STUDIA MATHEMATICA 116 (2) (1995)

Operator fractional-linear transformations: convexity and compactness of image; applications

by

V. KHATSKEVICH (Karmiel) and V. SHUL'MAN (Vologda)

Abstract. The present paper consists of two parts. In Section 1 we consider fractional-linear transformations (f.-l.t. for brevity) F in the space $\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)$ of all linear bounded operators acting from $\mathcal{X}_1$ into $\mathcal{X}_2$, where $\mathcal{X}_1$, $\mathcal{X}_2$ are Banach spaces. We show that in the case of Hilbert spaces $\mathcal{X}_1$, $\mathcal{X}_2$ the image $F(\mathcal{B})$ of any (open or closed) ball $\mathcal{B}\subset D(F)$ is convex, and if $\mathcal{B}$ is closed, then $F(\mathcal{B})$ is compact in the weak operator topology (w.o.t.) (Theorem 1.2). These results extend the corresponding results on compactness obtained in [3], [4] under some additional restrictions imposed on F. We also establish that the convexity of the image of f.-l.t. is a characteristic property of Hilbert spaces, that is, if for the f.-l.t. $F: K\to(I+K)^{-1}$ the image $F(\mathcal{K})$ of the open unit ball $\mathcal{K}$ of the space $\mathcal{L}(\mathcal{X})$ is convex, then $\mathcal{X}$ is a Hilbert space (Theorem 1.3).

In Section 2 we apply the compactness of $F(\overline{\mathcal{K}})$ for the closed unit operator ball $\overline{\mathcal{K}}$ to the study of the behavior of solutions to evolution problems in a Hilbert space $\mathcal{H}$. Namely, we establish the exponential dichotomy of solutions for the so-called hyperbolic case (such that the evolution operator is invertible). This is an extension of Theorem 1.1 of [5], where the corresponding assertion was established for the particular case of a Pontryagin space $\mathcal{H}$.

1. Fractional-linear transformations.

For Banach spaces $\mathcal{X}_1$, $\mathcal{X}_2$ let $\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)$ be the space of all bounded linear operators from $\mathcal{X}_1$ to $\mathcal{X}_2$. For any operator matrix $V=(V_{ij})$, $i,j=1,2$, $V_{ij}\in\mathcal{L}(\mathcal{X}_j,\mathcal{X}_i)$, the formula

$$F(K)=(V_{21}+V_{22}K)(V_{11}+V_{12}K)^{-1}$$

defines the f.-l.t. $F=F_V: D(F)\to\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)$, where

$$D(F)=\{K\in\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2): V_{11}+V_{12}K \text{ is invertible}\}$$

(see [2], [3]).

For any (open or closed) ball $\mathcal{B}\subset D(F)$ in $\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)$ we set $F(\mathcal{B})=\{F(K): K\in\mathcal{B}\}$.

1991 *Mathematics Subject Classification*: Primary 47B50, 47A53.
Key words and phrases: Hilbert space, fractional-linear transformation, evolution operator, indefinite metric.

LEMMA 1.1. *Let $\mathcal{X}_1$, $\mathcal{X}_2$ be Hilbert spaces and $\mathcal{Y}=\mathcal{Y}(R,P,Q)$ be the set of all operators $Y\in\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)$ satisfying the inequality*

$$(1)\qquad YRY^*+PY^*+YP^*+Q\le 0,$$

where $R\in\mathcal{L}(\mathcal{X}_1)$, $P\in\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)$, $Q\in\mathcal{L}(\mathcal{X}_2)$, $R\ge 0$ and $Q^*=Q$. Then $\mathcal{Y}$ is convex and closed in the w.o.t. of the space $\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)$.

Proof. The closedness of $\mathcal{Y}$ follows immediately from the fact that $(R\xi,\xi)\le\liminf(R\xi_n,\xi_n)$ for any sequence ξ_n weakly converging to ξ. To prove convexity we assume at first that R is invertible. Then (1) can be rewritten in the form

$$(YR^{1/2}+PR^{-1/2})(YR^{1/2}+PR^{-1/2})^*\le T,$$

where $T=PR^{-1}P^*-Q$. So $\mathcal{Y}$ is convex as the preimage of the convex set $\mathcal{S}=\{S:SS^*\le T\}$ under the affine transformation $Y\to YR^{1/2}+PR^{-1/2}$.

In the general case one can use the evident equality

$$\mathcal{Y}(R,P,Q)=\bigcup_{\lambda>0}\bigcap_{\varepsilon>0}\mathcal{Y}(R+\varepsilon I,P,Q-\varepsilon\lambda I).$$

THEOREM 1.2. *If $\mathcal{X}_1$, $\mathcal{X}_2$ are Hilbert spaces, then $F(\mathcal{B})$ is convex for any ball $\mathcal{B}\subseteq D(F)$. For $\mathcal{B}$ closed, $F(\mathcal{B})$ is w.o.t.-compact.*

Proof. Let $\mathcal{B}=\{K\in\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2):\|K-K_0\|\le r\}$ be a closed ball contained in $D(F)$. Dilating and translating $\mathcal{B}$ if necessary (and changing F accordingly) we may suppose that $\mathcal{B}$ is the unit ball:

$$\mathcal{B}=\mathcal{B}_0=\{K\in\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2):\|K\|\le 1\}.$$

Since $0\in\mathcal{B}_0\subset D(F)$ the operator V_{11} is invertible, hence

$$F(K)=(V_{21}+V_{22}K)(I+V_{11}^{-1}V_{12}K)^{-1}V_{11}^{-1}$$

and $\|V_{11}^{-1}V_{12}\|<1$ (in the other case $D(F)$ would not contain $\mathcal{B}_0$). So it is sufficient to prove the convexity and w.o.t.-compactness of the image of $\mathcal{B}_0$ under any map F of the form

$$F(K)=(B+CK)(I+DK)^{-1}$$

with $\|D\|\le 1$. In this case the equality $Y=F(K)$ is equivalent to

$$(2)\qquad (YD-C)K=B-Y.$$

Hence $Y\in F(\mathcal{B}_0)$ when (2) is satisfied for some $K\in\mathcal{B}_0$. It is known (see, for example, [8]) that this is the case when

$$(3)\qquad (YD-C)(YD-C)^*\ge(B-Y)(B-Y)^*.$$

But (3) coincides with (1) if we take $R=I-DD^*$, $P=CD^*-B$ and $Q=BB^*-CC^*$. Now Lemma 1 implies that $F(\mathcal{B}_0)$ is a convex and w.o.t.-closed subset of $\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)$. Since $\|F(K)\|\le(\|B\|+\|C\|)(1-\|D\|)^{-1}$ for $K\in\mathcal{B}_0$, $F(\mathcal{B}_0)$ is bounded and weakly compact (by the Banach–Alaoglu theorem). The convexity of the image of an open ball now follows from the fact that

$$\{K:\|K-K_0\|<r\}=\bigcup_{0<\lambda<1}\{K:\|K-K_0\|\le\lambda r\}.$$

Let us show that the convexity of the image of the open unit ball under the f.-l.t. $K\to(I+K)^{-1}$ characterizes Hilbert spaces in the class of all Banach spaces.

THEOREM 1.3. *If the image of the open unit ball of $\mathcal{L}(\mathcal{X})$ under the map $K\to(I+K)^{-1}$ is convex then $\mathcal{X}$ is a Hilbert space.*

Proof. Let $\mathcal{P}=\{(I+K)^{-1}:\|K\|<1\}$. Since $\mathcal{P}$ is convex by assumption, $\mathcal{P}\xi=\{A\xi:A\in\mathcal{P}\}$ is convex for any $\xi\in\mathcal{X}$. If $\xi\ne0$ then $\mathcal{P}\xi=\{\eta\in\mathcal{X}:(I+K)\eta=\xi,\|K\|<1\}=\{\eta\in\mathcal{X}:\|\xi-\eta\|<\|\eta\|\}$. The convexity of $\mathcal{P}\xi$ implies the convexity of the set

$$\mathcal{K}(\xi)=\{\eta\in\mathcal{X}:\|\xi-\eta\|<\|\xi+\eta\|\}$$

since $\mathcal{K}(\xi)=2\mathcal{P}\xi-\xi$.

Let $\eta\in\mathcal{K}(\xi)$ and $\lambda\in(0,1)$. Since $\mathcal{K}(\xi)$ is open there exists $0<\varepsilon<1$ such that the vector

$$\eta(\varepsilon)=\varepsilon\xi+\lambda^{-1}(\lambda\eta-\varepsilon\xi)$$

belongs to $\mathcal{K}(\xi)$. Since $\varepsilon\xi\in\mathcal{K}(\xi)$ the convexity of $\mathcal{K}(\xi)$ implies

$$\lambda\eta=\lambda\eta(\varepsilon)+(1-\lambda)\varepsilon\xi\in\mathcal{K}(\xi).$$

We have proved that the inequality

$$(4)\qquad \|\xi-\eta\|<\|\xi+\eta\|$$

implies

$$(5)\qquad \|\xi-\lambda\eta\|<\|\xi+\lambda\eta\|$$

for $\lambda\in(0,1)$. Interchanging the roles of ξ and η in (4) we get

$$\|\lambda\xi-\eta\|<\|\lambda\xi+\eta\|$$

for $\lambda\in(0,1)$. This is equivalent to the validity of (5) for $\lambda>1$. Hence $\lambda\mathcal{K}(\xi)=\mathcal{K}(\xi)$ for any $\lambda>0$. It follows that the set

$$\mathcal{T}(\xi)=\{\eta\in\mathcal{X}:\|\eta-\xi\|=\|\eta+\xi\|\}$$

is also invariant under multiplication by $\lambda>0$ (indeed, if $\lambda\eta$ does not belong to $\mathcal{T}(\xi)$, then $\lambda\eta\in\mathcal{K}(\xi)\cup\mathcal{K}(-\xi)$, hence $\eta\in\lambda^{-1}\mathcal{K}(\xi)\cup\lambda^{-1}\mathcal{K}(-\xi)=\mathcal{K}(\xi)\cup\mathcal{K}(-\xi)$, and therefore η does not belong to $\mathcal{T}(\xi)$). Hence $\lambda\mathcal{T}(\xi)\subset\mathcal{T}(\xi)$ for any $\lambda>0$. By James' Theorem [2] the last means that $\mathcal{X}$ is a Hilbert space.

COROLLARY 1.4. *If the image of the closed unit ball of $\mathcal{L}(\mathcal{X})$ under the mapping $K \to (\lambda I + K)^{-1}$ is convex for each $\lambda > 0$, then $\mathcal{X}$ is a Hilbert space.*

Proof. It is sufficient to notice that

$$\{(I+K)^{-1} : \|K\| < 1\} = \bigcup_{\varepsilon>0} \{(I+K)^{-1} : \|K\| \le 1-\varepsilon\} = \bigcup_{\lambda>1} \{\lambda(\lambda I + X)^{-1} : \|X\| < 1\}$$

is a convex set by Theorem 1.3.

2. Applications to evolution problems. Let

$$\frac{dx}{dt} = A(t)x \tag{6}$$

be a differential equation in a Hilbert space $\mathcal{H}$ with a scalar product $(\cdot,\cdot)$, and for $t \in \mathbb{R}^+ = [0,\infty)$, let $A(t)$ be selfadjoint operators in $\mathcal{H}$ with a common dense domain $\mathcal{D}$. The Cauchy problem for equation (6) is assumed to be uniformly well-posed. Therefore there exists a bounded linear operator $U(t)$ in $\mathcal{H}$ (called an *evolution operator*) such that for every solution $x(t)$ for (6) with $x(0) = x_0 \in \mathcal{D}$ we have $x(t) = U(t)x_0$. If y_0 does not belong to $\mathcal{D}$ we will call $y(t) = U(t)y_0$ a *generalized solution*.

Let $\mathcal{L}_{2,w}(\mathbb{R}^+, \mathcal{H})$ be the set of functions $x : \mathbb{R}^+ \to \mathcal{H}$ Bochner square integrable with respect to a strictly positive locally integrable weight $w = w(t)$. Let $\mathcal{N}$ denote the set of generalized solutions belonging to $\mathcal{L}_{2,w}(\mathbb{R}^+, \mathcal{H})$. Set $\mathcal{N}_0 = \{h \in \mathcal{H} : h = y(0),\ y \in \mathcal{N}\}$.

Consider the following indefinite metric on $\mathcal{H}$ depending on t:

$$[x,y]_t = (J(t)x, y),$$

where $J(t) = P_1(t) - P_2(t)$, $P_1(t) = \int_{+0}^{+\infty} dE_\lambda(t)$, $P_2(t) = \int_{-\infty}^{0} dE_\lambda(t)$, $E_\lambda(t)$ being the spectral function of $A(t)$.

The following sets (called *bicones*) will be used below:

$$\mathcal{C}_t^- = \{y_0 \in \mathcal{H} : [U(t)y_0, U(t)y_0]_t \le 0\}, \quad t \in \mathbb{R}^+.$$

A bicone $\mathcal{C}_t^-$ is said to be of *rank* $r \le \infty$ if it contains a subspace $\mathcal{L} \subset \mathcal{H}$ with $\dim \mathcal{L} = r$, and does not contain subspaces of greater dimensions (see [6]; note that in [6] the case of $r < \infty$ was studied only).

Suppose that $J(t)$ is strongly differentiable. Consider the derivative of the solution $x(t)$ for (6) along the trajectory:

$$(J(t)x(t), x(t))' = 2\,\mathrm{Re}(J(t)A(t)x(t), x(t)) + (J(t)'x(t), x(t)).$$

We will assume below that $(J(t)x(t), x(t))'$ is qualified positive and that the evolution problem is hyperbolic, that is, the operator $U(t)$ is invertible for all $t \in \mathbb{R}^+$.

THEOREM 2.1. *Suppose the Cauchy problem for the equation (6) is uniformly well-posed and the metric $[\cdot,\cdot]$ satisfies the following conditions:*

(a) *$J(t)$ is strongly differentiable, the limit $\lim_{t\to\infty} \dim P_2(t) = d_-$ exists and*

$$\inf_{\|\xi\|=1} \{\mathrm{Re}[A(t)\xi,\xi]_t + \tfrac{1}{2}(J(t)'\xi,\xi)\} \ge w(t), \quad t \in \mathbb{R}^+; \tag{7}$$

(b) *the evolution operator is invertible for each $t \in \mathbb{R}^+$ and*

$$[U^*(t)z, U^*(t)z]_0 \ge 0 \tag{8}$$

for every $t \in \mathbb{R}^+$ and for each $z \in \mathcal{H}$ such that $[z,z]_t \ge 0$.

Then the generalized solutions $y(t) = U(t)y_0$, $y_0 \in \mathcal{H}$, have the following properties:

1) *$\mathcal{N}_0 \supset \mathcal{C}_\infty^- = \bigcap_{t\in\mathbb{R}^+} \mathcal{C}_t^-$, where $\mathcal{C}_\infty^-$ is a bicone of rank d_-;*

2) *for any $y(t) \in \mathcal{N}$,*

$$\int_t^\infty w(s)\|y(s)\|^2\,ds \le I(y)\exp\Big(-2\int_0^t w(s)\,ds\Big), \tag{9}$$

where $I(y) = \int_0^\infty w(s)\|y(s)\|^2\,ds$;

3) *for any $y_0 \in \mathcal{H} \setminus \mathcal{C}_\infty^-$,*

$$\|y(t)\| \ge [y_0,y_0]_0 \exp\Big(2\int_0^t w(s)\,ds\Big), \quad t \in \mathbb{R}^+. \tag{10}$$

COROLLARY 2.1. *Under the conditions of Theorem 2.1 let*

$$\int_0^\infty w(t)\,dt = \infty. \tag{11}$$

Then all the statements 1)–3) are true, and moreover, $\mathcal{N}_0$ is a closed subspace of $\mathcal{H}$ with $\dim \mathcal{N}_0 = d_-$.

Before we prove Theorem 2.1 we should recall that in the case $\dim P_1(t) < \infty$, $t \in \mathbb{R}^+$, condition (8) is automatically satisfied (see [7]). Then if $J(t) = J = \mathrm{const}$, we have: (7) is equivalent to $\mathrm{Re}[A(t)\xi,\xi]_t \ge w(t)\|\xi\|^2$, $\xi \in \mathcal{D}$.

Proof of Theorem 2.1. With the help of (7) we get, for any $\tau < t (\in \mathbb{R}^+)$ and $y_0 \in \mathcal{D}$,

$$[U(t,\tau)y_0, U(t,\tau)y_0]_t - [y_0,y_0]_\tau \ge 2\int_\tau^t w(s)\|U(s,\tau)y_0\|^2\,ds, \tag{12}$$

where $U(t,\tau)$ is the operator assigning to each $y_0 \in \mathcal{D}$ the value $y(t,s)$ of the solution for equation (6) which satisfies the initial condition $y(\tau,\tau) = y_0$

(so that $U(t)$ is the brief notation for $U(t,0)$). By continuity of $U(t,\tau)$ the inequality (12) holds for any $y_0 \in \mathcal{H}$. Hence we obtain (keeping in mind $\|U(t)y_0\|^2 \geq [U(t)y_0, U(t)y_0]_t$ and setting $y(t) = U(t)y_0$)

$$\|y(t)\|^2 \geq 2 \int_0^t w(s)\|y(s)\|^2\,ds + [y_0, y_0]_0, \tag{13}$$

where $y(t) = U(t)y_0$. Taking $y_0 \in \mathcal{H} \setminus C_0^-$ and arguing as in the Bellman–Gronwall lemma (see [1], Chapt. II) we get (10). Namely, from (13) we have

$$\frac{\|y(t)\|^2 w(t)}{2\int_0^t w(s)\|y(s)\|^2\,ds + [y_0, y_0]_0} \geq w(t).$$

Hence integrating from 0 to t we obtain the required inequality.

From (12) it is easy to see that

$$\mathcal{C}_t^- \subset \mathcal{C}_\tau^- \quad \text{for } t > \tau,\ t, \tau \in \mathbb{R}^+.$$

It follows from the condition (b) that the operator $U^{-1}(t)$ generates a fractional-linear transformation

$$F_{U^{-1}(t)}(K(t)) = (U_{11}^{-1}(t) + U_{12}^{-1}(t)K(t))(U_{12}^{-1}(t) + U_{22}^{-1}K(t))^{-1}$$

of the closed unit operator ball $\mathcal{K}(t) \subset \mathcal{L}(P_2(t)\mathcal{H}, P_1(0)\mathcal{H})$ such that

$$U^{-1}(t) : \mathcal{H}_t \to \mathcal{H}_0,$$
$$\mathcal{H}_0 = P_1(0)\mathcal{H} \oplus P_2(0)\mathcal{H} = \mathcal{H}_1^0 \oplus \mathcal{H}_2^0,$$
$$\mathcal{H}_t = P_1(t)\mathcal{H} \oplus P_2(t)\mathcal{H} = \mathcal{H}_1^t \oplus \mathcal{H}_2^t,$$
$$U_{ij}^{-1}(t) : \mathcal{H}_j^t \to \mathcal{H}_j^0, \quad i, j = 1, 2; \quad K(t) \in \mathcal{K}(t).$$

By Lemma 1.1 all the bicones $\mathcal{C}_t^-$, $t \in \mathbb{R}^+$, are convex and closed. Since they are evidently bounded, they are w.o.t.-compact. Hence using the property of $\dim P_2(t)$ (see the condition (a)) it is easy to check by letting $t \to \infty$ that $\mathcal{C}_\infty^-$ is a bicone of rank d_-.

Now let us prove $\mathcal{C}_\infty^- \subset \mathcal{N}_0$. Take $z \in \mathcal{C}_\infty^-$. From (13) we obtain

$$2 \int_0^\infty \|y(t)\|^2\,ds \leq -[z, z]_0.$$

This means that $y(t) = U(t)z \in \mathcal{N}$. So 1) is proved. Then setting $y_0 = y(\tau)$ in (12) and letting $t \to \infty$ we get (9). So 2) is proved.

Proof of Corollary 2.1. Let (11) hold. From (10)–(11) it follows that $[y_0, y_0] \leq 0$ for all $y_0 \in \mathcal{N}_0$. In view of 1) we hence obtain $\mathcal{N}_0 = \mathcal{C}_\infty^-$ and the statement is proved.

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DEPARTMENT OF APPLIED MATHEMATICS
INTERNATIONAL COLLEGE OF TECHNOLOGY
P.O. BOX 78
KARMIEL 20101, ISRAEL

DEPARTMENT OF MATHEMATICS
VOLOGDA POLYTECHNICAL INSTITUTE
15 LENIN ST.
160008 VOLOGDA, RUSSIA

Received January 19, 1995
Revised version May 16, 1995
(3405)