Some counterexamples to subexponential growth of orthogonal polynomials

by

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Abstract. We give examples of polynomials $p(n)$ orthonormal with respect to a measure $\mu$ on $\mathbb{R}$ such that the sequence $(p(n, x))$ has exponential lower bound for some points $x$ of supp $\mu$. Moreover, the set of such points is dense in the support of $\mu$.

1. Introduction. Let $\mu$ be a probability measure on $\mathbb{R}$ with all moments finite. Applying the Gram-Schmidt procedure to $\{x^n\}$ with respect to the inner product $(f, g) = \int f g \, d\mu$ we get a system of polynomials $\{p(n, x)\}$ satisfying

$$xp(n, x) = \lambda_{n+1}p(n+1, x) + \beta_n p(n, x) + \lambda_n p(n-1, x),$$

where $\lambda_n > 0, \beta_n \in \mathbb{R}$.

In [5] J. Zhang has shown that for $\lambda_n$ and $\beta_n$ asymptotically periodic and $\lambda_n$ bounded away from 0 one has

$$\limsup_{n \to \infty} |p(n, x)|^{1/n} \leq 1$$

uniformly for $x \in \text{supp} \mu$. Zhang’s proof is a refinement of methods used in [3] where the case of convergent coefficients was considered (see also [1], [2]).

There were suggestions that asymptotic periodicity of $\lambda_n$ and $\beta_n$ is essential in the result above. In [4] R. Szwarc has constructed examples where for a point $x$ in supp $\mu$,

$$\liminf_{n \to \infty} |p(n, x)|^{1/n} > 1$$

with $\lambda_n$ and $\beta_n$ bounded, $\lambda_n$ bounded away from 0. One of his examples is $\lambda_n = 1/2, n \geq 0$, and $\beta_n = 0$ for $n \in N \setminus A$, where $A$ is a lacunary subset of $N$.

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In this paper we show that the set of points satisfying (1) can be dense in the support of $\mu$. Moreover, we are able to compute $\text{supp} \mu$ explicitly.

Our results are the following:

**Theorem 1.** Let $\{p(n, x)\}$ be the system of orthogonal polynomials satisfying

$$xp(n, x) = \frac{1}{2} p(n + 1, x) + b_n p(n, x) + \frac{1}{2} p(n - 1, x),$$

where

$$b_n = \begin{cases} 0, & n_{2k} \leq n < n_{2k+1}, \\ \pi, & n_{2k+1} \leq n < n_{2k+2}, \end{cases}$$

and $n_k = 2^k, k > 0, n_0 = 0$, and $\mu$ be the corresponding orthogonality measure. Then $\text{supp} \mu = [\pi - 1, \pi + 1]$ and the set of points satisfying (1), i.e.

$$\{x \in \text{supp} \mu \mid \liminf_{n \to \infty} |p(n, x)|^{1/n} > 1\},$$

is dense in $\text{supp} \mu$.

**Theorem 2.** Let $\{p(n, x)\}$ and $\mu$ be as above, where

$$n_k = \begin{cases} 2^j, & k = 2^j, \\ (j + 1)2^j, & k = 2j + 1. \end{cases}$$

Then

$$\limsup_{n \to \infty} |p(n, x)|^{1/n} \leq 1$$

uniformly for $x \in [\pi - 1, \pi + 1]$ and the set of points satisfying (1) is dense in $[-1, 1]$.

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2. Some useful lemmas. Denote by $p(n, x)$ polynomials satisfying

$$xp(n, x) = \frac{1}{2} p(n + 1, x) + b_n p(n, x) + \frac{1}{2} p(n - 1, x),$$

(2)

$$p(-1, x) = 0, \quad p(0, x) = 1,$$

where

$$b_n = \begin{cases} r_1, & n_{2k} \leq n < n_{2k+1}, \\ r_2, & n_{2k+1} \leq n < n_{2k+2}. \end{cases}$$

$\{n_k\}$ is an increasing sequence of integers, $n_0 = 0$, and $r_1, r_2$ are real numbers. We will write $p(n) = p(n, x)$ for fixed $x$.

With the polynomials $p(n, x)$ we associate the Jacobi matrix $J$

$$J = \begin{pmatrix} b_0 & \frac{1}{2} & \frac{1}{2} & \\ \frac{1}{2} & b_1 & \frac{1}{2} & \\ \frac{1}{2} & \frac{1}{2} & b_2 & \end{pmatrix}.$$  

If the coefficients $b_n$ are bounded then the orthogonality measure $\mu$ coincides with the spectral measure of $J$, hence $\text{supp} \mu$ and the spectrum $\sigma(J)$ are equal.

Let

$$J_0 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \\ \frac{1}{2} & 0 & \frac{1}{2} & \\ \frac{1}{2} & \frac{1}{2} & 0 & \end{pmatrix}.$$  

It is well known that $\sigma(J_0) = [-1, 1]$.

**Lemma 1.** Let $\{p(n, x)\}$ be the system of orthogonal polynomials satisfying (2), and

$$\limsup_{k \to \infty} (n_{2k+1} - n_{2k}) = \limsup_{k \to \infty} (n_{2k+2} - n_{2k+1}) = \infty.$$  

Then $\text{supp} \mu = [r_1 - 1, r_1 + 1] \cup [r_2 - 1, r_2 + 1]$, where $\mu$ is the orthogonality measure.

**Proof.** Without loss of generality we can assume $r_1 < r_2$. Hence we have

$$J_0 + r_1 I \leq J \leq J_0 + r_2 I.$$  

This gives $\text{supp} \mu = \sigma(J) \subseteq [r_1 - 1, r_2 + 1]$. From (3) we get

$$[r_1 - 1, r_1 + 1] \cup [r_2 - 1, r_2 + 1] \subseteq \sigma(J).$$  

(for example the sequence $x_k^\psi \in \ell^2(N)$, $x_k^\psi(n) = \psi n^\psi + r_1$ for $n_{2k} \leq n < n_{2k+1}$ and 0 otherwise, is an approximate eigenvector corresponding to the number $r_1 + \cos \psi \in \sigma(J)$).

Now if $r_2 - r_1 \leq 2$, then $\text{supp} \mu = [r_1 - 1, r_2 + 1]$. Assume $r_2 - r_1 > 2$.

Fix $x \in (r_1 + 1, r_2 - 1)$. There exists a constant $c > 1$ such that $|x - r_i| > c$. Note that $p(1) = 2(x - b_0) > 2c > c = \text{cp}(0)$, so $p(1) - \text{cp}(0) > 0$. From (2) we have

$$|p(n + 1)| \geq 2c|p(n)|,$$

hence

$$|p(n + 1)| - c|p(n)| \geq c|p(n)| - |p(n - 1)| \geq c|p(n)| - c^2|p(n - 1)|.$$
By induction we get $|p(n + 1)| - c|p(n)| \geq 0$, hence $|p(n)| \geq c^n$. Therefore
$$\forall x \in (r_1 + 1, r_2 - 1) \quad \lim_{n \to \infty} |p(n, x)| = \infty.$$ 
Hence $(r_1 + 1, r_2 - 1) \cap \text{supp} \mu = \emptyset$.■

Throughout the rest of this section we use the following notation:

(4)
$$r = r_2 - r_1,$$
(5)
$$x = r_2 + \cos \psi, \quad \psi \in [0, \pi]$$
(6)
$$p(n) = p(n, x).$$

We assume that $|r| > 2$. Then there is a unique real number $\gamma$ satisfying
$$\frac{1}{2}(\gamma + \gamma^{-1}) = r + \cos \psi, \quad |\gamma| > 1.$$ 

**Lemma 2.** Let $r = r_2 - r_1$ be a transcendental number. For any $\psi \in \mathbb{Q} \pi$, 
$$\frac{p(n + 1, x)}{p(n, x)} \neq \gamma^{-1}.$$ 

**Proof.** By substituting $x' = x - r_1$ we can reduce ourselves to the case $r_1 = 0$, $r_2 = r$. Moreover, without loss of generality we can take $r > 0$. Assume now a contrario that we have equality in (8), i.e.

(9)
$$\frac{p(n + 1, r + \cos \psi)}{p(n, r + \cos \psi)} = \gamma^{-1},$$

for some $n$ and $\psi$. From (7) we get
$$\gamma^{-1} = r + \cos \psi - \sqrt{(r + \cos \psi)^2 - 1}.$$ 

The left hand side of (9) is a rational function in $r$, as opposed to the right hand side. Hence (9) cannot be satisfied identically.

We can transform (9) to a polynomial equation in $r$ with algebraic coefficients because $\cos \psi$ is algebraic. This implies that $r$ is algebraic, which contradicts the assumptions.■

**Lemma 3.** We have

(10)
$$p(n) = \frac{\gamma p(n_{2k}) - p(n_{2k} - 1)}{\gamma^2 - 1} \gamma^{n-n_{2k}+1} + \frac{\gamma p(n_{2k} - 1) - p(n_{2k})}{\gamma^2 - 1} \gamma^{-(n-n_{2k})}$$

for $n_{2k} < n \leq n_{2k+1}$, and

(11)
$$p(n) = \left\{ \begin{array}{ll}
\frac{-\sin(n-n_{2k+1})\psi}{\sin \psi} p(n_{2k+1} - 1) & \psi \neq 0, \pi, \\
\frac{\sin(n-n_{2k+1}+1)}{\sin \psi} p(n_{2k+1}), & (n-n_{2k+1}+1) p(n_{2k+1}) - (n-n_{2k+1}) p(n_{2k+1} - 1), \\
(n-n_{2k+1}+1) p(n_{2k+1}) - (n-n_{2k+1}) p(n_{2k+1} - 1), & \text{otherwise},
\end{array} \right.$$

for $n_{2k+1} < n \leq n_{2k+2}$ ($\psi$ and $\gamma$ are given by (5), (7)). Moreover, if $\psi = \frac{q}{q_0} \pi$, $\psi \neq 0, \pi$, and $q_0 (n_{2k+2} - n_{2k+1})$ for any $k > k_0$, then for $k > k_0$ and $n_{2k} < n \leq n_{2k+1}$ one has

(12)
$$p(n) = \frac{\gamma p(n_{2k}) - p(n_{2k} - 1)}{\gamma^2 - 1} \gamma^{n_{2k+1}+1} + \frac{\gamma p(n_{2k} - 1) - p(n_{2k})}{\gamma^2 - 1} \gamma^{-n_{2k}},$$

where

(13)
$$m_n = n - n_{2k} + \sum_{i=k_0}^{k-1} (n_{2i+1} - n_{2i}).$$

**Proof.** From (2) we get

$$\frac{1}{2} p(n) + r_1 p(n - 1) + \frac{1}{2} p(n - 2) = \frac{1}{2} (\gamma + \gamma^{-1}) r_1 (p(n - 1) + \frac{1}{2} p(n - 2)),$$

and

(14)
$$p(n) = (\gamma + \gamma^{-1}) p(n - 1) - p(n - 2) \quad \text{for } n_{2k} < n \leq n_{2k+1},$$

so

$$\frac{1}{2} p(n) + r_2 p(n - 1) + \frac{1}{2} p(n - 2) = \frac{1}{2} (r_2 + \cos \psi) p(n - 1),$$

so

$$p(n) = 2p(n - 1) \cos \psi - p(n - 2) \quad \text{for } n_{2k+1} < n \leq n_{2k+2}.$$ 

Now we can get (10) and (11) by induction.

To prove (12) observe that by (11) we get

(15)
$$p(n_{2k}) = p(n_{2k-1}), \quad p(n_{2k} - 1) = p(n_{2k-1} - 1),$$

for $k > k_0$. Arrange the numbers $n$ satisfying

$$n \leq n_{2k_0+1} \quad \text{or} \quad n_{2k} < n \leq n_{2k+1} \quad \text{for } k > k_0$$

into an increasing sequence. A number $n$ will occupy the position $\vec{n}$ given by

$$\vec{n} = \left\{ \begin{array}{ll}
n & \text{for } n \leq n_{2k_0+1}, \\
n - \sum_{i=0}^{k-1} (n_{2i+2} - n_{2i+1}) = m_n + n_{2k_0} & \text{for } k > k_0, n_{2k} < n \leq n_{2k+1}.
\end{array} \right.$$ 

Define a sequence $\bar{p}(n)$ by $\bar{p}(\vec{n}) = p(n)$. By (14) and (15) we get

$$\bar{p}(\vec{n}) = (\gamma + \gamma^{-1}) \bar{p}(\vec{n} - 1) - \bar{p}(\vec{n} - 2) \quad \text{for } \vec{n} > n_{2k_0}.$$ 

By induction we obtain

$$\bar{p}(\vec{n}) = \frac{\gamma \bar{p}(n_{2k_0}) - \bar{p}(n_{2k_0} - 1)}{\gamma^2 - 1} \gamma^{n_{2k_0}+1} + \frac{\gamma \bar{p}(n_{2k_0} - 1) - \bar{p}(n_{2k_0})}{\gamma^2 - 1} \gamma^{-(n_{2k_0})},$$

Taking into account that $\vec{n} = n$ for $n \leq n_{2k_0+1}$ and that $\bar{p}(\vec{n}) = p(n)$ gives the conclusion.■
Proposition 1. Let $\psi = (q/q_0)\pi$, $\psi \neq 0, \pi$ and $q, q_0 \in \mathbb{N}$. Assume $q_0 \mid (n_{2k+2} - n_{2k+1})$ for all $k$ starting from some $k_0$. Then there is a constant $A > 0$ and an integer $N$ such that

$$|p(n)| > A|\gamma|^m$$

for $N \leq n_{2k} < n \leq n_{2k+2}$,

where

$$m = \min \left( n - n_{2k} + \sum_{i=k_0}^{k-1} (n_{2i+1} - n_{2i}), \sum_{i=k_0}^k (n_{2i+1} - n_{2i}) \right).$$

Proof. From (12) (Lemma 3) we get for $n_{2k} < n \leq n_{2k+1}$, $k \leq k_0$,

$$|p(n)| \geq \left| \frac{\gamma p(n_{2k_0}) - p(n_{2k_0} - 1)}{\gamma^2 - 1} \right| |\gamma|^{-m_n + 1}$$

$$- \left| \frac{\gamma (p(n_{2k_0} - 1) - p(n_{2k_0}))}{\gamma^2 - 1} \right| |\gamma|^{-m_n},$$

where $m_n$ is given by (13). By Lemma 2,

$$\left| \frac{\gamma p(n_{2k_0 + 1}) - p(n_{2k_0 + 1} - 1)}{\gamma^2 - 1} \right| \neq 0.$$

Since $|\gamma| > 1$ there exist constants $B > 0$ and $k_1$ such that

$$|p(n)| > B|\gamma|^m,$$

where $k > k_1$ and $n_{2k} < n \leq n_{2k+1}$.

From (11) we have for $n_{2k+1} < n \leq n_{2k+2}$,

$$|p(n)| = \left| \frac{p(n_{2k+1} - 1)}{\sin \psi} \right| \left| \frac{p(n_{2k+1})}{p(n_{2k+1} - 1)} \left| \sin(n - n_{2k+1}) \psi - \sin(n - n_{2k+1}) \psi \right| \right|.$$  

Examine now $\frac{p(n_{2k+1})}{p(n_{2k+1} - 1)}$. From (12) we have

$$p(n) = \frac{\gamma p(n_{2k_0 + 1}) - p(n_{2k_0 + 1} - 1)}{\gamma^2 - 1} \gamma^{-m_n + 1}$$

$$+ \frac{\gamma p(n_{2k_0 + 1} - 1) - p(n_{2k_0} + 1)}{\gamma^2 - 1} \gamma^{-m_n},$$

for $k > k_0$, $n = n_{2k+1}, n_{2k+1} - 1$. Lemma 2 states that

$$\forall n \in \mathbb{N}, \quad \gamma p(n_{2k_0 + 1}) - p(n_{2k_0 + 1} - 1) \neq 0,$$

so from (20) we get

$$\frac{p(n_{2k+1})}{p(n_{2k+1} - 1)} \to \gamma \quad \text{as} \quad k \to \infty.$$  

Now return to (19). We have assumed that $r$ is transcendental, hence so is $\gamma$. Since $\sin k\psi$ is an algebraic number,

$$\forall k \in \mathbb{N} \quad \gamma \sin k\psi \neq \sin(k - 1)\psi \neq 0.$$

Moreover, there exists a constant $C > 0$ (depending on $\psi$) such that

$$\forall k \in \mathbb{N} \quad |\gamma \sin k\psi - (k - 1)\psi| > C.$$

By (21) there exists $k_2$ such that $|p(n_{2k_1} + 1)/p(n_{2k_2} + 1) - 1| < C/2$ for $k > k_2$. Hence for $n_{2k+1} < n \leq n_{2k_2+2}$, $k > k_2$,

$$\left| \frac{p(n_{2k+1})}{p(n_{2k+1} - 1)} \sin(n - n_{2k+1}) \psi - \sin(n - n_{2k+1}) \psi \right|$$

$$\geq |\gamma \sin(n - n_{2k+1}) \psi - \sin(n - n_{2k+1}) \psi|$$

$$- \left| \frac{p(n_{2k+1})}{p(n_{2k+1} - 1)} \right| |\sin(n - n_{2k+1}) \psi| \geq C.$$

By (19) we thus get

$$|p(n)| \geq \left| \frac{p(n_{2k+1} + 1)}{p(n_{2k+1} - 1)} \right| \frac{C}{2},$$

and by (18),

$$|p(n)| > \frac{BC}{2} |\gamma|^m,$$

where $n_{2k} < n \leq n_{2k+2}$, $k > k_1, k_2$, and

$$m = n_{2k+1} - 1 - n_{2k} + \sum_{i=k_0}^{k-1} (n_{2i+1} - n_{2i}) < \sum_{i=k_0}^k (n_{2i+1} - n_{2i}).$$

Combining (18) and (22) gives the conclusion with

$$A = \min \left( B, \frac{BC}{2 \sin \psi} \right) \quad \text{and} \quad N = \max(n_{2k_0}, n_{2k_1}, n_{2k_2}).$$

Proposition 2. Let $n_{2k} < n \leq n_{2k+2}$. Then

$$|p(n)| \leq M 2^{-k+1} A^k |\gamma|^m,$$

where

$$m = \min \left( n - n_{2k} + \sum_{i=0}^{k-1} (n_{2i+1} - n_{2i}), \sum_{i=0}^k (n_{2i+1} - n_{2i}) \right),$$

$$M = \prod_{i=0}^k (n_{2i+1} - n_{2i} + 1) \quad \text{and} \quad A = \left| \frac{\gamma^2}{\gamma^2 - 1} \right|. $$
Proof of (10) gives
\[ p(n) = \frac{\gamma^2 n^m - \gamma^{-m}}{\gamma^2 - 1} p(n_{2k}) + \frac{\gamma^m n^m - \gamma^{-m}}{\gamma^2 - 1} p(n_{2k} - 1) \]
for \( n_{2k} < n \leq n_{2k+1} \), where \( m = n - n_{2k} \). Observe that \( \gamma^2 \gamma^m \) and \( \gamma^{-m} \) have the same sign, so
\[ \left| \frac{\gamma^2 n^m - \gamma^{-m}}{\gamma^2 - 1} \right| \leq \left| \frac{\gamma^2}{\gamma^2 - 1} \right| |\gamma|^m. \]
In the same way we get
\[ \left| \frac{\gamma^m n^m - \gamma^{-m}}{\gamma^2 - 1} \right| \leq \left| \frac{\gamma^2}{\gamma^2 - 1} \right| |\gamma|^m. \]
Hence
\[ |p(n)| \leq \left| \frac{\gamma^2}{\gamma^2 - 1} \right| |\gamma|^{n-n_{2k}} (|p(n_{2k} - 1)| + |p(n_{2k})|). \]

For \( n_{2k+1} < n \leq n_{2k+2} \) we have from (11) (note that \( |\sin n\psi| \leq n|\sin \psi| \))
\[ |p(n)| \leq (n - n_{2k+1})|p(n_{2k+1} - 1)| + (n - n_{2k+1} + 1)|p(n_{2k+1})| \]
\[ \leq (n_{2k+2} - n_{2k+1} + 1)(|p(n_{2k+1} - 1)| + |p(n_{2k+1})|). \]
Hence
\[ (25) \quad |p(n)| \leq (n_{2k+2} - n_{2k+1} + 1) \left| \frac{\gamma^2}{\gamma^2 - 1} \right| |\gamma|^{n_{2k+1} - n_{2k}} (|p(n_{2k} - 1)| + |p(n_{2k})|). \]
Combining (24) and (25) gives
\[ |p(n)| \leq (n_{2k+2} - n_{2k+1} + 1) \left| \frac{\gamma^2}{\gamma^2 - 1} \right| |\gamma|^m (|p(n_{2k} - 1)| + |p(n_{2k})|), \]
for \( n_{2k} < n \leq n_{2k+2} \), with \( m = \min(n - n_{2k}, n_{2k+1} - n_{2k}) \). Now we get the conclusion by induction. \( \star \)

Clearly the above two propositions can be proved for \( x = r_1 + \cos \psi \) using the same arguments. So set now \( r = r_1 - r_2 \). Moreover, define \( x, \psi, \gamma \) as in (5), (7). In this notation we have:

**Proposition 3.** Let \( q_k \left| (n_{2k+1} - n_{2k}) \right| \) for all \( k \) starting from some \( k_0 \) and \( \psi = (q/q_k) \psi \), \( \sin \psi \neq 0 \), \( q, q_k \in \mathbb{N} \). Then there are a constant \( A > 0 \) and an integer \( N \) such that
\[ |p(n)| > A |\gamma|^m \quad \text{for} \quad N \leq n_{2k+1} < n \leq n_{2k+3}, \]
where
\[ m = \min\left(n - n_{2k+1} + \sum_{k=0}^{k-1} (n_{2j+2} - n_{2j+1}), \sum_{k=0}^{k} (n_{2i+2} - n_{2i+1})\right). \]

**Proposition 4.** Let \( n_{2k+1} < n \leq n_{2k+3} \). Then
\[ |p(n)| \leq (|p(n_{1})| + |p(n_{0})|) M^{k-1} A^k |\gamma|^m, \]
where
\[ m = \min\left(n - n_{2k+1} + \sum_{i=0}^{k-1} (n_{2i+2} - n_{2i+1}), \sum_{i=0}^{k} (n_{2i+2} - n_{2i+1})\right), \]
\[ M = \prod_{i=1}^{k+1} (n_{2i+1} - n_{2i}) \quad \text{and} \quad A = \left| \frac{\gamma^2}{\gamma^2 - 1} \right|. \]

3. Proofs of the main theorems

**Proof of Theorem 1.** By Lemma 1 we obtain \( \text{supp} \mu = [-1, 1] \cup [-\pi, \pi] \). Let \( x = \pi - \cos \psi \), where \( \psi = (q/2\pi) \pi \), \( \psi \neq k\pi \) and \( q, q_0, k \in \mathbb{N} \). Observe that \( 2^k \left| (n_{2k+2} - n_{2k+1}) \right| \) for \( k > q_0/2 \). By Proposition 1 there exist a constant \( A > 0 \) and an integer \( N \) such that
\[ |p(n, x)| > A |\gamma|^m \quad \text{for} \quad N \leq n_{2k} < n \leq n_{2k+2}, \]
where
\[ m = \min\left(n - 2^{2k} + \sum_{i=k_0}^{k-1} 2^{2i}, \sum_{i=k_0}^{k} 2^{2i}\right). \]
Hence \( |p(n, x)|^{1/n} > A^{1/n} |\gamma|^m \), where
\[ m' := \frac{\sum_{i=k_0}^{k-1} 2^{2i}}{2^{2k+2}} + 1 \leq \frac{1}{12}. \]
So
\[ \liminf_{n \to \infty} |p(n, x)|^{1/n} \geq |\gamma|^{1/12} > 1. \]
In the same way for \( x = \cos \psi \) (using Proposition 3) we prove
\[ \liminf_{n \to \infty} |p(n, x)|^{1/n} \geq |\gamma|^{1/12} > 1. \]

**Proof of Theorem 2.** Let \( x = \cos \psi \), where \( \psi = (q/2\pi) \pi \), \( \psi \neq k\pi \) and \( q, q_0, k \in \mathbb{N} \). As above, \( 2^k \left| (n_{2k+1} - n_{2k}) \right| \) for \( k > q_0/2 \). By Proposition 3 we get a constant \( A > 0 \) and an integer \( N \) such that
\[ |p(n, x)| > A |\gamma|^m \quad \text{for} \quad N \leq n_{2k+1} < n \leq n_{2k+3}, \]
where
\[ m = \min\left(n - (k + 1)2^k + \sum_{j=k_0+1}^{k-1} (j + 1)2^i, \sum_{j=k_0+1}^{k} (j + 1)2^i\right). \]
Hence $|p(n, x)|^{1/n} > A^{1/n}|\gamma|^m$, where

$$m' = \frac{\sum_{j=0}^{k-1} (j+1)2^j}{(k+2)2^{k+1}} \leq \frac{1}{2}.$$  

So

$$\liminf_{n \to \infty} |p(n, x)|^{1/n} \geq |\gamma|^{1/2} > 1.$$

Now let $x \in [\pi - 1, \pi + 1]$ and $\psi, \gamma$ be as in (5), (7). By Proposition 2 we get

$$(28) \quad |p(n, x)| \leq M 2^{k-1} A^k |\gamma|^m \quad \text{for } n_{2k} < n \leq n_{2k+1},$$

where

$$m = \min \left( n - k2^k + \sum_{j=0}^{k-1} 2^j, \sum_{j=0}^{k-1} 2^j \right),$$

$$M = \prod_{j=0}^{k} (2^j (j + 1) + 1) \quad \text{and} \quad A = \left| \frac{\gamma^2}{\gamma^2 - 1} \right|.$$  

Hence $|p(n, x)|^{1/n} \leq M^{1/(k2^k)}(2A)^{1/2k}|\gamma|^m$, where $m' = \sum_{j=0}^{k-1} 2^j \leq k/2$. Moreover, $M \leq \prod_{j=0}^{k} 2^j (j + 2) \leq 2^k (k + 2)$, so $M^{1/(k2^k)} \leq 1$. Therefore

$$\limsup_{n \to \infty} |p(n, x)|^{1/n} \leq 1.$$

Observe that $|\gamma|$ is bounded away from 1 for all $x \in [\pi - 1, \pi + 1]$, so $A$ is bounded away from 0. Hence (28) holds with constant $A$ independent of $x$. So $\limsup_{n \to \infty} |p(n, x)|^{1/n} \leq 1$ uniformly for $x \in [\pi - 1, \pi + 1]$. \n
References


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