathbf{A}) - \varrho(\mathbf{A}_i)| \le \varrho(\mathbf{A} - \mathbf{A}_i) \le \|\mathbf{A} - \mathbf{A}_i\| \le \varepsilon/3$. Since $\|\mathbf{A} - \mathbf{A}_i\|_\varepsilon \le \|\mathbf{A} - \mathbf{A}_i\| \le \varepsilon/3$, we conclude that

$$\|\mathbf{A}\|_\varepsilon \le \|\mathbf{A} - \mathbf{A}_i\|_\varepsilon + \|\mathbf{A}_i\|_\varepsilon \le \varepsilon/3 + \varrho(\mathbf{A}_i) + \varepsilon/3 \le \varrho(\mathbf{A}) + \varepsilon. \ ∎$$

If we consider a continuous linear operator $\mathbf{B}$ with splitting spectrum as at the beginning, we get, in particular,

LEMMA 3. *For any $\varepsilon > 0$, there is an equivalent norm $\|\cdot\|_\varepsilon$ such that*

$$\|\mathbf{B}x\|_\varepsilon \le (r_0 + \varepsilon)\|x\|_\varepsilon \quad (x \in \mathcal{X}_0) \tag{5}$$

and

$$(6)\qquad (r_- - \varepsilon)\|x\|_\varepsilon \le \|\mathbf{B}x\|_\varepsilon \le (r_+ + \varepsilon)\|x\|_\varepsilon \quad (x \in \mathcal{X}^0).$$

Proof. Since the restrictions $\mathbf{B}_0$ and $\mathbf{B}^0$ of $\mathbf{B}$ leave respectively the subspaces $\mathcal{X}_0$ and $\mathcal{X}^0$ invariant, they commute with each other, as well as with the continuous linear operator $(\mathbf{B}^0)^{-1}$. By Lemma 2, we find an equivalent norm $\|\cdot\|_\varepsilon$ such that

$$\|\mathbf{B}x\|_\varepsilon = \|\mathbf{B}_0x\|_\varepsilon \le (r_0 + \varepsilon)\|x\|_\varepsilon \quad (x \in \mathcal{X}_0),$$
$$\|\mathbf{B}x\|_\varepsilon = \|\mathbf{B}^0x\|_\varepsilon \le (r_+ + \varepsilon)\|x\|_\varepsilon \quad (x \in \mathcal{X}^0),$$

and

$$\|(\mathbf{B}^0)^{-1}z\|_\varepsilon \le (r_- - \varepsilon)^{-1}\|z\|_\varepsilon \quad (z \in \mathcal{X}^0),$$

i.e.

$$\|\mathbf{B}x\|_\varepsilon \ge (r_- - \varepsilon)\|x\|_\varepsilon \quad (x \in \mathcal{X}^0). \ \blacksquare$$

From Lemma 3 and the definition of equivalent norms we get in turn

LEMMA 4. *For sufficiently small $\varepsilon > 0$ there are positive constants m_- and m_+ such that*

$$(7)\qquad m_-(r_- - \varepsilon)^n\|\mathbf{P}^0x\| - (r_0 + \varepsilon)^n\|\mathbf{P}_0x\| \le \|\mathbf{B}^nx\| \le m_+(r_+ + \varepsilon)^n\|\mathbf{P}^0x\| + m_+(r_0 + \varepsilon)^n\|\mathbf{P}_0x\|$$

for $n = 0, 1, \ldots$ and $x \in \mathcal{X}$.

Proof. Given $\varepsilon > 0$ sufficiently small, consider the equivalent norm $\|\cdot\|_\varepsilon$ satisfying the estimates (5) and (6). We rewrite these estimates in the form

$$(8)\qquad \|\mathbf{B}_0\mathbf{P}_0x\|_\varepsilon \le (r_0 + \varepsilon)\|\mathbf{P}_0x\|_\varepsilon,$$

and

$$(9)\qquad (r_- - \varepsilon)\|\mathbf{P}^0x\|_\varepsilon \le \|\mathbf{B}^0\mathbf{P}^0x\|_\varepsilon \le (r_+ + \varepsilon)\|\mathbf{P}^0x\|_\varepsilon.$$

Consider now the norm $\|x\|_1 = \|\mathbf{P}_0x\|_\varepsilon + \|\mathbf{P}^0x\|_\varepsilon$ in $\mathcal{X}$, which is obviously equivalent to the original norm $\|\cdot\|$ on $\mathcal{X}$, as well as to the norm $\|\cdot\|_\varepsilon$ on both $\mathcal{X}_0$ and $\mathcal{X}^0$. From (8) and (9) we get

$$(10)\qquad \|\mathbf{B}x\|_1 = \|\mathbf{BP}_0x\|_\varepsilon + \|\mathbf{BP}^0x\|_\varepsilon \le (r_0 + \varepsilon)\|\mathbf{P}_0x\|_1 + (r_+ + \varepsilon)\|\mathbf{P}^0x\|_1$$

and

$$(11)\qquad \|\mathbf{B}x\|_1 = \|\mathbf{BP}_0x\|_\varepsilon + \|\mathbf{BP}^0x\|_\varepsilon \ge (r_- - \varepsilon)\|\mathbf{P}^0x\|_1 - (r_0 + \varepsilon)\|\mathbf{P}_0x\|_1.$$

Suppose that

$$(12)\qquad (r_- - \varepsilon)^k\|\mathbf{P}^0x\|_1 - (r_0 + \varepsilon)^k\|\mathbf{P}_0x\|_1 \le \|\mathbf{B}^kx\|_1 \le (r_+ + \varepsilon)^k\|\mathbf{P}^0x\|_1 + (r_0 + \varepsilon)^k\|\mathbf{P}_0x\|_1$$

for fixed $k \in \mathbb{N}$; we show that (12) is then also true for $k+1$. In fact, since $\mathbf{P}^0\mathbf{B} = \mathbf{BP}^0$, $\mathbf{P}_0\mathbf{B} = \mathbf{BP}_0$, and $\mathbf{P}_0\mathbf{P}^0 = \mathbf{P}^0\mathbf{P}_0 = 0$, we see from (10) that

$$\begin{aligned}\|\mathbf{B}^{k+1}x\|_1 &\le (r_0 + \varepsilon)\|\mathbf{P}_0\mathbf{B}^kx\|_1 + (r_+ + \varepsilon)\|\mathbf{P}^0\mathbf{B}^kx\|_1\\ &= (r_0 + \varepsilon)\|\mathbf{B}^k\mathbf{P}_0x\|_1 + (r_+ + \varepsilon)\|\mathbf{B}^k\mathbf{P}^0x\|_1\\ &\le (r_0 + \varepsilon)^{k+1}\|\mathbf{P}_0x\|_1 + (r_+ + \varepsilon)^{k+1}\|\mathbf{P}^0x\|_1.\end{aligned}$$

Similarly, one shows that

$$\|\mathbf{B}^{k+1}x\|_1 \ge (r_- - \varepsilon)^{k+1}\|\mathbf{P}^0x\|_1 - (r_0 - \varepsilon)^{k+1}\|\mathbf{P}_0x\|_1.$$

Consequently, (12) holds for any $n \in \mathbb{N}$. Since the norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent, we have $m\|x\|_1 \le \|x\| \le M\|x\|_1$ for any $x \in \mathcal{X}$, and conclude from (12) (with $k \le n$) that on the one hand,

$$\begin{aligned}\|\mathbf{B}^nx\| &\ge m\|\mathbf{B}^nx\|_1 \ge m(r_- - \varepsilon)^n\|\mathbf{P}^0x\|_1 - m(r_0 + \varepsilon)^n\|\mathbf{P}_0x\|_1\\ &\ge m_-[(r_- - \varepsilon)^n\|\mathbf{P}^0x\| - (r_0 + \varepsilon)^n\|\mathbf{P}_0x\|],\end{aligned}$$

where $m_- = mM^{-1}$, and, on the other hand,

$$\begin{aligned}\|\mathbf{B}^nx\| &\le M\|\mathbf{B}^nx\|_1 \le M(r_+ + \varepsilon)^n\|\mathbf{P}^0x\|_1 + M(r_0 + \varepsilon)^n\|\mathbf{P}_0x\|_1\\ &\le m_+[(r_+ + \varepsilon)^n\|\mathbf{P}^0x\| + (r_0 + \varepsilon)^n\|\mathbf{P}_0x\|],\end{aligned}$$

where $m_+ = m^{-1}M$. ■

Choosing, in particular, $\varepsilon > 0$ small enough in Lemma 4 such that $r_- - \varepsilon > r_0 + \varepsilon$ yields

THEOREM 1. *Let $\mathbf{P}^0x \ne 0$. Then the sequences $\mathbf{B}^nx$ and $\mathbf{B}^n\mathbf{P}^0x$ are equivalent as $n \to \infty$.*

2. Asymptotics of iterates and localization of the spectral radius. In what follows, we shall assume that $r_- = r_+ = \varrho(\mathbf{B})$ and $\dim \mathcal{X}^0 < \infty$. These hypotheses may be satisfied, for instance, if $\mathbf{B}$ is completely continuous.

As a matter of fact, it follows from the Banach–Steinhaus theorem [1] that the equality (2) fails only on a meager set (a set of first category). In this section we shall characterize more explicitly those $x \in \mathcal{X}$ for which this equality holds. To this end, we have to study the leading term in the asymptotic expansion, as $n \to \infty$, of $\mathbf{B}^nx$; by Theorem 1, we can study equivalently the leading term in the asymptotic expansion of $\mathbf{B}^n\mathbf{P}^0x$ if $\mathbf{P}^0x \ne 0$. By our hypotheses, the subspace $\mathcal{X}^0$ admits a representation [2] as a direct sum $\mathcal{X}^0 = \mathcal{V}_1 + \ldots + \mathcal{V}_s$ of $\mathbf{B}$-invariant subspaces $\mathcal{V}_1, \ldots, \mathcal{V}_s$ in such a way that, for an appropriate choice of bases [3], the matrix corresponding to the restriction $\mathbf{B}_j$ of $\mathbf{B}$ to $\mathcal{V}_j$ is a Jordan block with eigenvalue $\lambda_j = \varrho(\mathbf{B})e^{i\phi_j}$ and normalized eigenvector e_j $(j = 1, \ldots, s)$. We write $\mathbf{P}_j$ for the projection of

$\mathcal{X}$ onto $\mathcal{V}_j$, m_j for the dimension of $\mathcal{V}_j$, and

(13) $$k_j(x) = \sup\{k : \xi_k^j \neq 0\} \quad (\mathbf{P}_j x = (\xi_1^j, \xi_2^j, \ldots, \xi_{m_j}^j)),$$

(14) $$\phi_j(x) = [\lambda_j^{k_j(x)-1}(k_j(x)-1)!]^{-1}\xi_{k_j(x)}^j \quad (j = 1, \ldots, s).$$

LEMMA 5. *Suppose that* $\mathbf{P}_j x \neq 0$ *for some* $j = 1, \ldots, s$. *Then*

(15) $$\lim_{n\to\infty} e^{-n\phi_j}\|\mathbf{B}^n\mathbf{P}_j x\|^{-1}\mathbf{B}^n\mathbf{P}_j x = e_j.$$

Proof. Fix $x \in \mathcal{X}$ with $\mathbf{P}_j x = (\xi_1^j, \ldots, \xi_{k_j}^j, 0, \ldots, 0) \neq 0$, $\xi_{k_j}^j \neq 0$, where we drop the dependence on x in $k_j = k_j(x)$. For n sufficiently large we have

$$\mathbf{B}^n\mathbf{P}_j x = \lambda_j^n w_{j,n},$$

where

$$w_{j,n} = \Big(\sum_{k=0}^{k_j-1}\lambda_j^{-k}C_n^k\xi_{k+1}^j, \sum_{k=1}^{k_j-1}\lambda_j^{-k+1}C_n^{k-1}\xi_{k+1}^j, \ldots \\ \ldots, \xi_{k_j-2}^j + \lambda_j^{-1}\xi_{k_j}^j, \xi_{k_j}^j, 0, \ldots, 0\Big).$$

Since

$$\lim_{n\to\infty} C_n^i n^{-k_j+1} = \begin{cases} 0 & \text{for } i < k-1, \\ ((k_j-1)!)^{-1} & \text{for } i = k_j - 1, \end{cases}$$

the sequence $n^{-k_j+1}w_{j,n}$ tends to $\phi_j(x)e_j$ (see (14)) as $n \to \infty$. This means that $w_{j,n} \asymp n^{k_j-1}\phi_j(x)e_j$. Moreover, since $\lambda_j^n = (\varrho(\mathbf{B}))^n e^{in\phi_j}$, we have

(16) $$\|\mathbf{B}^n\mathbf{P}_j x\| \sim n^{k_j-1}(\varrho(\mathbf{B}))^n\phi_j(x),$$

hence

$$\lim_{n\to\infty} e^{-n\phi_j}\|\mathbf{B}^n\mathbf{P}_j x\|^{-1}\mathbf{B}^n\mathbf{P}_j x = \lim_{n\to\infty}\frac{(\varrho(\mathbf{B}))^n v_{j,n}}{\phi_j(x)n^{k_j-1}(\varrho(\mathbf{B}))^n} = e_j. \ ■$$

Now let

(17) $$k(x) = \max_j k_j(x), \quad J(x) = \{j : k_j(x) = k(x)\}.$$

THEOREM 2. *Let* $J(x) \neq \emptyset$. *Then*

(18) $$\mathbf{B}^n x = n^{k(x)-1}(\varrho(\mathbf{B}))^n \sum_{j\in J(x)} e^{in\phi_j}\phi(x)e_j + o(n^{k(x)-1}(\varrho(\mathbf{B}))^n).$$

In particular,

(19) $$\varrho(\mathbf{B}) = \limsup_{n\to\infty}\|\mathbf{B}^n x\|^{1/n}.$$

Proof. For $x \in \mathcal{X}$ with $J(x) \neq \emptyset$ we have, by (15) and (16),

$$\mathbf{B}^n\mathbf{P}_j x = e^{in\phi_j}(\varrho(\mathbf{B}))^n n^{k_j(x)-1}[\phi_j(x)e_j + o(n^{k_j(x)-1}(\varrho(\mathbf{B}))^n)].$$

Furthermore,

$$\begin{aligned}\mathbf{B}^n\mathbf{P}^0 x &= \sum_{j=1}^s \mathbf{B}^n\mathbf{P}_j x \\ &= \sum_{j=1}^s e^{in\phi_j}(\varrho(\mathbf{B}))^n n^{k_j(x)-1}[\phi_j(x)e_j + o(n^{k_j(x)-1}(\varrho(\mathbf{B}))^n)] \\ &= n^{k(x)-1}(\varrho(\mathbf{B}))^n\sum_{j=1}^s e^{in\phi_j}n^{k_j(x)-k(x)}[\phi_j(x)e_j + o(n^{k_j(x)-k(x)})].\end{aligned}$$

Since, by (17), $k_j(x) \le k(x)$, we have $n^{k_j(x)-k(x)} \to 0$ $(n \to \infty)$ for $j \notin J(x)$, and $n^{k_j(x)-k(x)} = 1$ for $j \in J(x)$. Consequently,

$$\mathbf{B}^n\mathbf{P}^0 x = n^{k(x)-1}(\varrho(\mathbf{B}))^n\sum_{j\in J(x)} e^{in\phi_j}\phi(x)e_j + o(n^{k(x)-1}(\varrho(\mathbf{B}))^n).$$

By Theorem 1, we conclude that (18) holds. The relation (19) is a straightforward consequence of (18). ■

As was communicated to the authors by A. B. Antonevich, statements as those given in Theorem 2 are of interest in the theory of invariant measures on discrete linear systems.

3. The nonlinear case. We pass to nonlinear operators. Suppose that $\mathbf{A}$ is a nonlinear operator which is Fréchet differentiable at a fixed point x_* and admits a representation

(20) $$\mathbf{A}x = \mathbf{A}x_* + \mathbf{B}(x - x_*) + \Omega(x - x_*),$$

where $\mathbf{B} = \mathbf{A}'(x_*)$ is linear and completely continuous, and

(21) $$\|\Omega h\| \le \gamma(r)\|h\| \quad (\|h\| \le r),$$

with $\gamma(r)$ increasing, nonnegative, and continuous ($\gamma(0) = 0$).

Let $\varrho = \varrho(\mathbf{B}) < 1$. As was shown above, the sequence $\mathbf{B}^n x_0$ converges, for $x_0 \notin \mathcal{X}_0$, to zero at least as fast as a geometric progression with ratio ϱ. It is natural to expect that this remains "nearly" valid also in the nonlinear case, with the modification that the space $\mathcal{X}_0$ is "deformed" into some set which is "tangent" to $\mathcal{X}_0$. For $r, \alpha > 0$, consider the set

(22) $$\Pi(r,\alpha) = \{x \in \mathbf{B}(x_*, r) : \|\mathbf{P}^0(x - x_*)\| \ge \alpha\|\mathbf{P}_0(x - x_*)\|\}.$$

THEOREM 3. *Suppose that the peripherical spectrum of the continuous operator* $\mathbf{A}'(x_*)$ *is Fredholm, and that* $\varrho(\mathbf{A}'(x_*)) < 1$. *Then for each* $\alpha > 0$ *there exists an* $r > 0$ *such that*

(23) $$\lim_{n\to\infty}\|x_n - x_*\|^{1/n} = \varrho(\mathbf{A}'(x_*))$$

and

(24) $$\lim_{n\to\infty} \|\mathbf{P}_0(x_n - x_*)\| \cdot \|\mathbf{P}^0(x_n - x_*)\|^{-1} = 0,$$

provided that $x_0 \in \Pi(r,\alpha)$.

Proof. Without loss of generality, we suppose throughout that $x_* = 0$ and $\mathbf{A}x_* = 0$ (this can be ensured by passing, if necessary, from $\mathbf{A}$ to the shifted operator $\widetilde{\mathbf{A}}x = \mathbf{A}(x_* + x) - \mathbf{A}x_*$).

Since the relations (23) and (24) are invariant under taking equivalent norms, we may suppose that the norm in $\mathcal{X}$ has the properties considered in the previous two sections.

Let $\varepsilon > 0$ be sufficiently small such that $\varrho - \varepsilon < r_0 + \varepsilon$. According to [9], we have

(25) $$\|x_n - x_*\| \le C(\varrho+\varepsilon)^n \|x_0 - x_*\|$$

with some constant $C = C(\varepsilon)$. Choose $r > 0$ such that

(26) $$\frac{r_0 + \varepsilon + \gamma(r)(1+M)}{\varrho - \varepsilon - \gamma(r)(1+\alpha^{-1})} < q_0 < 1,$$

(27) $$\varrho + \varepsilon + \gamma(r) < 1,$$

(28) $$\frac{\varrho - \varepsilon - \gamma(r)(1+\alpha^{-1})}{r_0 + \varepsilon + \gamma(r)(1+\alpha)} > 1,$$

and

(29) $$\gamma(r)(1+\alpha^{-1}) < 1$$

($\gamma(r)$ as in (21)). We claim that the set $\Pi(r,\alpha)$ is invariant under the operator $\mathbf{A}$. In fact, for $x \in \Pi(r,\alpha)$ we have

(30) $$\begin{aligned}\|\mathbf{P}^0\mathbf{A}x\| &\ge \|\mathbf{P}^0\mathbf{B}x\| - \|\mathbf{P}^0\Omega x\| \\ &\ge (\varrho-\varepsilon)\|\mathbf{P}^0 x\| - \gamma(\|x\|)(\|\mathbf{P}^0x\| + \|\mathbf{P}_0 x\|)\end{aligned}$$

and

(31) $$\begin{aligned}\|\mathbf{P}_0\mathbf{A}x\| &\le \|\mathbf{P}_0\mathbf{B}x\| + \|\mathbf{P}_0\Omega x\| \\ &\le (r_0+\varepsilon)\|\mathbf{P}_0 x\| + \gamma(\|x\|)(\|\mathbf{P}^0x\| + \|\mathbf{P}_0 x\|),\end{aligned}$$

hence, by (28),

(32) $$\|\mathbf{P}^0\mathbf{A}x\| \ge \alpha\|\mathbf{P}_0\mathbf{A}x\|.$$

Moreover,

$$\|\mathbf{A}x\| \le \|\mathbf{B}x\| + \|\Omega\mathbf{A}x\| \le (\varrho + \varepsilon + \gamma(r))\|x\|.$$

Consequently, the ball $\mathbf{B}(x_*, r)$ is invariant under $\mathbf{A}$, and thus also the set $\Pi(r,\alpha)$, by (32).

We have shown that, if we take the initial approximation x_0 from $\Pi(r,\alpha)$, then all successive approximations (1) belong to $\Pi(r,\alpha)$ as well. By (30) we may find, in particular, an index n_0 such that $\|\mathbf{P}^0 x_n\| \ge (\varrho - 2\varepsilon)^n \|\mathbf{P}^0 x_{n-1}\|$ for any $n > n_0$ and $x_0 \in \Pi(r,\alpha)$, $x_0 \ne 0$. Consequently,

$$\|x_n\| \ge \frac{\|\mathbf{P}^0 x_n\|}{\|\mathbf{P}^0\|} \ge (\varrho - 2\varepsilon)^n \frac{\|\mathbf{P}^0 x_n\|}{(\varrho-2\varepsilon)^n\|\mathbf{P}^0\|} \quad (n = 1, 2, \ldots),$$

and (23) follows from (25), since $\varepsilon > 0$ is arbitrary.

To prove (24), we consider the subset

$$\Pi(r,\alpha,M) = \{x \in \Pi(r,\alpha) : \|\mathbf{P}^0(x - x_*)\| \le M\|\mathbf{P}_0(x - x_*)\|\}$$

of $\Pi(r,\alpha)$ and fix $x_0 \in \Pi(r,\alpha,M)$. For $x_1 = \mathbf{A}x_0$ we then get

$$\begin{aligned}\|\mathbf{P}^0 x_1\| &\ge (\varrho-\varepsilon)\|\mathbf{P}^0 x_0\| - \gamma(\|x_0\|)(\|\mathbf{P}^0 x_0\| + \|\mathbf{P}_0 x_0\|) \\ &\ge [r_0 + \varepsilon + \gamma(r)(1+M)]\|\mathbf{P}_0 x\|,\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{P}_0 x_1\| &\le (r_0+\varepsilon)\|\mathbf{P}_0 x_0\| + \gamma(\|x_0\|)(\|\mathbf{P}^0 x_0\| + \|\mathbf{P}_0 x_0\|) \\ &\le [r_0 + \varepsilon + \gamma(r)(1+M)]\|\mathbf{P}_0 x_0\|,\end{aligned}$$

hence, by (26),

$$\frac{\|\mathbf{P}_0 x_1\|}{\|\mathbf{P}^0 x_1\|} \le q_0 \frac{\|\mathbf{P}_0 x_0\|}{\|\mathbf{P}^0 x_0\|}.$$

This shows that the sequence $\|\mathbf{P}_0 x_n\| \cdot \|\mathbf{P}^0 x_n\|^{-1}$ decreases at least as fast as a geometric progression with ratio $q_0 < 1$, as long as $x_n \in \Pi(r,\alpha,M)$. Consequently, $\|\mathbf{P}^0 x_{n_0}\| \ge M\|\mathbf{P}_0 x_{n_0}\|$ for some index n_0. But then

$$\begin{aligned}\frac{\|\mathbf{P}_0 x_{n_0+1}\|}{\|\mathbf{P}^0 x_{n_0+1}\|} &\le \frac{(r_0+\varepsilon)\|\mathbf{P}_0 x_{n_0}\| + \gamma(\|x_{n_0}\|)(\|\mathbf{P}^0 x_{n_0}\| + \|\mathbf{P}_0 x_{n_0}\|)}{(\varrho-\varepsilon)\|\mathbf{P}^0 x_{n_0}\| - \gamma(\|x_{n_0}\|)(\|\mathbf{P}^0 x_{n_0}\| + \|\mathbf{P}_0 x_{n_0}\|)} \\ &\le \frac{M^{-1}(r_0+\varepsilon) + \gamma(r)(1+M^{-1})}{\varrho - \varepsilon - \gamma(r)(1+\alpha^{-1})} \cdot \frac{\|\mathbf{P}_0 x_{n_0}\|}{\|\mathbf{P}^0 x_{n_0}\|} \le M^{-1},\end{aligned}$$

hence $\|\mathbf{P}^0 x_{n_0+1}\| \ge M\|\mathbf{P}_0 x_{n_0+1}\|$. This shows in turn that the set $\{x : \|\mathbf{P}^0 x\| \ge M\|\mathbf{P}_0 x\|\}$ is invariant under $\mathbf{A}$, and thus, together with x_{n_0}, all successive approximations x_n $(n > n_0)$ remain in this set. Consequently, the sequence $\|\mathbf{P}_0 x_n\| \cdot \|\mathbf{P}^0 x_n\|^{-1}$ eventually becomes (and remains) smaller than the (arbitrarily small) positive number M^{-1}. We conclude that (24) holds, and the proof is complete. ■

The relation (24) admits a simple geometric interpretation: the sequence $x_n - x_*$ is "tangent" to the eigenspace $\mathcal{X}^0$ of $\mathbf{B} = \mathbf{A}'(x_*)$ which corresponds to the eigenvalues on the circumference $|\lambda| = \varrho(\mathbf{B})$. Observe that Theorem 3 yields no statements on the asymptotics of the sequence of iterates. Under some additional hypotheses, however, such statements are possible. This will be carried out in the remaining part of this paper.

4. Asymptotics of iterates of nonlinear operators. We say that a (nonlinear) operator $\mathbf{A}$ satisfies the *Φ-condition* at x_* if $\mathbf{A}$ admits a Fréchet derivative $\mathbf{B} = \mathbf{A}'(x_*)$ at x_*, and the operator $\Omega h = \mathbf{A}(x_* + h) - \mathbf{A}x_* - \mathbf{B}h$ satisfies an estimate

$$\|\Omega h\| \le K\|h\|^{1+\delta}$$

for some $K, \delta > 0$.

We adopt the notation of the preceding section. For $j = 1, \ldots, s$, let $T_j = \lambda_j I - \mathbf{B}$, where $\mathbf{B} = \mathbf{A}'(x_*)$. We assume that the eigenvectors e_j and generalized eigenvectors g_j of $\mathbf{B}$ corresponding to the eigenvalue λ_j of $\mathbf{B}$ satisfy $e_j = T_j^{m_j-1} g_j$. Similarly, we assume that the (normalized) eigenfunctionals f_j and generalized eigenfunctionals l_j of $\mathbf{B}^*$, also corresponding to the eigenvalue λ_j, satisfy $f_j(e_j) = 1$ and $l_j(g_j) = 1$ and, of course, $f_j(e_k) = 0$, $l_j(g_k) = 0$ $(j, k = 1, \ldots, s,\ j \ne k)$. For $j = 1, \ldots, s$ and $r, \alpha > 0$ we define

$$\Pi_j(r, \alpha) = \{x \in B(x_*, r) : |l_j(x - x_*)| \ge \alpha\|x - x_*\|\}.$$

The following auxiliary statement will be fundamental in what follows.

LEMMA 6. *Suppose that the operator $\mathbf{A}$ satisfies the Φ-condition at a fixed point x_*. Then the functionals*

$$\theta_j^{(n)}(z) = l_j(z + \lambda_j^{-n}(\mathbf{A}^n z - \mathbf{B}^n z)) \quad (n = 1, 2, \ldots) \tag{33}$$

are defined on $\mathbf{B}(x_*, r)$ for sufficiently small $r > 0$. Moreover, the limit

$$\theta_j^*(z) = \lim_{n\to\infty} \theta_j^{(n)}(z) \tag{34}$$

exists for any $z \in \Pi_j(r, \alpha)$ and is different from zero.

Proof. Choose $\sigma > 0$ sufficiently small such that

$$(\varrho + \sigma)^{1+\delta} < \varrho, \tag{35}$$

where $\varrho = \varrho(\mathbf{B}) = \varrho(\mathbf{A}'(x_*))$. It is easy to see that

$$l_j(\mathbf{A}^n z - \mathbf{B}^n z) = \lambda_j l_j\Big(\sum_{k=0}^{n-1} \lambda_j^{1-k} \Omega \mathbf{A}^k z\Big),$$

hence

$$\theta_j^{(n)}(z) = l_j\Big(z + \sum_{k=0}^{n-1} \lambda_j^{1-k} \Omega\mathbf{A}^k z\Big).$$

This implies that

$$|\theta_j^{(n+p)}(z) - \theta_j^{(n)}(z)| \le \varrho \sum_{k=n}^{n+p-1} \varrho^{-k} \|\Omega \mathbf{A}^k z\|$$

and further, by (25), for $x_* = 0$,

$$|\theta_j^{(n+p)}(z) - \theta_j^{(n)}(z)| \le KC^{1+\delta}\varrho\|z\|^{1+\delta} \sum_{k=0}^{n-1} \varrho^{-k}(\varrho+\sigma)^{(1+\delta)k}.$$

In view of (35), this means that $\theta_j^{(n)}(z)$ is a Cauchy sequence, and hence the limit (34) exists. Similarly, one can obtain the lower estimate

$$|\theta_j^{(n)}(z)| \ge |l_j(z)| - KC^{1+\delta}\varrho\|z\|^{1+\delta} \sum_{k=0}^{n-1} \varrho^{-k}(\varrho+\sigma)^{(1+\delta)k}.$$

Defining

$$S = \sum_{k=0}^{\infty} \varrho^{-k}(\varrho+\sigma)^{(1+\delta)k},$$

for $z \in \Pi_j(r, \alpha)$ we then get

$$|\theta_j^{(n)}(z)| \ge [\alpha - K\varrho(C\|z\|)^{1+\delta} S]\|z\|.$$

The term in square brackets is bounded from below by some $\beta > 0$, provided that $r > 0$ is sufficiently small. Consequently, $|\theta_j^{(n)}(z)| \ge \beta\|z\|$, and hence the limit (34) is different from zero for any $z \in \Pi_j(r, \alpha)$. ∎

Observe that iterating the representation $\mathbf{A}z = \mathbf{B}z + \Omega z$ we get

$$\mathbf{A}^n z = \mathbf{B}^n z + \sum_{k=0}^{n-1} \mathbf{B}^{n-k-1} \Omega \mathbf{A}^k z, \tag{36}$$

hence

$$\begin{aligned} l_j(\mathbf{A}^n z) &= l_j(\mathbf{B}^n z) + \sum_{k=0}^{n-1} l_j(\mathbf{B}^{n-k-1}\Omega\mathbf{A}^k z) \\ &= \lambda_j^n l_j(z) + \sum_{k=0}^{n-1} \lambda_j^{n-k-1} l_j(\Omega\mathbf{A}^k z) \\ &= \lambda_j^n \Big[l_j(z) + \lambda_j^{-1} \sum_{k=0}^{n-1} \lambda_j^{-k} l_j(\Omega\mathbf{A}^k z)\Big]. \end{aligned} \tag{37}$$

On the other hand, from (36) we get

$$l_j(\mathbf{A}^n z - \mathbf{B}^n z) = \lambda_j^n \lambda_j^{-1} \sum_{k=0}^{n-1} \lambda_j^{-k} l_j(\Omega\mathbf{A}^k z),$$

hence

$$\lambda_j^{-1} \sum_{k=0}^{n-1} \lambda_j^{-k} l_j(\Omega\mathbf{A}^k z) = \lambda_j^{-n} l_j(\mathbf{A}^n z - \mathbf{B}^n z).$$

Equality (37) further implies that

$$l_j(\mathbf{A}^n z) = \lambda_j^n [l_j(z) + \lambda_j^{-n} l_j(\mathbf{A}^n z - \mathbf{B}^n z)] = \lambda_j^n \theta_j^{(n)}(z),$$

by the definition (33) of $\theta_j^{(n)}(z)$. By Lemma 6 we conclude that, for the successive approximations (1), the coordinate $l_j(x_n) = \lambda_j^n \theta_j^{(n)}(x_0)$, corresponding to the generalized eigenvector g_j of highest order in the Jordan block $\mathbf{B}_j$, changes with rate ϱ^n, provided that we take the initial approximation x_0 from $\Pi_j(r,\alpha)$ for $r > 0$ sufficiently small.

Before proceeding further, let us prove yet another auxiliary result.

Lemma 7. *Let $0 < q < 1$ and $m \in \mathbb{N}$ be fixed. Then*

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} C_{n-k-1}^{m-1} n^{1-m} q^m = \frac{1}{(1-q)(m-1)!}.$$

Proof. Defining

$$s_{n-m} = \sum_{k=0}^{n-m} q^k, \qquad \sigma_{n-m} = \sum_{k=0}^{n-m} C_{n-k-1}^{m-1} n^{1-m} q^k (m-1)!,$$

we get

$$\begin{aligned} s_{n-m} - \sigma_{n-m} &= \sum_{k=0}^{n-1} \left[1 - \frac{(n-k-1)\ldots(n-k-m-1)}{n^{m-1}}\right] q^k \\ &= n^{-1} \sum_{k=0}^{n-1} \frac{\alpha_1(k) n^{m-2} + \ldots + \alpha_{m-1}(k)}{n^{m-2}} q^k \\ &= n^{-1} \sum_{k=0}^{n-1} [\alpha_1(k) + n^{-1}\alpha_2(k) + \ldots + n^{-m+1}\alpha_{m-1}(k)] q^k, \end{aligned}$$

where $\alpha_j(k)$ $(j = 1, \ldots, m-1)$ are polynomials of degree j in k. Consequently, the series

$$\sum_{k=0}^{n-1} [\alpha_1(k) + n^{-1}\alpha_2(k) + \ldots + n^{-m+1}\alpha_{m-1}(k)] q^k$$

converges absolutely, and thus $s_{n-m} - \sigma_{n-m}$ tends to zero as $n \to \infty$. From this we get

$$\lim_{n\to\infty} \sigma_{n-m} = (1-q)^{-1},$$

and the assertion follows easily. ∎

Consider now the coordinate $f_j(x_n)$ corresponding to the eigenvector e_j $(j = 1, \ldots, s)$. By Lemmas 6 and 7,

$$\begin{aligned} (38) \qquad f_j(x_n) &= C_n^{m_j-1} \lambda_j^{n-m_j-1} l_j(x_0) \\ &\quad + \sum_{k=0}^{n-1} C_{n-k-1}^{m_j-1} \lambda_j^{n-k-m_j} l_j(\Omega \mathbf{A}^k x_0) + o(\varrho^n n^{m_j-1}) \\ &= n^{m_j-1} \lambda_j^n \phi_j(x_0) l_j \Big[x_0 + \lambda_j \sum_{k=0}^{n-m} \lambda_j^{-k} \Omega \mathbf{A}^k x_0\Big] \\ &\quad + o(\varrho^n n^{m_j-1}) \\ &= n^{m_j-1} \lambda_j^n \phi_j(x_0) \theta_j^{(n)}(x_0) + o(\varrho^n n^{m_j-1}). \end{aligned}$$

Let

$$k = \max_j m_j, \quad J^* = \{j : m_j = k\}, \quad \Pi^*(r,\alpha) = \bigcap_{j\in J^*} \Pi_j(r,\alpha).$$

From (38) we finally obtain our main result:

Theorem 4. *Suppose that the hypotheses of Theorem 3 hold, and that the operator $\mathbf{A}$ satisfies the Φ-condition at a fixed point x_*. Then*

$$\mathbf{A}^n x_0 = n^{k-1} [\varrho(\mathbf{A}'(x_*))]^n \sum_{j\in J^*} \frac{e^{in\phi_j} \theta_j^{(n)}(x_0) e_j}{(k-1)! \lambda_j^{k-1}} + o(n^{k-1}\varrho^n)$$

for any $\alpha > 0$ and sufficiently small $r > 0$.

The statements of Theorems 3 and 4 imply, in particular, that the "portion" of initial values $x_0 \in B(x_*, r)$ for which equality (2) holds tends to 100% as $r \to 0$. In fact, by (23) we may take the elements of $\Pi(r,\alpha)$ as these initial values, where α may be chosen arbitrarily small for r sufficiently small. From (22) we conclude, in turn, that, as $\alpha \to 0$, the set $\Pi(r,\alpha)$ ultimately "exhausts" the whole ball $B(x_*, r)$. (More precisely, this means the following. If $\mathcal{X}$ is finite-dimensional, the ratio of the Lebesgue measure of $\Pi(r,\alpha)$ and the Lebesgue measure of $B(x_*, r)$ tends to 1 as $\alpha \to 0$. If $\mathcal{X}$ is infinite-dimensional, an analogous statement holds with the Lebesgue measure replaced by the measure "concentrated" near the subspace $\mathcal{X}^0$. In general, one can show that the set of all directions h, $\|h\| = 1$, such that (2) fails for all initial approximations $x_0 = x_* + th$ with t sufficiently small, is an "almost meager" subset of the unit sphere.)

Similarly, the statement of Theorem 4 implies that, for initial values taken from $\Pi^*(r,\alpha)$, the vectors $x_n - x_*$ converge "directionally" towards the subspace $\mathcal{X}^{00} = \operatorname{span}\{e_j : j \in J^*\}$. To conclude, we remark that, following the usual reasoning (see, e.g., [6, 9]), one can obtain formulas for convergence acceleration, or formulas for probabilistic convergence estimates for successive approximations.

References

[1] J. Daneš, *On the local spectral radius*, Časopis Pěst. Mat. 112 (1987), 177–187.
[2] N. Dunford and J. Schwartz, *Linear Operators I*, Interscience Publ., Leyden, 1963.
[3] F. R. Gantmacher, *The Theory of Matrices*, Chelsea, New York, 1959.
[4] B. A. Godunov, *The behavior of successive approximations for nonlinear operators*, Dokl. Akad. Nauk Ukrain. SSR 4 (1971), 294–297 (in Russian).
[5] —, *Convergence acceleration in the method of successive approximations*, in: Operator Methods for Differential Equations, Voronezh, 1979, 18–25 (in Russian).
[6] P. Hartman, *Ordinary Differential Equations*, Wiley, New York, 1964.
[7] N. M. Isakov, *On the behavior of continuous operators near a fixed point in the critical case*, in: Qualitative and Approximate Methods for Investigation of Operator Equations, Yaroslavl', 1978, 74–89 (in Russian).
[8] L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, Nauka, Moscow, 1977 (in Russian); English transl.: Pergamon Press, Oxford, 1982.
[9] M. A. Krasnosel'skiĭ, G. M. Vaĭnikko, P. P. Zabreĭko, Ya. B. Rutitskiĭ and V. Ya. Stetsenko, *Approximate Solution of Operator Equations*, Nauka, Moscow, 1969 (in Russian); English transl.: Noordhoff, Groningen, 1972.
[10] M. A. Krasnosel'skiĭ, E. A. Lifshits and A. V. Sobolev, *Positive Linear Systems*, Nauka, Moscow, 1985 (in Russian); English transl.: Heldermann, Berlin, 1989.
[11] P. P. Zabreĭko and N. M. Isakov, *Reduction principle in the method of successive approximations and invariant manifolds*, Sibirsk. Mat. Zh. 20 (1979), 539–547 (in Russian).

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On Dirichlet–Schrödinger operators with strong potentials

by

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Abstract. We consider Schrödinger operators $H = -\Delta/2 + V$ ($V \geq 0$ and locally bounded) with Dirichlet boundary conditions, on any open and connected subdomain $D \subset \mathbb{R}^n$ which either is bounded or satisfies the condition $d(x, D^c) \to 0$ as $|x| \to \infty$. We prove exponential decay at the boundary of all the eigenfunctions of H whenever V diverges sufficiently fast at the boundary ∂D, in the sense that $d(x, D^c)^2 V(x) \to \infty$ as $d(x, D^c) \to 0$. We also prove bounds from above and below for $\mathrm{Tr}(\exp[-tH])$, and in particular we give criterions for the finiteness of such trace. Applications to pointwise bounds for the integral kernel of $\exp[-tH]$ and to the computation of expected values of the Feynman–Kac functional with respect to Doob h-conditioned measures are given as well.

1. Introduction. Let D be an open and connected proper subset of $\mathbb{R}^n$. On D, one can consider the *Dirichlet Laplacian*, Δ_D, or the positive operator $H_0 = -\Delta_D/2$, which is the self-adjoint operator (in $\mathrm{L}^2(D)$) associated with the closure of the quadratic form

$$Q_0(f) = \frac{1}{2} \int_D |\nabla f(x)|^2 \, dx, \qquad f \in C_0^\infty(D). \tag{1.1}$$

If $V : D \to [0, \infty)$ is a locally bounded measurable function (the *potential* function), then one can also consider the Schrödinger operator $H = -\Delta/2 + V = H_0 + V$, with Dirichlet boundary conditions, which is the self-adjoint operator associated, by the above procedure, with the quadratic form

$$Q(f) = Q_0(f) + \int_D V(x)|f(x)|^2 \, dx, \qquad f \in C_0^\infty(D). \tag{1.2}$$

The reason for the factor 1/2 in (1.1), (1.2) is simply that this is the usual normalization for the generator of the semigroup associated with Brownian motion, and we prove some of the main results of this paper by probabilistic methods. It should also be noted here, once for all, that a negative part of

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