Contents of Volume 116, Number 3

STUDIA MATHEMATICA

Executive Editors: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

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Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by

PRINTED IN POLAND

ISSN 0039-3223

On the multiplicity function of ergodic group extensions. II

by

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Abstract. For an arbitrary set $A \subseteq \mathbb{N}^+$ containing 1, an ergodic automorphism T whose set of essential values of the multiplicity function is equal to A is constructed. If A is additionally finite, T can be chosen to be an analytic diffeomorphism on a finite-dimensional torus.

1. Introduction. Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an automorphism of a standard Borel space. It induces a unitary operator $U_T : L^2(X, \mu) \to L^2(X, \mu)$, $U_T f = fT$. Define $Z(f) = \operatorname{span}\{fT^k : k \in \mathbb{Z}\}$, $f \in L^2(X, \mu)$. By the *spectral measure* σ_f of f we mean the unique Borel measure on the circle $\mathbb{T}$ given by

$$\widehat{\sigma}_f(n) = \int_{\mathbb{T}} z^n \, d\sigma_f(z), \qquad n \in \mathbb{Z}.$$

A number $m \in \mathbb{N}^+ \cup \{\infty\}$ ($\mathbb{N}^+ = \{1, 2, \ldots\}$) is said to belong to the set of *essential values* $E(T)$ of T if there exist $f_1, \ldots, f_m \in L^2(X, \mu)$ with $Z(f_i) \perp Z(f_j)$, $i \neq j$, such that $\sigma_{f_1}, \ldots, \sigma_{f_m}$ are all equivalent and for no element $f \in L^2(X, \mu)$ for which $Z(f) \perp Z(f_1) \oplus \ldots \oplus Z(f_m)$, σ_f is equivalent to σ_{f_1}. The greatest element of $E(T)$ is called the *maximal spectral multiplicity* of T and will be denoted by $\operatorname{msm}(T)$. Among spectral measures there is one, say σ_f, such that $\sigma_g \ll \sigma_f$ for all $g \in L^2(X, \mu)$; call it the *maximal spectral type* of T. For more spectral theory of unitary operators on separable Hilbert spaces see [19].

Our goal is to describe what kind of subsets $A \subset \mathbb{N}^+$ can be realized as $E(T)$ for some ergodic T. Although no restrictions to reach an arbitrary subset of $\mathbb{N}^+$ are likely to exist, constructions to obtain a concrete set as $E(T)$ are quite complicated. Before the paper of Oseledec [18], the only known values for $E(T)$ were: $\{1\}$, $\{\infty\}$, $\{1, \infty\}$ (the latter in the case of quasi-discrete spectrum automorphisms [1]). Oseledec constructed an ergodic T with

$$2 \leq \operatorname{msm}(T) < 30.$$

1991 *Mathematics Subject Classification*: Primary 28D05.

As shown by Vershik, for Gaussian automorphisms either $E(T) = \{1\}$ or $E(T)$ is infinite (see [4], part III). It follows from Katok–Stepin theory of cyclic approximation that a "typical" automorphism T (with respect to the weak topology of automorphisms) of $(X, \mathcal{B}, \mu)$ has $E(T) = \{1\}$ ([12]) with the maximal spectral type being singular. Other generic results, like describing $E(T \times T)$ for "typical" T, are studied in [11]. More information about the history of the spectral multiplicity in ergodic theory can be found in [21]. A real progress has been made by Robinson [21], where for given $n \in \mathbb{N}^+$, an ergodic (even weakly mixing) automorphism T with $\mathrm{msm}(T) = n$ (in fact $E(T) = \{1, n\}$) has been constructed. Extending the result of Mathew and Nadkarni [16], in [2] and [15] ergodic automorphisms with $E(T) = \{1, 2n\}$, $n \geq 1$, have been found, where furthermore $2n$ corresponds to the multiplicity of the Lebesgue component (Robinson's examples have singular spectra); Banach's famous problem of finding an ergodic T with $E(T) = \{1\}$ and Lebesgue spectral type is still open.

Let $A \subseteq \mathbb{N}^+$ satisfy

$$1 \in A, \quad m, n \in A \Rightarrow \mathrm{lcm}(m, n) \in A. \tag{1}$$

Extending the main result from [21], in [22], given a finite A satisfying (1), an ergodic T for which $A = E(T)$ has been constructed. The paper [8] is devoted to extending this result to arbitrary (not necessarily finite) subsets of $\mathbb{N}^+$ satisfying (1). In [3], Robinson's result from [22] is reproved, though this time with T being additionally an analytic diffeomorphism on a finite-dimensional torus.

In this note, based on the methods from [8] and [3], by some algebraic perturbation of the main idea from [8], we will prove the following

THEOREM 1. *Given a set $A \subseteq \mathbb{N}^+$ with $1 \in A$, there exists an ergodic T such that $E(T) = A$. Moreover, T can be constructed to be weakly mixing. If A is additionally finite then T can be constructed as an analytic diffeomorphism on a finite-dimensional torus.*

For other recent papers where the multiplicity problem is treated, see [5], [6], [7], [13], [14], [17], [20].

The authors would like to thank Jan Kwiatkowski for some valuable discussions.

2. Description of the method. The properties announced in Theorem 1 will, in fact, be properties of a cocycle. We briefly recall the method from [8].

Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic automorphism. Assume that G is a compact metric abelian group with Haar measure m. Any measurable map $\varphi : X \to G$ is called a *cocycle*; in fact, the cocycle generated by φ is

given by

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x)\varphi(Tx)\ldots\varphi(T^{n-1}x), & n > 0, \\ 1, & n = 0, \\ (\varphi(T^n x)\ldots\varphi(T^{-1}x))^{-1}, & n < 0. \end{cases} \tag{2}$$

The cocycle φ gives rise to an automorphism

$$T_\varphi : (X \times G, \widetilde{\mu}) \to (X \times G, \widetilde{\mu}), \quad T_\varphi(x, g) = (Tx, \varphi(x)g),$$

called a *group extension* of T, where $\widetilde{\mu}$ stands for the product measure $\mu \times m$. Clearly,

$$L^2(X \times G, \widetilde{\mu}) = \bigoplus_{\chi \in \widehat{G}} L_\chi,$$

where $L_\chi = \{f \otimes \chi : f \in L^2(X, \mu)\}$. The spaces L_χ are closed and T_φ-invariant; moreover, $U_{T_\varphi} : L_\chi \to L_\chi$ is unitarily equivalent to

$$V_{\varphi,T,\chi} : L^2(X, \mu) \to L^2(X, \mu), \quad V_{\varphi,T,\chi}(f) = \chi\varphi \cdot fT.$$

The maximal spectral type of $V_{\varphi,T,\chi}$ will be denoted by ϱ_χ (note that if $\chi \equiv 1$ then $V_{\varphi,T,\chi}$ is unitarily equivalent to U_T).

Let $v : G \to G$ be a continuous group automorphism and let $\widehat{v} : \widehat{G} \to \widehat{G}$ denote its dual, $\widehat{v}(\chi) = \chi \circ v$. If $\chi \in \widehat{G}$ then its $\widehat{v}$-*trajectory* is the set $\{\chi, \widehat{v}(\chi), \widehat{v}^2(\chi), \ldots\}$. Put

$$E(v) = \{\mathrm{card}(\{\chi, \widehat{v}(\chi), \widehat{v}^2(\chi), \ldots\}) : \chi \in \widehat{G}\}.$$

In [8] the following theorem has been proved:

THEOREM 2. *Given a continuous group automorphism $v : G \to G$, there exist an ergodic automorphism $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ and a cocycle $\varphi : X \to G$ such that*

(a) *$V_{\varphi,T,\chi}$ has simple spectrum for each $\chi \in \widehat{G}$,*

(b) *if $\chi, \gamma \in \widehat{G}$ are in the same $\widehat{v}$-trajectory then ϱ_χ and ϱ_γ are equivalent,*

(c) *if $\chi, \gamma \in \widehat{G}$ are in different $\widehat{v}$-trajectories then ϱ_χ and ϱ_γ are mutually singular,*

(d) *T_φ is ergodic.*

If T_φ satisfies (a)–(d) above then obviously $E(T_\varphi) = E(v)$. In [22], [8] and [3], given a set A satisfying (1), some compact abelian metric group G and its automorphism v with $E(v) = A$ are constructed. However, as we were recently informed by Professor B. Weiss, the sets $E(v)$ always satisfy (1). Therefore by Theorem 2 we can only reach sets satisfying (1) as sets of essential values of the multiplicity function.

If $H \subset G$ is a compact subgroup then it determines a natural factor $T_{\varphi,H}$ of T_φ given by

$$T_{\varphi,H} : (X \times G/H, \widetilde{\mu}_H) \to (X \times G/H, \widetilde{\mu}_H), \quad T_{\varphi,H}(x, gH) = (Tx, \varphi(x)gH),$$

where $\widetilde{\mu}_H$ is the image of $\widetilde{\mu}$ via the map $(x,g) \to (x, gH)$. Define

$$\mathcal{H} = \operatorname{ann} H = \{\chi \in \widehat{G} : \chi(h) = 1 \text{ for each } h \in H\}.$$

We have $\operatorname{ann} \mathcal{H} = H$. Notice that (up to a natural identification)

$$L^2(X \times G/H, \widetilde{\mu}_H) = \bigoplus_{\chi \in \mathcal{H}} L_\chi \tag{3}$$

and L_χ is now $U_{T_{\varphi,H}}$-invariant. Let $\mathcal{H} \subset \widehat{G}$ be a (countable) subgroup. Set

$$E(v, \mathcal{H}) = \{\operatorname{card}(\{\chi, \widehat{v}(\chi), \widehat{v}^2(\chi), \ldots\} \cap \mathcal{H}) : \chi \in \mathcal{H}\}.$$

THEOREM 3. *Given a continuous group automorphism $v : G \to G$ and a subgroup $\mathcal{H} \subset \widehat{G}$, there exist an ergodic automorphism $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ and a cocycle $\varphi : X \to G$ such that T_φ is ergodic and*

$$E(T_{\varphi,H}) = E(v, \mathcal{H}),$$

where $H = \operatorname{ann} \mathcal{H}$.

Proof. Take T and φ satisfying the assertion of Theorem 2. Notice that the quotient action $U_{T_{\varphi,H}}$ on L_χ is exactly the action of U_{T_φ} on L_χ; now, by (3) and (a)–(d) of Theorem 2, the result directly follows.

The most difficult part of Theorem 1 will be proved if we show the following

ALGEBRAIC LEMMA. *Given a subset $A \subseteq \mathbb{N}^+$ with $1 \in A$, there exist a compact metric abelian group G, a continuous group automorphism $v : G \to G$ and a subgroup $\mathcal{H} \subset \widehat{G}$ such that*

$$E(v, \mathcal{H}) = A. \tag{4}$$

Proof. We will show that on $G = \mathbb{T} \times \mathbb{T} \times \ldots$ we can find v and $H = \operatorname{ann} \mathcal{H}$ satisfying (4). We have $\widehat{G} = \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots$ Suppose that

$$A = \{1 < n_1 < n_2 < \ldots\} \subset \mathbb{N}^+.$$

First, define sequences of natural numbers $l_1^{(k)} < l_2^{(k)} < \ldots < l_{n_k}^{(k)} < l_k$, $k = 1, 2, \ldots$, as follows:

$$l_0 = 1, \quad l_k = 2^{n_k} l_{k-1}, \quad k = 2, 3, \ldots,$$
$$l_p^{(k)} = 2^{p-1} l_{k-1}, \quad p = 1, \ldots, n_k, \ k = 1, 2, \ldots$$

Put

$$\mathcal{H} = \{h \in \widehat{G} : h = (h_1, h_2, \ldots), h_n = 0 \text{ for } n \in \mathbb{N} \setminus \{l_1^{(1)}, \ldots, l_{n_1}^{(1)}, l_1 + l_1^{(2)}, \ldots, l_1 + l_{n_2}^{(2)}, \ldots, l_1 + \ldots + l_{k-1} + l_1^{(k)}, \ldots, l_1 + \ldots + l_{k-1} + l_{n_k}^{(k)}, \ldots\}\}.$$

Define $\widehat{v} : \widehat{G} \to \widehat{G}$ to be the following permutation:

$$\widehat{v}((\underbrace{h_1, \ldots, h_{l_1}}, \underbrace{h_{l_1+1}, \ldots, h_{l_1+l_2}}, \ldots, \underbrace{h_{l_1+\ldots+l_{k-1}+1}, \ldots, h_{l_1+\ldots+l_{k-1}+l_k}}, \ldots))$$

$$= (\underbrace{h_{l_1}, h_1, \ldots, h_{l_1-1}}, \underbrace{h_{l_1+l_2}, h_{l_1+1}, \ldots, h_{l_1+l_2-1}}, \ldots, \underbrace{h_{l_1+\ldots+l_{k-1}+l_k}, h_{l_1+\ldots+l_{k-1}+1}, \ldots, h_{l_1+\ldots+l_{k-1}+l_k-1}}, \ldots).$$

Notice that $A \subset E(v, \mathcal{H})$ because $\operatorname{card}(\{0, \widehat{v}(0), \ldots\} \cap \mathcal{H}) = 1$ and $\operatorname{card}(\{h, \widehat{v}(h), \ldots\} \cap \mathcal{H}) = n_k$ for $h \in \mathcal{H}$ defined by $h_{l_1+\ldots+l_{k-1}+l_1^{(k)}} = 1$, $h_n = 0$ for $n \neq l_1 + \ldots + l_{k-1} + l_1^{(k)}$.

It remains to show that $E(v, \mathcal{H}) \subset A$. Take $h \in \mathcal{H}$. Then only a finite number of h_n are different from zero. Let $i \in \mathbb{N}$ be largest such that $h_i \neq 0$. There exist $k, p \in \mathbb{N}$ for which $i = l_1 + \ldots + l_{k-1} + l_p^{(k)}$ $(1 \le p \le n_k)$. Then $\widehat{v}^{l_k}(h) = h$ because $l_i \mid l_k$ for $i \le k$. Assume that there exists $q \in \mathbb{N}$ with $q < p$ such that for $j = l_1 + \ldots + l_{k-1} + l_q^{(k)}$ we have $h_j \neq 0$. Now, if we look at the "distance" between the nonzero values h_i, h_j in $\widehat{v}^m(h)$ $(m \in \mathbb{N})$, we will always find either $i - j$ or $(l_k - 1) - (i - j)$. Moreover, whenever $\widehat{v}^m(h) \in \mathcal{H}$, the nonzero values h_i, h_j will be at positions with indices in the set

$$L_k = \{l_1 + \ldots + l_{k-1} + l_1^k, \ldots, l_1 + \ldots + l_{k-1} + l_{n_k}^{(k)}\}.$$

However, the numbers $l_1^{(k)}, \ldots, l_{n_k}^{(k)}$ are chosen so that $l_{p_1}^{(k)} - l_{p_2}^{(k)} \neq l_{q_1}^{(k)} - l_{q_2}^{(k)}$ and $(l_k - 1) - (l_{p_1}^{(k)} - l_{p_2}^{(k)}) \neq l_{q_1}^{(k)} - l_{q_2}^{(k)}$ for $\{p_1, p_2\} \neq \{q_1, q_2\}$. Thus $\widehat{v}^m(h) \in \mathcal{H}$ implies $l_k \mid m$, so $\widehat{v}^m(h) = h$ and hence

$$\operatorname{card}(\{h, \widehat{v}(h), \ldots\} \cap \mathcal{H}) = 1 \quad (\in A).$$

Assume now that h_i is the only nonzero value among the coordinates with indices from $l_1 + \ldots + l_{k-1} + 1$ to $l_1 + \ldots + l_k$. No matter what the other nonzero values of h are, we will show that $\widehat{v}^m(h) \in \mathcal{H}$ iff h_i (in $\widehat{v}^m(h)$) stands at a position whose coordinate belongs to L_k. The necessity is obvious. The "distances" (in $\{l_1 + \ldots + l_{k-1} + 1, \ldots, l_1 + \ldots + l_k\}$) between the coordinates from L_k are multiples of l_{k-1}. Therefore $l_{k-1} \mid m$. Since $l_s \mid l_{k-1}$ for $s \le k - 1$,

$$(\widehat{v}^m(h))_n = h_n \quad \text{for } n \le l_1 + \ldots + l_{k-1},$$

whence $\widehat{v}^m(h) \in \mathcal{H}$. We have shown that in this case

$$\operatorname{card}(\{h, \widehat{v}(h), \ldots\} \cap \mathcal{H}) = \operatorname{card} L_k = n_k \in A.$$

Remark. Notice that in the case of A finite, the construction needed for the proof of the Algebraic Lemma can be accomplished on a finite-dimensional torus, and moreover for some $N \ge 1$, $v^N = \mathrm{Id}$.

Therefore, for A finite, there exist $d \ge 1$, $v : \mathbb{T}^d \to \mathbb{T}^d$ with $v^N = \mathrm{Id}$ for some $N \ge 1$ and $\mathcal{H} \subset \mathbb{Z}^d$ such that

$$A = E(v, \mathcal{H}). \tag{5}$$

Now, the methods from [3] of producing analytic diffeomorphisms of the form T_φ (where T is a one-dimensional irrational rotation and $\varphi : \mathbb{T} \to \mathbb{T}^d$), ergodic with respect to Lebesgue measure and having $E(T_\varphi)=E(v)$, can be applied. Therefore, by (5),

$$E(T_{\varphi,H})=E(v,\mathcal{H})=A,$$

for $H=\operatorname{ann}\mathcal{H}$.

3. Weakly mixing case. The method from [8] can be directly applied only to T which is an ergodic rotation. Therefore, our constructions, though ergodic, are not weakly mixing. We will now show that these can be adapted to T admitting some fast cyclic approximation (hence weak mixing can be achieved).

Let $V:H\to H$ be an isometry on a separable Hilbert space. It is said to be *rigid* if for some sequence (m_t),

$$(\forall f\in H)\quad V^{m_t}f\to f.$$

The sequence (m_t) is then called a *rigidity time* for V (in case $V=U_T$ a *rigidity time* for T). Based on an idea from [11], call V *α-weakly mixing along a sequence* (m_t) if

$$(\forall f\in H)\quad (V^{m_t}f,f)\to\alpha\|f\|^2$$

($\alpha\in\mathbb{C}; 0\le|\alpha|\le1$). Notice that if (m_t) is a rigidity time then V is 1-weakly mixing along (m_t). Moreover, if V is α-weakly mixing and $|\alpha|<1$ then V has no eigenvalues (since eigenvalues of V are of modulus 1). As shown in [8],

(6) if $U_i:H_i\to H_i$, $i=1,2$, are unitary and α_i-weakly mixing along a common subsequence (m_t) then the maximal spectral types of U_i are mutually singular whenever $\alpha_1\ne\alpha_2$.

We will need a criterion for α-weak mixing of unitary operators of the form $V_{\varphi,T,\chi}$, where T is rigid (compare with Proposition 5 of [8].

PROPOSITION 1. *If $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is ergodic, $\varphi:X\to G$ is a cocycle and (m_t) a rigidity time for T then $V_{\varphi,T,\chi}$ is α-weakly mixing along (m_t) whenever*

$$\int_X \chi(\varphi^{(m_t)})\,d\mu\to\alpha.$$

Moreover, if $|\alpha|<1$, then $V_{\varphi,T,\chi}$ has no eigenvalues.

Proof. First, we show that if $g=fT-f$, $f\in L^2(X,\mu)$ (i.e. if g is a coboundary) then

(7)
$$\int_X \chi(\varphi^{(m_t)})g\,d\mu\to 0.$$

Indeed,

$$\begin{aligned}\int_X \chi(\varphi^{(m_t)})fT\,d\mu-\int_X\chi(\varphi^{(m_t)})f\,d\mu
&=\int_X \chi\varphi^{(m_t)}(x)\chi(\varphi(T^{m_t}x))\overline{\chi(\varphi(x))}f(Tx)\,d\mu(x)\\
&\quad-\int_X\chi(\varphi^{(m_t)}(x))f(x)\,d\mu(x)+\int_X\chi(\varphi^{(m_t)}(x))f(Tx)\,d\mu(x)\\
&\quad-\int_X \chi\varphi^{(m_t)}(x)\chi(\varphi(T^{m_t}x))\overline{\chi(\varphi(x))}f(Tx)\,d\mu(x)\\
&=\int_X\chi\varphi^{(m_t)}(Tx)f(Tx)\,d\mu(x)-\int_X\chi(\varphi^{(m_t)}(x))f(x)\,d\mu(x)\\
&\quad-\int_X\chi\varphi^{(m_t)}(Tx)f(Tx)[1-\chi(\varphi(T^{m_t}x))\overline{\chi(\varphi(x))}]\,d\mu(x).\end{aligned}$$

Hence, by the Schwarz inequality

$$\Big|\int_X\chi\varphi^{(m_t)}g\,d\mu\Big|^2\le\|f\|^2_{L^2}\int_X|1-\chi(\varphi(T^{m_t}x))\overline{\chi(\varphi(x))}|^2\,d\mu\to0$$

since (m_t) is a rigidity time for T. Hence (7) is established.

Now, since T is ergodic, the coboundaries are dense in the space of zero mean functions, so obviously (7) is valid if g is replaced by any zero mean function. Consequently, if $g\in L^2(X,\mu)$ then

(8)
$$\int_X\chi(\varphi^{(m_t)})g\,d\mu\to\alpha\int_X g\,d\mu.$$

Take an arbitrary $f\in L^2(X,\mu)$; then

$$\begin{aligned}&\Big|\int_X\chi\varphi^{(m_t)}(x)f(T^{m_t}x)\overline{f(x)}\,d\mu-\int_X\chi(\varphi^{(m_t)}(x))|f(x)|^2\,d\mu\Big|\\
&\qquad=\Big|\int_X\chi\varphi^{(m_t)}\bar f(fT^{m_t}-f)\,d\mu\Big|\le\|f\|_{L^2}\Big(\int_X|fT^{m_t}-f|^2\,d\mu\Big)^{1/2}\to0\end{aligned}$$

and directly by (8),

$$\int_X\chi(\varphi^{(m_t)}(x))|f(x)|^2\,d\mu(x)\to\alpha\|f\|^2.$$

Recall that the disjointness criterion (6) for spectral measures is used to construct cocycles for which ϱ_χ and ϱ_γ are mutually singular whenever χ

and γ are in different $\widehat{v}$-trajectories (for nontrivial χ with $|\alpha_\chi| < 1$ to assure weak mixing of T_φ provided that T is weakly mixing).

On the other hand, to show that ϱ_χ and ϱ_γ are equivalent if they are in the same $\widehat{v}$-trajectory one solves a functional equation

$$\varphi(Sx)/v\varphi(x) = f(Tx)/f(x) \tag{9}$$

for some

$$S : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu), \qquad ST = TS, \tag{10}$$

and a measurable $f : X \to G$. (Indeed, if we have $\widetilde{S}(x, g) = S_{f,v}(x, g) = (Sx, f(x)v(g))$ then (9) and (10) are equivalent to saying that $\widetilde{S}T_\varphi = T_\varphi\widetilde{S}$, $\widetilde{S} : (X \times G, \widetilde{\mu}) \to (X \times G, \widetilde{\mu})$; moreover, $U_{\widetilde{S}}(L_\chi) = L_{\widehat{v}\chi}$ for each $\chi \in \widehat{G}$.)

Now, given G and a continuous group automorphism $v : G \to G$, we can apply the whole machinery of [8] (for constructing appropriate cocycles) to the del Junco–Rudolph rank-1, rigid, weakly mixing example (see [10]), where the centralizer of T (in the sense of (10)) can be explicitly described. We thus get weakly mixing cocycles $\varphi : X \to G$ for which (9) can be solved (with an appropriate S) and such that $V_{\varphi,T,\chi}$ are α_χ-weakly mixing with $\alpha_\chi \neq \alpha_\gamma$ whenever χ and γ are from different $\widehat{v}$-trajectories. This completes the proof of Theorem 1.

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Received April 25, 1994
Revised version March 31, 1995 (3270)