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A fixed point theorem for demicontinuous pseudocontractions in Hilbert space

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Abstract. Let C be a closed, bounded, convex subset of a Hilbert space. Let $T : C \to C$ be a demicontinuous pseudocontraction. Then T has a fixed point. This is shown by a combination of topological and combinatorial methods.

In a Hilbert space, a *pseudocontraction* is a map satisfying

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(Tx - x) - (Ty - y)\|^2$$

or equivalently,

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2.$$

Clearly, this is equivalent to

$$\langle (I - T)x - (I - T)y, x - y\rangle \ge 0.$$

That is, $(I - T)$ is monotone. Monotone mappings appeared in the proof of Browder's fixed point theorem.

A map T is *demicontinuous* if $\{x_n\}$ converging to x in the norm implies that $\{Tx_n\}$ converges weakly to Tx.

Our main result is:

If C is a closed, bounded, convex subset of a Hilbert space, and $T : C \to C$ is a demicontinuous pseudocontraction, then T has a fixed point.

Martin [5] showed that if T is a norm continuous pseudocontraction, then T has a fixed point (this was actually sort of a byproduct of some other work). Browder [2] showed that it would be enough for T to be demicontinuous if C were a ball. We were able to get the result for any closed, bounded, convex subset by the use of combinatorial methods (Ramsey's Theorem) and by the use of a class of maps similar to the pseudocontractions. In fact, our actual result is for a class of maps slightly more general

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than pseudocontractions. We called them pseudocontractive type, and they satisfy the inequality

(D) $$\|Tx - Ty\|^2 \le \|x-y\|^2 + \max\{\|Tx - x\|^2 + \|Ty - y\|^2, \|(Tx-x)-(Ty-y)\|^2\}.$$

This inequality arises in the following manner:

Consider the two inequalities:

(E) $$\|Tx - Ty\|^2 \le \|x-y\|^2 + \|Tx-x\|^2 + \|Ty-y\|^2$$

and

(F) $$\|Tx - Ty\|^2 \le \|x-y\|^2 + \|(Tx-x)-(Ty-y)\|^2.$$

Observe that, for a pair x and y, if $\langle Tx-x, Ty-y\rangle \ge 0$, then (E) implies (F), and thus (D) implies (F). Again, if $\langle Tx-x, Ty-y\rangle \le 0$, then (F) implies (E), hence (D) implies (E). We shall show by combinatorial methods (Ramsey's Theorem) that for any infinite set $R \subset \operatorname{dom} T$, and for any $\delta > 0$, there is an infinite subset $S \subset R$ such that for x and y in S,

$$2\langle Tx-x, Ty-y\rangle \ge -\delta\|Tx-x\|^2 - \delta\|Ty-y\|^2.$$

(And this does not depend on T at all.) Thus, for x and y in S and satisfying (F),

(E(δ)) $$\|Tx - Ty\|^2 \le \|x-y\|^2 + (1+\delta)\|Tx-x\|^2 + (1+\delta)\|Ty-y\|^2.$$

This is very useful because for a closed ball B and a closed convex set $C \subset B$, if $\varrho : B \to C$ is the "closest point" projection and $T : C \to C$ satisfies (E) or (E(δ)), then $T \circ \varrho : B \to B$ will satisfy the same inequality.

We start with the map $T \circ \varrho : B \to B$ and by topological methods, namely Brouwer's fixed point theorem, we find a sequence of points $\{x_n\}$ and projections $\{\Pi_n\}$ such that $\Pi_n T \circ \varrho x_n = x_n$. These projections will be such that for $n > j$, the range of Π_n will contain the range of Π_j, as well as $T \circ \varrho x_j$ and x_j. Given $\delta > 0$, an infinite subsequence will satisfy (E(δ)) for T, hence for $T \circ \varrho$. Then, by an argument essentially trigonometric, we show that a weak limit of the $\{x_n\}$ must be a fixed point.

THEOREM 1. *Let C be a closed, bounded, convex subset of a Hilbert space. Let T be a demicontinuous mapping, $T : C \to C$. Let T satisfy*

$$\|Tx-Ty\|^2 \le \|x-y\|^2 + \max\{\|(Tx-x)-(Ty-y)\|^2, \|Tx-x\|^2 + \|Ty-y\|^2\}.$$

Then T has a fixed point.

THEOREM 1#. *Let C be a closed, bounded, convex subset of a Hilbert space. Let $T : C \to C$ be a demicontinuous pseudocontraction. Then T has a fixed point.*

It is clear that Theorem 1# is a special case of Theorem 1.

Before proving our theorem, we need the following definitions and lemmas.

NOTATION 1. $\mathcal{H}$ will denote a Hilbert space. B will always denote a closed ball with center at zero.

DEFINITION 1. A mapping $T : C \to \mathcal{H}$ is said to be *hemicontinuous* if the map $t \to T[(1-t)x + ty]$ is continuous from $[0,1]$ into the weak topology of $\mathcal{H}$. T is *demiclosed at zero* if $\{x_n\}$ converging weakly to x and $\{Tx_n\}$ converging strongly to 0 imply $Tx = 0$.

DEFINITION 2. A map $T : \mathcal{H} \to \mathcal{H}$ is of *pseudocontractive type* if it satisfies the inequality

$$\|Tx - Ty\|^2 \le \|x-y\|^2 + \max\{\|Tx-x\|^2 + \|Ty-y\|^2, \|(Tx-x)-(Ty-y)\|^2\}$$

for all x and y.

LEMMA 1. *Let B be a closed ball in $\mathcal{H}$ and let $T : B \to B$ be demicontinuous. Then there exists a sequence $\{z_n\} \subset B$ such that*

(a) *$\langle Tz_n - z_n, Tz_m - z_m\rangle = 0$ for $m \ne n$,*
(b) *$\langle Tz_n - z_n, z_n\rangle = 0$, and*
(c) *$\{Tz_n - z_n\}$ converges weakly to zero.*

Proof. If the mapping T has a fixed point x in B, we complete the proof by letting $z_n = x$ for all n. Assuming no fixed point, we let M_1 be a 1-dimensional subspace of $\mathcal{H}$, and $\Pi_1 : B \to M_1 \cap B$ be the well defined orthogonal projection. Thus, the mapping $\Pi_1 \circ T : M_1 \cap B \to M_1 \cap B$ is continuous, and has a fixed point by the well known Brouwer theorem. Since $\Pi(Tz_1 - z_1) = \Pi \circ Tz_1 - \Pi z_1 = 0$, we have $(Tz_1 - z_1) \in M_1^{\perp}$. Let $M_2 = M_1 \oplus (Tz_1 - z_1)$ and $\Pi_2 : B \to M_2 \cap B$ be the orthogonal projection. As above, the mapping $\Pi_2 \circ T$ has a fixed point z_2 in $M_2 \cap B$. Generally, we construct a family of finite rank subspaces $\{M_n\}$ and a sequence $\{z_n\}$ with $z_n \in M_n \cap B$ such that $(Tz_n - z_n) \in M_n^{\perp}$, $M_{n+1} = M_n \oplus (Tz_n - z_n)$ and $\Pi_n \circ T(z_n) = z_n$. To show that the sequence $\{z_n\}$ satisfies (a)–(c), we observe that $\langle Tz_n - z_n, h\rangle = 0$ for all $h \in M_n$ and $(Tz_m - z_m) \in M_n$ for $m < n$. Thus, $\langle Tz_n - z_n, z_n\rangle = 0$ and, by the symmetry of the inner product, $\langle Tz_n - z_n, Tz_m - z_m\rangle = 0$ for all $m \ne n$. Since $\{Tz_n - z_n\}$ is an orthogonal set of vectors in $\mathcal{H}$, by Bessel's inequality we have $\langle Tz_n - z_n, h\rangle \to 0$ for all $h \in \mathcal{H}$. ■

LEMMA 2. *Let C be a convex subset of $\mathcal{H}$, and let $T : C \to C$ be a hemicontinuous pseudocontractive type mapping. Suppose $\{x_n\}$ to be a sequence in C such that $(Tx_n - x_n) \to 0$ (weakly), $\langle Tx_n - x_n, x_n\rangle \to 0$, and $x_n \to x$ (weakly). Then x is a fixed point of T.*

Proof. First, by the law of cosines, the definition of pseudocontractive type is equivalent to the mapping T satisfying

$$\langle Tw - Ty, w - y\rangle \le \|w - y\|^2 + \max\{0, \langle Tw - w, Ty - y\rangle\}, \qquad \forall w, y \in C.$$

Let $z = (1-b)x + bTx$ for some b, $0 < b < 1$. Since T is hemicontinuous, we can find a number b such that

$$\langle Tz - z, Tx - x\rangle \ge (1/2)\langle Tx - x, Tx - x\rangle.$$

For any $\varepsilon > 0$, there exists n large enough such that $|\langle Tx_n - x_n, z - x_n\rangle| < \varepsilon$, $|\langle Tx_n - x_n, Tz - z\rangle| < \varepsilon$, and $|\langle Tz - z, x - x_n\rangle| < \varepsilon$. Thus,

$$\begin{aligned}\langle Tz - Tx_n, z - x_n\rangle &= \langle Tz - z, z - x_n\rangle + \langle z - x_n, z - x_n\rangle + \langle x_n - Tx_n, z - x_n\rangle\\ &= \|z - x_n\|^2 + \langle x_n - Tx_n, z - x_n\rangle\\ &\quad + \langle Tz - z, b(Tx - x)\rangle + \langle Tz - z, x - x_n\rangle\\ &\ge \|z - x_n\|^2 + (b/2)\|Tx - x\|^2 - 2\varepsilon.\end{aligned}$$

On the other hand, T is of pseudocontractive type, so

$$\begin{aligned}\langle Tz - Tx_n, z - x_n\rangle &\le \|z - x_n\|^2 + \max\{0, \langle Tz - z, Tx_n - x_n\rangle\}\\ &\le \|z - x_n\|^2 + \varepsilon.\end{aligned}$$

Finally, we have $3\varepsilon \ge (b/2)\|Tx - x\|^2$, from which we have $Tx = x$. ■

Combining Lemma 2 with the definition of demiclosed, we immediately have the following:

LEMMA 3. *Let C be a closed, bounded, convex subset of $\mathcal{H}$ and let $T : C \to C$ be a hemicontinuous pseudocontractive type mapping. Then the operator $I - T$ is demiclosed at zero.*

The next two lemmas will help us use Lemma 1's results in the case where C is not a ball.

LEMMA 4 (Ramsey's Theorem). *Let $\mathcal{V} \subset \mathbb{N} \times \mathbb{N}$ be the set of all ordered pairs (m, n) such that $m > n$. Let $\phi : \mathcal{V} \to \{0, 1\}$ be any function. Then there exists $A \subseteq \mathbb{N}$ such that ϕ restricted to $\mathcal{V} \cap (A \times A)$ is a constant.*

For the proof, see [7].

Ramsey's Theorem is often expressed in terms of graph theory. In that case (m, n) is the edge from m to n, and $\{0, 1\}$ is {red, green} or {adjacent, not adjacent}.

LEMMA 5. *Given any infinite set of vectors W in a Hilbert space, and any $\delta > 0$, there exists V, an infinite subset of W, such that for all $y_n, y_m \in V$,*

$$\|y_n - y_m\|^2 \le (1 + \delta)\|y_n\|^2 + (1 + \delta)\|y_m\|^2.$$

Proof. We shall use Lemma 4. We define ϕ in this manner:

$$\phi(m, n) = \begin{cases} 1 & \text{if } y_m \text{ and } y_n \text{ satisfy the inequality,}\\ 0 & \text{if they do not.}\end{cases}$$

Thus, either there exists the infinite subset V we desire, or there exists an infinite subset S such that for all $y_n, y_m \in S$,

$$\|y_n - y_m\|^2 \ge (1 + \delta)\|y_n\|^2 + (1 + \delta)\|y_m\|^2,$$

which, by the law of cosines, is equivalent to

$$-2\langle y_n, y_m\rangle \ge \delta\|y_n\|^2 + \delta\|y_m\|^2.$$

This easily implies that the ratio of the norms is bounded by $2/\delta$; and, without loss of generality, we can say that $1 \ge \|y_m\| \ge \delta/2$ for any $y_m \in S$.

We now complete the proof by a Gram–Schmidt process. We renumber the elements of S as $\{z_n\}$. We then construct an orthonormal basis $\{u_n\}$ so that

$$z_1 = a_1 u_1, \quad z_2 = b_{12} u_1 + a_2 u_2, \ \ldots, \quad z_n = b_{1n} u_1 + b_{2n} u_2 + \ldots + a_n u_n,$$

where all the a_j's are positive. It is clear that $b_{ij} \le -(\delta/2)\|z_j\|$ for all i and j. Thus, when $j > [(2/\delta)^2 + 1]$, we get $\|z_j\| > 1$. This contradiction proves the lemma.

Proof of Theorem 1. To prove that the operator T has at least one fixed point in C, it suffices, by Lemma 2 and the weak compactness of C, to show that there exists a sequence $\{x_n\}$ in C such that

(i) $\{Tx_n - x_n\}$ converges weakly to zero, and
(ii) $\langle Tx_n - x_n, x_n\rangle \to 0$.

Let B be a ball containing C and let $\varrho : \mathcal{H} \to C$ be the retraction sending $x \in \mathcal{H}$ to the closest point to x in C. It is well known that ϱ will satisfy

$$\begin{aligned}\|\varrho x - \varrho y\| &\le \|x - y\| \quad \text{for all } x, y \in \mathcal{H}, \text{ and}\\ \|x - \varrho y\|^2 &\le \|x - y\|^2 - \|\varrho y - y\|^2 \quad \text{for all } x \in C,\ y \in \mathcal{H}.\end{aligned}$$

For the demicontinuous mapping $T \circ \varrho : B \to B$, there exists, by Lemma 1, a sequence $\{z_n\}$ having the properties (a)–(c) listed in Lemma 1. Set $x_n = \varrho z_n$. Note that $\{x_n\} \subset C$, $Tx_n - x_n = T \circ \varrho z_n - z_n + z_n - \varrho z_n$, and

$$\begin{aligned}\langle Tx_n - x_n, x_n\rangle &= \langle T \circ \varrho z_n - z_n, z_n\rangle + \langle T \circ \varrho z_n - z_n, \varrho z_n - z_n\rangle\\ &\quad + \langle z_n - \varrho z_n, \varrho z_n - z_n\rangle + \langle z_n - \varrho z_n, z_n\rangle.\end{aligned}$$

Now $\{T \circ \varrho z_n - z_n\}$ weakly converges to 0, and $\langle T \circ \varrho z_n - z_n, z_n\rangle = 0$. Thus, if $\liminf \|\varrho z_n - z_n\| = 0$, a subsequence of $\{x_n\}$ will satisfy (i) and (ii), completing the proof.

We assume toward a contradiction that there exists $b > 0$ such that $\|\varrho z_n - z_n\| \ge b$ for all n. By Lemma 5, there exists V, an infinite subset of

$\{T \circ \varrho z_n - \varrho z_n\}$ such that for two elements of V,

$$\|(T\circ\varrho z_n-\varrho z_n)-(T\circ\varrho z_j-\varrho z_j)\|^2\le\|T\circ\varrho z_n-\varrho z_n\|^2+\|T\circ\varrho z_j-\varrho z_j\|^2+b^2.$$

We will let W denote the corresponding subset of $\{z_n\}$. Since T is of pseudocontractive type, we have for $z_n, z_j \in W$,

$$\begin{aligned}\|T\circ\varrho z_n-T\circ\varrho z_j\|^2 &\le \|\varrho z_n-\varrho z_j\|^2+\|T\circ\varrho z_n-\varrho z_n\|^2+\|T\circ\varrho z_j-\varrho z_j\|^2+b^2\\ &\le \|z_n-z_j\|^2+\|T\circ\varrho z_n-z_n\|^2-\|\varrho z_n-z_n\|^2\\ &\quad+\|T\circ\varrho z_j-z_j\|^2-\|\varrho z_j-z_j\|^2+b^2\\ &\le \|z_n-z_j\|^2+\|T\circ\varrho z_n-z_n\|^2+\|T\circ\varrho z_j-z_j\|^2+b^2-2b^2\\ &\le \|z_n-z_j\|^2+q\|T\circ\varrho z_n-z_n\|^2+q\|T\circ\varrho z_j-z_j\|^2\end{aligned}$$

for some $q<1$. On the other hand, by the law of cosines, we have

$$\begin{aligned}\|T\circ\varrho z_n-T\circ\varrho z_j\|^2 &\ge \|z_n-z_j\|^2+\|T\circ\varrho z_n-\varrho z_n\|^2+\|T\circ\varrho z_j-\varrho z_j\|^2\\ &\quad-2|\langle T\circ\varrho z_n-z_n,T\circ\varrho z_j-z_j\rangle|-2|\langle T\circ\varrho z_n-z_n,z_n-z_j\rangle|\\ &\quad-2|\langle T\circ\varrho z_j-z_j,z_n-z_j\rangle|.\end{aligned}$$

Combining the two inequalities above, and noting that $\langle T\circ\varrho z_n-z_n, T\circ\varrho z_j-z_j\rangle=0$ for $j\ne n$, and $\langle T\circ\varrho z_j-z_j,z_n-z_j\rangle=0$ for $j>n$, we have

$$\begin{aligned}2|\langle T\circ\varrho z_n-z_n,z_n-z_j\rangle| &\ge (1-q)\|T\circ\varrho z_n-z_n\|^2+(1-q)\|T\circ\varrho z_j-z_j\|^2\\ &\ge (1-q)\|\varrho z_n-z_n\|^2+(1-q)\|\varrho z_j-z_j\|^2\ge 2(1-q)b^2.\end{aligned}$$

Let $S=\{z_n\}$ be a subsequence of W; and let z be the weak limit of S. Observe that

$$\begin{aligned}\langle T\circ\varrho z_n-z_n,z_n-z_j\rangle &= \langle T\circ\varrho z_n-z_n,z_n-\Pi_n z\rangle\\ &\quad+\langle T\circ\varrho z_n-z_n,\Pi_n z-z\rangle\\ &\quad+\langle T\circ\varrho z_n-z_n,z-z_j\rangle.\end{aligned}$$

Recall that $\langle T\circ\varrho z_n-z_n,z_n-\Pi_n z\rangle=0$, that for sufficiently large n, $\|\Pi_n z-z\|$ is as small as necessary, and that for any fixed n, there exists $j>n$ such that $|\langle T\circ\varrho z_n-z_n,z-z_j\rangle|$ is also as small as desired. This gives the desired contradiction. ■

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