STUDIA MATHEMATICA 116 (2) (1995)

The basic sequence problem

by

N. J. KALTON (Columbia, Mo.)

Abstract. We construct a quasi-Banach space X which contains no basic sequence.

1. Introduction. It is a classical result in Banach space theory, known to Banach himself [1], that every (infinite-dimensional) Banach space contains a closed linear subspace with a basis, or, in other words, a basic sequence. The corresponding question for quasi-Banach spaces (and more general F-spaces) has, however, remained open. A number of equivalent formulations are known ([11], [14], [16], [17]); the question is also raised in a slightly disguised form in [28], p. 114.

In [11] and [17] it is shown that a quasi-Banach space X contains a basic sequence if and only if there is a strictly weaker Hausdorff vector topology on X. Thus the existence of a space with no basic sequence is equivalent to the existence of a (topologically) *minimal* space (i.e. one on which there is no strictly weaker Hausdorff vector topology). See [3] and [4] for a discussion of minimal spaces. It further follows that X contains a basic sequence if and only if there is some infinite-dimensional closed subspace with separating dual ([11], Theorem 4.4). Several positive results are known. For example, the work of Bastero [2] implies that every subspace of $L_p[0,1]$ $(0<p<1)$ contains a basic sequence, while the author's results in [12] imply that every quotient of $L_p[0,1]$ contains a basic sequence. Bastero's result can be lifted to the wider class of so-called natural spaces and has further been extended by Tam [30] who shows that every complex quasi-Banach space with an equivalent plurisubharmonic norm contains a basic sequence. These results suggest that almost all "reasonable" spaces contain a basic sequence.

In this paper, we will prove

THEOREM 1.1. *There is a quasi-Banach space Y with a one-dimensional subspace L so that*

1991 *Mathematics Subject Classification*: Primary 46A16.

The author was supported in part by NSF-grant DMS-9201357.

(1) *if Y_0 is a closed infinite-dimensional subspace of Y then $L \subset Y_0$, and*

(2) *Y/L is isomorphic to the Banach space ℓ_1.*

In particular, Y contains no basic sequence and is minimal.

It is clear that (1) would make it impossible for Y to contain a basic sequence.

There are other applications of this space. A topological vector space X is said to have the *Hahn–Banach Extension Property* (HBEP) if whenever X_0 is a closed subspace of X and f is a continuous linear functional on X_0 then f can be extended to a continuous linear functional on X. The author showed in [11], answering a question raised by Duren, Romberg and Shields [5] (see also [25], [29]) that for an F-space (complete metric linear space) (HBEP) is equivalent to local convexity. It was very well known that metrizability is necessary in this theorem, but some partial results of Ribe [25] suggested that completeness might not be required. Ribe showed that if X is a metric linear space so that X is isomorphic to $X \oplus X$ then if X has (HBEP) it must be locally convex. More recently, the author [14] extended Ribe's result to show

THEOREM 1.2. *Let X be a decomposable quasi-Banach space (i.e. there is a bounded projection P on X so that neither P nor $I-P$ has finite rank). Suppose X_0 is a dense subspace of X. Then X_0 has* (HBEP) *if and only if X is locally convex.*

A proof of Theorem 1.2 is included in Section 6. The Hahn–Banach extension property for metrizable spaces is also discussed in [10].

However, if Y is the space constructed above, we will show that any algebraic complement Y_0 of L has (HBEP). Thus we have

THEOREM 1.3. *There is a non-locally convex metric linear space Y_0 with the Hahn–Banach Extension Property.*

In 1962, Klee [18] asked whether for every topological vector space (X,τ), the topology τ can be expressed as the supremum of two not necessarily Hausdorff vector topologies τ_1 and τ_2 so that (the Hausdorff quotient of) (X,τ_1) has a separating dual (i.e. is *nearly convex*) and (X,τ_2) has trivial dual. Recently Peck [22] has shown this to be true for certain twisted sums of a Banach space and a one-dimensional space (see also [23]). The space constructed here, Y, turns out to be a counterexample to Klee's problem.

THEOREM 1.4. *There is a quasi-Banach space Y so that the topology on Y is not the supremum of a trivial dual topology and a nearly convex topology.*

The construction of our example depends heavily on the recent remarkable developments in infinite-dimensional Banach spaces due to Gowers, Maurey, Odell and Schlumprecht [7], [8], [9], [20], [21]. It is perhaps a little ironic that the basic sequence question for quasi-Banach spaces turns out to be so closely related to the *unconditional* basic sequence problem for Banach spaces. However, it should be stressed that we use an example of a Banach space with an unconditional basis, very similar to that used by Gowers in [7]; the fundamental estimates we need are in [9].

Let us conclude this introduction by explaining the shortcomings of the example. It is still an open question whether every quasi-Banach space (or F-space) must contain a proper closed infinite-dimensional subspace. A space with no proper closed infinite-dimensional subspace is called *atomic*. The existence of an atomic quasi-Banach space is known to be equivalent to the existence of a *quotient minimal* quasi-Banach space, i.e. a space X so that every quotient is minimal (this concept is due to Drewnowski [3]). See [14] or [16] for a discussion. Our example is quite far from an atomic space, and it is not clear at present whether it can be used towards making such a monster. We remark that Reese [24] has constructed an example of an "almost" atomic F-space, i.e. a space X with a sequence of finite-dimensional subspaces V_n with $\dim V_n > n$ so that if $x_n \in V_n$ is any sequence which is non-zero infinitely often then $[x_n] = X$. It is still unknown whether even this phenomenon can be reproduced in a quasi-Banach space. We suspect, however, that an atomic quasi-Banach space will eventually be found.

We would like to thank several colleagues for helpful comments and remarks during the course of this work, in particular P. Casazza, D. Kutzarova, M. Lammers, M. Mastyło and N. T. Peck. We also want to thank B. Maurey for a substantial simplification of the last part of the argument which we have incorporated into the proof. We also wish to thank the referee for many very helpful suggestions and comments on improving the presentation of the paper.

2. Idea of the construction. In this section, we introduce the basic ideas and notation and prove that the space Y which will be constructed in Sections 3–5 yields solutions to the problems mentioned in the introduction.

We denote by c_{00} the space of all finitely non-zero (real) sequences. If $x \in c_{00}$ we denote its co-ordinates by $\{x(j)\}_{j=1}^{\infty}$. We let $a(x) = \min\{j : x(j) \neq 0\}$ and $b(x) = \max\{j : x(j) \neq 0\}$. If A is a subset of $\mathbb{N}$ then $Ax(j) = x(j)\chi_A(j)$ where χ_A is the characteristic function of A. If E_1, E_2 are subsets of $\mathbb{N}$ we write $E_1 < E_2$ if $\max E_1 < \min E_2$. We shall also write for $x, y \in c_{00}$ that $x < y$ if $b(x) < a(y)$. On the other hand, the natural co-ordinatewise order on c_{00} will be denoted by $x \le y$, i.e. $x \le y$ if and only if $x(j) \le y(j)$ for all $j \in \mathbb{N}$. Let $c_{00}^+ = \{x \in c_{00} : x \ge 0\}$.

For $x, y \in c_{00}$ we will write $\langle x, y\rangle = \sum_{j=1}^\infty x(j)y(j)$. We will also use the same terminology when $x \in c_{00}^+$ and $y = \log v$ for some sequence $v \in c_{00}^+$; in this case it will be understood that the pairing can take the value $-\infty$ and that $0 \log 0 = 0$.

By a *sequence space* X we will mean a subspace X of the space ω of all sequences equipped with a lattice norm $\| \ \|_X$ so that

(1) $c_{00} \subset X$,

(2) if $|x| \le |y| \in X$ then $x \in X$ and $\|x\|_X \le \|y\|_X$, and

(3) if $0 \le x_n \uparrow x$ and $x_n \in X$ with $\sup \|x_n\|_X < \infty$ then $x \in X$ with $\|x\|_X = \sup \|x_n\|_X$ (the Fatou property).

The canonical basis vectors $\{e_n\}_{n=1}^\infty$ then form a 1-unconditional basis for the closure X_0 of c_{00}. For convenience we will write X^* for the Köthe dual of X, which coincides with the Banach space dual of X_0. We will denote the closed unit ball of a Banach space X by B_X. We denote the canonical norm on ℓ_p by $\| \ \|_p$ for the cases $p = 1$ and $p = \infty$.

Consider a map $\Phi : c_{00} \to \mathbb{R}$. For any $u_1, \ldots, u_n$ we define $\Delta_\Phi(u_1, \ldots, u_n) = \sum_{i=1}^n \Phi(u_i) - \Phi(\sum_{i=1}^n u_i)$. Φ is called *quasilinear* if

(4) $\Phi(\alpha u) = \alpha u$ for $\alpha \in \mathbb{R}$, $u \in c_{00}$, and

(5) for a constant $\delta = \delta(\Phi)$ we have $|\Delta(u, v)| \le \delta(\|u\|_1 + \|v\|_1)$ whenever $u, v \in c_{00}$.

Given a quasilinear map Φ we can form the twisted sum $Y = \mathbb{R} \oplus_\Phi \ell_1$, which is defined to be the completion of $\mathbb{R} \oplus c_{00}$ under the quasinorm

$$\|(\alpha, u)\|_\Phi = |\alpha - \Phi(u)| + \|u\|_1.$$

It is readily verified that if L is the span of the vector $e_0 = (1, 0)$ then Y/L is isomorphic to ℓ_1. This construction was first used in [13] and [26] with explicit non-trivial twisted sums of $\mathbb{R}$ and ℓ_1 to deduce that local convexity is not a three-space property; see also [27].

THEOREM 2.1. *Let $\Phi : c_{00} \to \mathbb{R}$ be a quasilinear map and let $Y = \mathbb{R} \oplus_\Phi \ell_1$. Then the following conditions are equivalent*:

(1) *Y contains no basic sequence.*

(2) *If Y_0 is an infinite-dimensional closed subspace of Y then Y_0 contains e_0.*

(3) *The quotient map $\pi : Y \to \ell_1$ is strictly singular.*

(4) *Y is topologically minimal.*

(5) *There is no infinite-dimensional subspace F of c_{00} so that for some constant K we have $|\Phi(u)| \le K\|u\|_1$ for all $u \in F$.*

(6) *If $T : \ell_1 \to Y$ is a bounded operator then T is compact.*

(7) *If $T : Y \to Y$ is a bounded operator then $T = \lambda I + S$ where $\lambda \in \mathbb{R}$ and S is compact.*

Proof. The equivalence of (1) and (4) is well known (see Theorem 4.2 of [11] and Theorem 3.2 of [17], or see [16]). (2) is clearly equivalent to (3) and implies (1). Conversely, if (3) fails then there is an infinite-dimensional closed subspace isomorphic to a subspace of ℓ_1. Thus (1)–(4) are all equivalent.

Next we prove (2) implies (5). Suppose F is an infinite-dimensional subspace of c_{00} so that $|\Phi(u)| \le K\|u\|_1$ for $u \in E$. Let Y_0 be the closure of the subspace of all $(0, x)$ for $x \in E$. Suppose $(0, x_n)$ converges to e_0. Then $|1 - \Phi(x_n)|$ and $\|x_n\|_1$ converge to zero, which is a contradiction.

Next assume (5) and suppose Y contains a basic sequence. By a perturbation argument we can suppose it contains a normalized basic sequence of the form (α_n, u_n) where $u_n \in c_{00}$. By passing to a subsequence we can suppose that $u_1 < u_2 < \ldots$ and that e is not in the closed linear span of (α_n, u_n). It follows that π is an isomorphism on the span of this basic sequence so that for some K we have

$$\Big|\sum_{i=1}^n \alpha_i t_i - \Phi\Big(\sum_{i=1}^n t_i u_i\Big)\Big| \le K\Big\|\sum_{i=1}^n t_i u_i\Big\|_1$$

for all $t_1, \ldots, t_n$. Let F_0 be the subspace of the linear span of the $(u_n)_{n=1}^\infty$ consisting of all $\sum_{i=1}^n t_i u_i$ with $\sum_{i=1}^n \alpha_i t_i = 0$. Then $|\Phi(u)| \le K\|u\|_1$ for $u \in F_0$. Thus (5) implies (1).

(3) implies (6). If $T : \ell_1 \to Y$ is bounded then πT is strictly singular and hence compact. If (x_n) is a sequence in the unit ball of ℓ_1 then by passing to a subsequence we can suppose that $\pi T x_n$ converges. Hence there exist $y_n \in Y$ so that (y_n) converges and $\pi T x_n = \pi y_n$. But then $Tx_n - y_n \in L$ and so has a convergent subsequence.

(6) implies (7). If $T : Y \to Y$ is a bounded operator then since L is the intersection of the kernels of all continuous linear functionals on Y we must have $T(L) \subset L$. Thus $Te = \lambda e$ for some λ. Let $S = T - \lambda I$; then $S = S_0\pi$ where $S_0 : Y/L \to Y$ is compact by (6).

(7) implies (3). If π is not strictly singular, there is a subspace Y_0 of Y of infinite codimension and isomorphic to ℓ_1. Hence there is an isomorphic embedding $V : \ell_1 \to Y$. Then suppose $V\pi = \lambda I + S$ where S is compact. Let $\pi_0 : Y \to Y/Y_0$ be the quotient map. Then $\lambda\pi_0 = -S\pi_0$ is compact. Hence $\lambda = 0$, but this contradicts the fact that V is an isomorphism.

THEOREM 2.2. *If Y satisfies the equivalent conditions of Theorem 2.1 then any algebraic complement of L has the Hahn–Banach Extension Property.*

Proof. Let Z be an algebraic complement of L. The continuous linear functionals on Z separate points, so that any linear functional on a finite-dimensional subspace can be extended continuously to Z. Now let Z_0 be a closed infinite-dimensional subspace of Z and suppose f is a continuous linear functional on Z_0. Let W be the closure of Z_0 in Y and let f denote the extension of f to W. Then W and $f^{-1}(0)$ contain L by (2) and so f factors to a continuous linear functional on $W/L \subset Y/L$, which is a Banach space. Hence by the Hahn–Banach theorem f can be extended continuously to Y and hence also to Z.

THEOREM 2.3. *If Y satisfies the conditions of Theorem 2.1 then the topology τ on Y cannot be the supremum of two vector topologies τ_1, τ_2 so that (Y, τ_1) is nearly convex and (Y, τ_2) has trivial dual.*

Proof. Clearly e_0 must be in the closure of $\{0\}$ for τ_1. Let E be the closure of $\{0\}$ for τ_2. If $e_0 \notin E$ then Theorem 2.1 implies that E is finite-dimensional and that Y^* separates the points of E. Hence $Y = Y_0 \oplus E$ for some closed subspace Y_0 of Y. Now Y_0 contains no basic sequence and so its topology is minimal; however, τ_2 is Hausdorff on Y_0 so that it must agree with the original topology. This implies that $Y_0^* = \{0\}$, but in fact Y_0^* is infinite-dimensional. This contradiction establishes the theorem.

We now review the method of approach to the example. Theorem 2.1 reduces the problem to a type of distortion question expressed by (4). The recent results of the author [15] show that there is a close relationship between quasilinear maps on c_{00} and sequence spaces (see Theorem 6.8 of [15]). We will explain the connection in the next section and show how the recent spaces discovered by Gowers and Maurey ([7] and [9]) enable us to construct a pathological Φ.

3. Indicators of sequence spaces. We now introduce some ideas from [15]. Suppose X is a sequence space. We define the *indicator* Φ_X (called the *entropy map* in [21]) on c_{00} by $\Phi_X(u) = \langle u, \log x\rangle$ where $u = x^*x$ is the (unique) *Lozanovskiĭ factorization* of u, i.e. $x \in B_X^+$ and $x^* \in X^*$ satisfy $\langle x, x^*\rangle = \|x^*\|_{X^*} = \|u\|_1$ and $\operatorname{supp} x, \operatorname{supp} x^* \subset \operatorname{supp} u$. The Lozanovskiĭ factorization originates in [19].

Clearly $\Phi_X(\alpha u) = \alpha\Phi_X(u)$ for $u \in c_{00}$. Furthermore, if $u, v \in c_{00}$ we also have

(1) $$|\Delta(u,v)| \le \frac{4}{e}(\|u\|_1 + \|v\|_1)$$

where $\Delta = \Delta_{\Phi_X}$ (see Lemma 5.6 of [15]). If $u \in c_{00}^+$ then we can characterize the Lozanovskiĭ factorization as the solution of an optimization problem so that

(2) $$\Phi_X(u) = \max_{x \in B_X^+} \langle u, \log x\rangle.$$

This idea originates with Gillespie [6]. Furthermore, for $u_1, \ldots, u_n \in c_{00}^+$ we have the inequalities

(3) $$0 \le \Delta(u_1, \ldots, u_n) \le \sum_{i=1}^n \|u_i\|_1 \log \frac{S}{\|u_i\|_1}$$

where $S = \sum_{i=1}^n \|u_i\|_1$; see [15], Lemma 5.5.

Suppose $f : [1,\infty) \to [1,\infty)$ is any increasing map with $f(1) = 1$ and so that $f(t) \le t$ for all $t \ge 1$. We will say that a sequence space X has a *lower f-estimate on blocks* if, whenever $x_1 < \ldots < x_n \in c_{00}$, then

$$\|x_1 + \ldots + x_n\|_X \ge \frac{1}{f(n)} \sum_{i=1}^n \|x_i\|_X,$$

and an *upper f-estimate on blocks* if, whenever $x_1 < \ldots < x_n \in c_{00}$, then

$$\|x_1 + \ldots + x_n\|_X \le f(n) \max_{1\le i\le n} \|x_i\|_X.$$

LEMMA 3.1. *Suppose X satisfies an upper f-estimate on blocks. Then for $u_1 < \ldots < u_n$ in c_{00}^+ we have*

$$\Delta(u_1, \ldots, u_n) \le \log f(n)(\|u_1\|_1 + \ldots + \|u_n\|_1).$$

Proof. Let $u_i = x_i x_i^*$ be the Lozanovskiĭ factorizations. Then since $f(n)^{-1}(x_1 + \ldots + x_n) \in B_X$ we have by (2),

$$\Phi_X(u_1 + \ldots + u_n) \ge \Big\langle \sum_{i=1}^n u_i, \log\Big(f(n)^{-1}\sum_{i=1}^n x_i\Big)\Big\rangle$$

so that the lemma follows.

The following is a special case of Lemma 5.8 of [15]. Unfortunately, as the referee has pointed out, Lemma 5.8 in [15] is misstated with the inequality reversed, and in the proof the maximum should be replaced by the minimum. This lemma is used in Theorem 5.7 of [15], which is correct although an inequality is again reversed. In view of this we will sketch a simple direct proof.

LEMMA 3.2. *Suppose $s_1, \ldots, s_n, t_1, \ldots, t_n \ge 0$ and let $\sum_{i=1}^n s_i = S$ and $\sum_{i=1}^n t_i = T$. Then*

$$\sum_{i=1}^n \Big(s_i \log \frac{s_i + t_i}{s_i} + t_i \log \frac{s_i + t_i}{t_i}\Big) \le S \log \frac{S+T}{S} + T \log \frac{S+T}{T}.$$

Remark. The summand is zero if either s_i or t_i vanishes.

Proof. We will seek to maximize the function

$$u(s_1, \ldots, s_n, t_1, \ldots, t_n) = \sum_{i=1}^n \Big(s_i \log \frac{s_i + t_i}{s_i} + t_i \log \frac{s_i + t_i}{t_i}\Big)$$

subject to the constraints $\sum_{i=1}^n s_i = S$ and $\sum_{i=1}^n t_i = T$ and $s_i \ge 0,\ t_i \ge 0$ for $1 \le i \le n$. By continuity, there is a point where the maximum is attained. We can suppose $s_i t_i > 0$ for $1 \le i \le m$ and $s_i t_i = 0$ if $m+1 \le i \le n$. By the method of Lagrange multipliers it is easy to show that s_i/t_i is constant for $1 \le i \le m$. But then

$$u(s_1, \ldots, s_n, t_1, \ldots, t_n) = S_0 \log \frac{S_0 + T_0}{S_0} + T_0 \log \frac{S_0 + T_0}{T_0}$$

where $S_0 = \sum_{i=1}^m s_i \le S$ and $T_0 = \sum_{i=1}^m t_i \le T$. This expression is monotone increasing in S_0 and T_0 and so the result follows.

Let $D = B_{\ell_1} \cap c_{00}^+$.

LEMMA 3.3. *Suppose X satisfies an upper f-estimate on blocks and suppose $u \in D$. Let $u = \sum_{i=1}^n u_i$ where $u_1 < \ldots < u_n$. Let A be any subset of $\mathbb{N}$ and let $t = \|Au\|_1$. Then*

$$\begin{aligned}\Delta(u_1, \ldots, u_n) - (1-t)\log f(n) - \varphi(t) &\le \Delta(Au_1, \ldots, Au_n) \\ &\le \Delta(u_1, \ldots, u_n) + \varphi(t),\end{aligned}$$

where $\varphi(t) = t\log(1/t) + (1-t)\log(1/(1-t))$ $(\le \log 2)$.

Proof. Let $\mathbb{N} \setminus A = B$. Then

$$\begin{aligned}&\Delta(Au_1, \ldots, Au_n, Bu_1, \ldots, Bu_n) \\ &\qquad = \Delta(Au_1, \ldots, Au_n) + \Delta(Bu_1, \ldots, Bu_n) + \Delta(Au, Bu).\end{aligned}$$

Similarly

$$\Delta(Au_1, \ldots, Au_n, Bu_1, \ldots, Bu_n) = \Delta(u_1, \ldots, u_n) + \sum_{i=1}^n \Delta(Au_i, Bu_i).$$

Since $\Delta(Bu_1, \ldots, Bu_n), \Delta(Au, Bu) \ge 0$ we deduce

$$\Delta(Au_1, \ldots, Au_n) \le \Delta(u_1, \ldots, u_n) + \sum_{i=1}^n \Delta(Au_i, Bu_i).$$

Now we use (3) and Lemma 3.2. We have

$$\begin{aligned}\sum_{i=1}^n \Delta(Au_i, Bu_i) &\le \sum_{i=1}^n \left(\|Au_i\|_1 \log \frac{\|u_i\|_1}{\|Au_i\|_1} + \|Bu_i\|_1 \log \frac{\|u_i\|_1}{\|Bu_i\|_1} \right) \\ &\le t \log \frac{1}{t} + (1-t) \log \frac{1}{1-t}.\end{aligned}$$

For the former inequality we observe that $\Delta(Bu_1, \ldots, Bu_n) \le \log f(n)\|Bu\|_1$. Hence

$$\Delta(Au_1, \ldots, Au_n) \ge \Delta(u_1, \ldots, u_n) - (1-t)\log f(n) - \Delta(Au, Bu),$$

and the second inequality follows.

LEMMA 3.4. *Suppose $u \in c_{00}^+$ with $\|u\|_1 \le 1$. Suppose $u = xx^*$ where $x \in B_X^+$, $x^* \in B_{X^*}^+$. Then $\Phi_X(u) - \langle u, \log x\rangle \le \|u\|_1 \log(1/(\|u\|_1))$ $(\le 1/e)$.*

Proof. We can suppose that the supports of x, x^* coincide with the support of u. Define Z by $\|z\|_Z = \max(\|z\|_X, \|u\|_1^{-1}\langle |z|, x^*\rangle)$. Then $\|z\|_X \le \|z\|_Z \le \|u\|^{-1}\|z\|_X$ so that $\Phi_X(v) + \|v\|_1 \log \|u\|_1 \le \Phi_Z(v) \le \Phi_X(v)$ for $v \in c_{00}^+$. However, $\|x\|_Z \le 1$ and $\|x^*\|_{Z^*} \le \|u\|_1$ so that $u = xx^*$ is the Lozanovskiĭ factorization for u. Thus $\Phi_Z(u) = \langle u, \log x\rangle$ and the lemma follows.

The next lemma is essentially due to Odell and Schlumprecht [21].

LEMMA 3.5. *Given $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists $\eta > 0$ so that if $u_1 < \ldots < u_n$ are in D, $u = (1/n)(u_1 + \ldots + u_n)$ and $\delta = (1/n)\Delta(u_1, \ldots, u_n) < \eta$ then for the Lozanovskiĭ factorizations $u = xx^*$ and $u_i = x_i x_i^*$ we have $\|Au\|_1 < \varepsilon$ where $A = \{j : y(j) > (1+\varepsilon)x(j)\}$ and $y = x_1 + \ldots + x_n$.*

Proof. By Proposition 2.3 of Odell and Schlumprecht [21], given $\varepsilon > 0$ there exists $\nu > 0$ so that if $v \in D$ and $z \in B_X^+$ are such that $\langle v, \log z\rangle > \Phi_X(v) - \nu$ then if $v = z_0 z_0^*$ is the Lozanovskiĭ factorization then $\|Bv\|_1 < \varepsilon$ where $B = \{j : z_0(j) > (1+\varepsilon)z(j)\}$. Let $\eta = \nu/n$. Then if $\delta < \eta$ we have

$$\sum_{i=1}^n (\Phi_X(u_i) - \langle u_i, \log x\rangle) < \nu$$

and since each term is positive we conclude that $\|A_i u_i\|_1 < \varepsilon$ where $A_i = \{j : x_i(j) > (1+\varepsilon)x(j)\}$. This quickly implies that $\|Au\|_1 < \varepsilon$.

4. The Gowers–Maurey space. At this point we let $f(x) = \log_2(x+1)$ and introduce as in [9] the class $\mathcal{F}$ of functions $g : [1,\infty) \to [1,\infty)$ having the following properties:

(1) $g(1) = 1$ and $g(x) < x$ for $x > 1$.
(2) g is strictly increasing and unbounded.
(3) $\lim_{x\to\infty} x^{-q}g(x) = 0$ for any $q > 0$.
(4) $x/g(x)$ is concave and non-decreasing.
(5) g is submultiplicative, i.e. $g(xy) \le g(x)g(y)$ for $x, y \ge 1$.

Clearly $f \in \mathcal{F}$ and so is $\sqrt{f}$.

Now suppose X is a sequence space. If $n \in \mathbb{N}$ and $\kappa > 1$ we define $\lambda_X(n,\kappa)$ to be the set of $x \in c_{00}^+$ so that $\|x\|_X = 1$ and $x = (1/n)(x_1 + \ldots + x_n)$ where $x_1 < \ldots < x_n$ and $\|x_i\|_X \le \kappa$ for $1 \le i \le n$. (Thus x is an ℓ_{1+}^n average with constant κ, in the sense of [9]: note that we restrict ourselves to non-negative sequences and to spaces X for which the canonical basis is unconditional.)

We then define $\mathrm{RIS}_X(n;\kappa)$ to be the collection of sequences $x_1 < \ldots < x_n$ in c_{00}^+ satisfying $x_i \in \lambda_X(M_i,\kappa)$ where $M_1 \geq 4\kappa\varrho^{-1}2^{36n^2\varrho^{-2}}$ and $M_{k+1} \geq 2^{4b(x_k)^2\varrho^{-2}}$ for $k \geq 1$ where $\varrho = \min(\kappa-1,1)$. We next define $\Lambda_X(n;\kappa)$ to be the collection of $x \in c_{00}^+$ of the form $x = \|x_1 + \ldots + x_n\|_X^{-1}(x_1 + \ldots + x_n)$ where $(x_1,\ldots,x_n) \in \mathrm{RIS}_X(n,\kappa)$. This definition differs slightly but inessentially from that of [9]. In fact, we will only really require the case $\kappa \geq 2$ when $\varrho = 1$; this is in contrast to [9] where values of κ close to one are important.

At the same time if $g \in \mathcal{F}$ we define $\mathcal{H}_X(g;m)$ to be the collection of (m,g)-forms, i.e. $x^* \in \mathcal{H}_X(g;m)$ if and only if $x^* = g(m)^{-1}(x_1^* + \ldots + x_m^*)$ where $x_1^* < \ldots < x_m^*$ are in c_{00}^+ and $\|x_i^*\|_{X^*} \leq 1$ for $1 \leq i \leq m$.

We will require certain lemmas from [9].

Lemma 4.1 (Lemma 4 of [9]). *Suppose $x \in \lambda_X(N,\kappa)$ and $x^* \in \mathcal{H}_X(g;M)$ where $g \in \mathcal{F}$. Then $\langle x,x^*\rangle \leq \kappa(1+2M/N)g(M)^{-1}$.*

Lemma 4.2 (Lemma 5 of [9]). *Suppose X satisfies a lower f-estimate on blocks and $g \in \mathcal{F}$ with $g \geq f^{1/2}$. Suppose $N \in \mathbb{N}$ and $\kappa > 1$. Suppose $M \geq 2^{36N^2\varrho^{-2}}$ and that $x \in \Lambda(N,\kappa)$, $x^* \in \mathcal{H}_X(g,M)$. Then $\langle x,x^*\rangle \leq (\kappa+\varrho)f(N)/N \leq (\kappa+1)f(N)/N$.*

Remark. For our statement of Lemma 4.2, observe that since X has a lower f-estimate, for any $\{x_i\}_{i=1}^N \in \mathrm{RIS}_X(N,\kappa)$ we have $\|\sum_{i=1}^N x_i\|_X \geq Nf(N)^{-1}$.

Our next lemma is a slight modification of Lemma 7 of [9].

Lemma 4.3. *Suppose X satisfies a lower f-estimate on blocks and $g \in \mathcal{F}$ with $g \geq f^{1/2}$. Suppose $\kappa \geq 2$ and $(x_1,\ldots,x_N) \in \mathrm{RIS}_X(N,\kappa)$. Let $x = \sum_{i=1}^N x_i$ and suppose that for every interval E with $\|Ex\|_X \geq 1$ we have*

$$(*)\qquad \|Ex\|_X \leq \sup\{\langle Ex,x^*\rangle : x^* \in \mathcal{H}_X(g;M),\ M \geq 2\}.$$

Then $\|x\|_X \leq (\kappa+1)N/g(N)$.

Proof. We introduce the length of an interval E as in [9]. Let $x_i \in \lambda_X(n_i,\kappa)$ for $1 \leq i \leq N$. Suppose x_i is written as $(1/n_i)\sum_{j=1}^{n_i} x_{ij}$ where $x_{i1} < \ldots < x_{in_i}$ and $\|x_{ij}\|_X \leq \kappa n_i^{-1}$. If E is any interval which intersects the support of $\sum_{i=1}^N x_i$ we let $k \leq l$ be the least and greatest indices i such that $Ex_i \neq 0$. Then we let p be the least index such that $Ex_{kp} \neq 0$ and q the greatest index such that $Ex_{lq} \neq 0$. Define $\ell(E) = l-k+qn_l^{-1} - pn_k^{-1}$. If E does not meet the support of $\sum_{i=1}^N x_i$ then $\ell(E) = 0$.

Now our hypotheses differ from Lemma 7 of [9] in that we assume $(*)$ whenever $\|Ex\|_X \geq 1$, while [9] assumes $(*)$ whenever $\ell(E) \geq 1$; we, however, assume $\kappa \geq 2$. Our hypotheses imply that $(*)$ holds if $\ell(E) \geq 2$ since then there exists at least one x_i with support contained entirely in E. As in [9] let $G(t) = t/g(t)$ for $t \geq 1$ and $G(t) = t$ for $t \leq 1$. Then if $\kappa n_1^{-1} \leq \ell(E) \leq 1$ we have $\|Ex\|_X \leq (\kappa+1)G(\ell(E))$ as in [9]. We claim the same inequality if $1 \leq \ell(E) \leq 2$; in fact, in this situation we can see that E intersects the supports of at most three x_i and so $\|Ex\|_X \leq 3 \leq (\kappa+1)G(\ell(E))$. The proof can now be completed by applying Lemma 7 of [9].

We will now define a Gowers–Maurey space Z, very similar to the construction in [9]; in fact, essentially the same space is considered by Gowers in [7] as a counterexample to the hyperplane problem, and also as a space in which all operators are strictly singular perturbations of a diagonal map. We will suppose that $\mathcal{P} = \{p_k\}_{k=1}^\infty$ is an increasing sequence of natural numbers satisfying $f(p_1) > 256$, $\log\log\log p_k > 4p_{k-1}^2$, $p_k > k^6 2^{100k^2}$, for all k. We shall also require that $f(p_{2k})p_{2k}^{-1} \leq k^{-3}/2$, which doubtless follows from our other hypotheses. For convenience we suppose each p_k is a square. We partition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ where $\mathcal{P}_1 = \{p_{2k-1}\}_{k=1}^\infty$ and $\mathcal{P}_2 = \{p_{2k}\}_{k=1}^\infty$.

Let $\mathbb{Q}_+$ denote the countable collection of $u \in c_{00}^+$ which have only rational coefficients and let σ be an injection from the collection of all finite subsets of $\mathbb{Q}_+$, $\{z_1,\ldots,z_s\}$ where $z_1 < \ldots < z_s$, to $\mathcal{P}_2$ which satisfy the condition $\sigma(z_1,\ldots,z_s) \geq 2^{10b(z_s)^2}$.

We then define Z implicitly by the formula

$$\|x\|_Z = \max(\|x\|_\infty, \|x\|_\alpha, \|x\|_\beta)$$

where

$$\|x\|_\alpha = \sup\{\langle |x|,x^*\rangle : x^* \in \mathcal{H}_Z(f;M),\ M \geq 2\},$$

$$\|x\|_\beta = \sup\Big\{f(k)^{-1/2}\sum_{i=1}^k \langle |x|,x_i^*\rangle\Big\},$$

with the supremum being over all $k \in \mathcal{P}_1$ and *special sequences* $(x_1^*,\ldots,x_k^*)$, i.e. such that $x_1^* < \ldots < x_k^*$, with $x_1^* \in \mathbb{Q}_+ \cap \mathcal{H}_Z(f;p_{2k})$ and then for $j \geq 1$, $x_{j+1}^* \in \mathbb{Q}_+ \cap \mathcal{H}_Z(f;\sigma(x_1^*,\ldots,x_j^*))$.

This implicit definition can be justified by an inductive construction as in [9]. Precisely we set $\|x\|_{Z_0} = \|x\|_\infty$ for $x \in c_{00}$ and then define for $N \geq 1$,

$$\|x\|_{Z_N} = \max(\|x\|_{Z_{N-1}}, \|x\|_{\alpha_{N-1}}, \|x\|_{\beta_{N-1}})$$

where

$$\|x\|_{\alpha_N} = \sup\{\langle |x|,x^*\rangle : x^* \in \mathcal{H}_{Z_N}(f;M),\ M \geq 2\},$$

$$\|x\|_{\beta_N} = \sup\Big\{f(k)^{-1/2}\sum_{i=1}^k \langle |x|,x_i^*\rangle\Big\}$$

with the supremum being over all $k \in \mathcal{P}_1$ and $(x_1^*,\ldots,x_k^*)$, i.e. such that $x_1^* < \ldots < x_k^*$, with $x_1^* \in \mathbb{Q}_+ \cap \mathcal{H}_{Z_N}(f;p_{2k})$ and then for $j \geq 1$, $x_{j+1}^* \in$

$\mathbb{Q}_+ \cap \mathcal{H}_{Z_N}(f;\sigma(x_1^*,\dots,x_j^*))$. It is then easily verified that $\|\ \|_{Z_N}$ is an increasing sequence of norms, bounded above by the ℓ_1-norm, and that the sets $H_{Z_N}(f;M)$ also increase in N. We set $\|x\|_Z = \lim_{N\to\infty}\|x\|_{Z_N}$.

We emphasize that this space is an unconditional version of the counterexample constructed in [9], but shares some of the same features. We will need versions for Z of certain lemmas proved in [9] for the Gowers–Maurey space. Fortunately the same basic techniques go through more or less unchanged.

Let us note first that Z satisfies a lower f-estimate. This follows immediately from the definition of $\|x\|_\alpha$. We also note that, by induction, it follows that $\|e_n\|_Z = 1$ for all n.

LEMMA 4.4. *Suppose* $(x_j)_{j=1}^n \in \mathrm{RIS}_Z(n;\kappa)$ *where* $\kappa \ge 1$. *Then* $\|\sum_{j=1}^n x_j\|_\infty < 1$.

Proof. We have $x_j \in \lambda_Z(M_j,\kappa)$ where $M_j \ge 4\kappa$ by the definition of $\mathrm{RIS}_Z(n;\kappa)$. Hence $\|x_j\|_\infty \le M_j^{-1}\kappa < 1$ and the lemma follows.

It now follows as in Lemma 10 of [9]:

LEMMA 4.5. *Suppose* $\kappa \ge 2$. *Suppose* $N \in \mathcal{P}_2$ *and* $\log N \le n \le \exp N$. *Then if* $\{x_1,\dots,x_n\} \in \mathrm{RIS}_Z(n,\kappa)$ *we have* $\|\sum_{i=1}^n x_i\|_Z \le (\kappa+1)nf(n)^{-1}$.

Proof. The key point, proved in [9], Lemma 9, is that there exists $g \in \mathcal{F}$ with $f^{1/2} \le g \le f$ such that $g(x) = f(x)$ for $\log N \le x \le \exp N$ and $g(k) = f^{1/2}(k)$ when $k \in \mathcal{P}_1$. Thus if $x \in c_{00}$ and $\|x\|_Z > \|x\|_\infty$ then

$$\|x\|_Z = \sup\{\langle Ex, x^*\rangle :\ x^* \in \mathcal{H}_Z(g;M),\ M \ge 2\}.$$

Now, by the preceding lemma if $x = \sum_{j=1}^n x_j$ and E is any interval then $\|Ex\|_\infty < 1$. We can therefore apply Lemma 4.3 to obtain the result.

The next lemma is simply a cruder form of Lemma 11 from [9].

LEMMA 4.6. *Suppose* $\kappa \ge 2$ *and* $N \in \mathcal{P}_2$. *If* $x \in \Lambda_Z(N,\kappa)$ *then* $x \in \lambda(\sqrt{N}, 2(\kappa+1))$.

Proof. Suppose $\{x_i\}_{i=1}^N \in \mathrm{RIS}_Z(N,\kappa)$ and that $x = \|\sum_{i=1}^N x_i\|_Z^{-1} \times \sum_{i=1}^N x_i$. We break $[1,N]$ into $\sqrt{N}$ intervals E_j each of length $\sqrt{N}$, which is an integer by hypothesis. Note that $\{x_i\}_{i\in E_j} \in \mathrm{RIS}_Z(\sqrt{N},\kappa)$. If $y_j = \sum_{i\in E_j} x_i$ then, by Lemma 4.5, $\|y_j\|_Z \le (\kappa+1)\sqrt{N}$. Also $\|\sum_{j=1}^N x_j\|_Z \ge N/f(N)$, by the lower f-estimate on X. Now $x = (1/\sqrt{N})\sum_{j=1}^{\sqrt{N}} z_j$ where $z_j = (\|\sum_{i=1}^N x_i\|_Z)^{-1}\sqrt{N} y_j$. But $\|z_j\|_Z \le (\kappa+1)(Nf(N))/(Nf(\sqrt{N})) \le 2(\kappa+1)$.

Our next result is a modification of Lemma 12 of [9]. In fact, this lemma appears to be incorrectly stated in [9] and so some modification is necessary. In the proof of the lemma in [9] it is claimed without justification that $\{x_1,\dots,x_k\}$ is a RIS of length k and constant $1+\varepsilon$. For the applications some modification similar to that given below seems adequate, however.

LEMMA 4.7. *Let* $\kappa \ge 2$. *Suppose* $k \in \mathcal{P}_1$ *with* $f(k) > 100\kappa^2$. *Suppose* $E_1,\dots,E_k$ *are intervals with* $E_1 < \dots < E_k$. *Let* $\{x_1^*,\dots,x_k^*\}$ *be a special sequence with* $\operatorname{supp} x_j^* \subset E_j$. *Let* $M_1 = p_{2k}$ *and* $M_{j+1} = \sigma(x_1^*,\dots,x_j^*)$ *for* $1 \le j \le k-1$. *Let* A *be any subset of* $\{1,2,\dots,k\}$ *and suppose for each* $j \in A$ *we have* $x_j \in c_{00}^+$ *with* $\operatorname{supp} x_j \subset E_j$ *so that* x_j, x_j^* *are disjoint and* $x_j \in \Lambda(M_j,\kappa)$. *Then*

$$\Big\|\sum_{i\in A} x_i\Big\|_Z \le 16\kappa k f(k)^{-1}.$$

Proof. We have $x_j \in \lambda_Z(\sqrt{M_j}, 4\kappa)$, by Lemma 4.6. Note that $\sqrt{M_1} = \sqrt{p_{2k}} \ge 4\kappa 2^{36k^2}$. We also have $\sqrt{M_{j+1}} > 2^{4b(x_j^*)^2}$.

Now assume A contains no two consecutive integers. Then if $j \in A$ we have $\sqrt{M_j} \ge 2^{4b(x_{j-2})^2}$ for $j \ge 2$ and so $\{x_j\}_{j\in A} \in \mathrm{RIS}_Z(|A|, 4\kappa)$. As in [9] we use Lemma 4.3.

Note first that there exists $h \in \mathcal{F}$ with $\sqrt{f} \le h \le f$, so that $h(n) = \sqrt{f(n)}$ if $n \in \mathcal{P}_1 \setminus \{k\}$ while $h(n) = f(n)$ if $n \in \mathcal{P}_2 \cup \{k\}$. This fact follows from Lemma 9 of [9].

Let $x = \sum_{i\in A} x_i$ and suppose that for some interval E we have $\|Ex\|_Z \ge 1$, and

$$\|Ex\|_Z > \sup\{\langle Ex, x^*\rangle : x^* \in \mathcal{H}_Z(h,m),\ m \ge 2\}.$$

Since $h \le f$ this implies that $\|Ex\|_Z > \|Ex\|_\alpha$. On the other hand, since $\{x_j\}_{j\in A} \in \mathrm{RIS}_Z(|A|, 4\kappa)$ we can apply Lemma 4.4 to deduce that $\|Ex\|_Z > \|Ex\|_\infty$. The conclusion is that $\|Ex\|_Z = \|Ex\|_\beta$. Thus there is a special sequence $\{z_1^*,\dots,z_l^*\}$, with $l \in \mathcal{P}_1$, so that

$$\|Ex\|_Z = f(l)^{-1/2}\Big\langle Ex, \sum_{i=1}^l z_i^*\Big\rangle.$$

However, $f(l)^{1/2} = h(l)$ unless $l = k$. We conclude $l = k$ and

$$1 \le \|Ex\|_Z \le f(k)^{-1/2}\sum_{i\in A}\sum_{j=1}^k \langle x_i, z_j^*\rangle.$$

Let t be the greatest integer so that $z_t^* = x_t^*$ (with $t = 0$ if no such integer exists). If $i < t$ it is clear that $\langle x_i, z_j^*\rangle = 0$ for all j. Similarly if $j \le t$ it is also clear that $\langle x_i, z_j^*\rangle = 0$ for all i. If $i = t$, then $\langle x_i, z_j^*\rangle = 0$ unless $j = t+1$ when of course $\langle x_t, z_{t+1}^*\rangle \le 1$. If $t+1 \le i \in A$ and $t+1 \le j \le k$ then, unless $t+1 = i = j$, we have $x_i \in \Lambda_Z(M_i,\kappa)$ and $z_j^* \in \mathcal{H}_Z(g;M_j')$ where $M_i, M_j' \in \mathcal{P}_2$ are not equal. It follows from the separation conditions on $\mathcal{P}_2$ that we can apply either Lemma 4.1 or Lemma 4.2; if $M_j' < M_i$, then by Lemma 4.1,

$$\langle x_i, z_j^* \rangle \le 24\kappa f(M_j')^{-1} \le 24\kappa f(p_{2k})^{-1},$$

or if $M_j' > M_i$, then $M_j' \ge 2^{36M_i^2}$ and by Lemma 4.2,

$$\langle x_i, z_j^* \rangle \le 2\kappa f(M_i)/M_i \le 2\kappa f(p_{2k})p_{2k}^{-1}.$$

In either case we have $\langle x_i, z_j^* \rangle \le \kappa k^{-2}$. If $i = j = t+1$ then $\langle x_i, z_j^* \rangle \le 1$.

Hence

$$\Big\langle \sum_{i\in A} x_i, \sum_{j=1}^{k} z_j^* \Big\rangle \le 2 + \kappa \le 3\kappa.$$

This implies that

$$\|Ex\|_Z \le 3\kappa f(k)^{-1/2} < 3/10$$

contrary to assumption. The conclusion from Lemma 4.3 is then that

$$\|x\|_Z \le 8\kappa |A| h(|A|)^{-1} \le 8\kappa k f(k)^{-1}.$$

The general result follows by splitting A into two subsets obeying the condition that no two consecutive integers are contained in either.

5. The main result. We now let $X = Z^*$ and consider the indicator Φ_X. We will need the elementary fact, which follows from duality, that X satisfies an upper f-estimate, i.e. if $x_1 < \ldots < x_n \in c_{00}$ then $\|x_1 + \ldots + x_n\|_X \le f(n) \max_{1\le i\le n} \|x_i\|_X$. It also follows from the definition of Z that if $x_1, \ldots, x_n$ is a special sequence (with $n \in \mathcal{P}_1$) then $\|x_1 + \ldots + x_n\|_X \le f(n)^{1/2}$.

Our main result, which combined with the results of Section 2 establishes Theorems 1.1, 1.3 and 1.4, is the following:

THEOREM 5.1. *For every infinite-dimensional subspace G of c_{00} we have* $\sup\{|\Phi_X(u)| : \|u\|_1 = 1,\ u \in G\} = \infty$.

Remark. The following proof has been substantially simplified according to a suggestion of B. Maurey.

Proof of Theorem 5.1. We will start from the assumption that there is a subspace G of infinite dimension so that $|\Phi_X(u)| \le K\|u\|_1$ for $u \in G$. We may suppose that if $u \in G$ then $\langle u, \chi \rangle = 0$ where χ is the constantly one sequence. Then by induction we can pick $\xi_1 < \xi_2 < \xi_3 < \ldots$ in G with $\|\xi_j\|_1 = 2$. We split ξ_i into positive and negative parts: $\xi_i = \xi_i' - \xi_i''$, where ξ_i', ξ_i'' are disjoint and non-negative. Then $\xi_i', \xi_i'' \in D$. We let R be the union of the supports of the ξ_i' and S be the union of the supports of the ξ_i''. Let W be the linear span of $\{|\xi_i|\}_{i=1}^{\infty}$.

Notice first that X satisfies an upper f-estimate on blocks where $f(x) = \log_2(x+1)$. If $\gamma > 0$ and $n \in \mathbb{N}$ we define $\Gamma(n,\gamma)$ to be the set of $w \in D$ such that there exist $w_1 < \ldots < w_n \in D$ with $w = (1/n)(w_1 + \ldots + w_n)$ and $(1/n)\Delta(w_1, \ldots, w_n) < \gamma$.

LEMMA 5.2. *Given any $m, n \in \mathbb{N}$ and $\delta > 0$ there exists $w \in W \cap \Gamma(n,\delta)$ with $m < a(w)$.*

Proof. For $n \in \mathbb{N}$ let c_n be the infimum of all constants γ so that if $m \in \mathbb{N}$ there exists $w \in W \cap \Gamma(n,\gamma)$ with $m < a(w)$. It is easy to see that $c_{np} \ge c_n + c_p$ for any n, p and that from Lemma 3.1, $c_n \le \log f(n)$. Hence $pc_n \le c_{n^p} \le \log f(n^p)$ and so letting $p \to \infty$ we obtain $c_n = 0$ for all n and the lemma follows.

We now turn to estimates on the Lozanovskiĭ factorization of $w \in \Gamma(n,\delta)$.

LEMMA 5.3. *For fixed n and $0 < \varepsilon < 1/2$ there exists $\eta > 0$ so that if $w \in \Gamma(n,\eta)$ and $w = xx^*$ is the Lozanovskiĭ factorization of w, then there exists $A \subset [a(w), b(w)]$ with $\|Aw\|_1 > 1 - \varepsilon$ and such that $Ax^*/\|Ax^*\|_Z \in \lambda_Z(n,2)$.*

Proof. If $w \in \Gamma(n,\delta)$ then $w = (1/n)\sum_{i=1}^{n} w_i$ where $w_1 < \ldots < w_n \in D$ are such that $(1/n)\Delta(w_1, \ldots, w_n) \le \delta$. Let $w_i = x_i x_i^*$ be the Lozanovskiĭ factorizations of each. Let $y = x_1 + \ldots + x_n$. If $c > 1$ let $A = \{j : y(j) \le cx(j),\ x(j) > 0\}$. Then $Ax^* \le cn^{-1}A(x_1^* + \ldots + x_n^*)$ and hence if $A_i = A \cap [a(w_i), b(w_i)]$ then $\|A_i x^*\|_Z \le c/n$. Now $\|Ax^*\|_Z \ge \|Aw\|_1$ and so $\|Ax^*\|_Z^{-1} Ax^* \in \lambda_Z(n,c')$ where $c' \le c\|Aw\|_1^{-1}$. Now, according to Lemma 3.5, if $\delta > 0$ is sufficiently small we can choose c close enough to 1 so that the conclusions follow.

Using the preceding lemma we describe a construction. Suppose $N \in \mathcal{P}_2$ and $\varepsilon > 0$. Then given any $m \in \mathbb{N}$ and any $M_1 > 2^{36N^2+4}$ we can construct two sequences $\{w_j\}_{j=1}^N$ and $\{\zeta_j\}_{j=1}^N$ and a sequence of integers $(M_j)_{j=1}^N$ so that

(1) $m < a(w_1)$,

(2) $w_1 < \zeta_1 < w_2 < \zeta_2 < \ldots < w_N < \zeta_N$,

(3) $w_j \in \Gamma(M_j, \eta_j) \cap W$ where $0 < \eta_j < \varepsilon$ is sufficiently small so that there exists $A_j \subset [a(w_j), b(w_j)]$ with $\|A_j w_j\|_1 > 1 - \varepsilon$ and $z_j = \|A_j x_j^*\|_Z^{-1} A_j x_j^* \in \lambda_Z(M_j, 2)$ where $w_j = x_j x_j^*$ is the Lozanovskiĭ factorization of w_j,

(4) $\zeta_j \in \lambda_Z(M_j, 2)$,

(5) $M_{j+1} > 2^{4b(\zeta_j)^2}$.

We will call the resulting sequence $\{w_j\}_{j=1}^N$ an *(N,ε)-sequence*, and $w = (1/N)(w_1 + \ldots + w_N)$ the associated *(N,ε)-average*. The sequence $\{\zeta_j\}_{j=1}^N$ is called the *ballast sequence*; it is present simply for technical reasons to provide ballast in the argument. Let H be the union of the supports of the ballast sequence.

LEMMA 5.4. *Suppose* $\{w_1,\dots,w_N\}$ *is an* (N,ε)*-sequence as above with associated* (N,ε)*-average* w *and ballast* $\{\zeta_j\}_{j=1}^N$. *Then there is a subset* A *of* $[a(w),b(w)]$ *and* $x\in\mathcal{H}_Z(f;N)\cap\mathbb{Q}_+$ *with* $\operatorname{supp}x\subset\operatorname{supp}w$ *so that*

(6) $\|Aw\|_1>1-\varepsilon$,

(7) *if* $B\subset A$ *there exists* $z\in\Lambda_Z(N,4)$ *supported in* $B\cup H$ *so that* $Bw\le 10xz$,

(8) *if* $B\subset A$ *then* $\langle Bw,\log x\rangle>\Phi_X(Bw)-4$.

Proof. Notice that $y=(1/f(N))(x_1+\dots+x_N)\in\mathcal{H}_Z(f;N)$ and $\|y\|_X\le 1$, since X has an upper f-estimate. Choose x with rational coefficients so that $y/2\le x\le y$. Let $A=A_1\cup\dots\cup A_N$ so that (6) immediately holds.

We recall that $z_j\in\lambda_Z(M_j,2)$ (condition (3)) for $1\le j\le N$. It follows easily that if B is a subset of A then we can find $0\le\alpha_j\le 1$ so that $\|Bx_j^*+\alpha_j\zeta_j\|_Z=1$ and then $Bx_j^*+\alpha_j\zeta_j\in\lambda_Z(M_j,4)$. The sequence $\{Bx_j^*+\alpha_j\zeta_j\}_{j=1}^N$ thus belongs to $\mathrm{RIS}_Z(N,4)$ (since $M_1>2^{36N^2+4}$) and so $\|\sum_{j=1}^N(Bx_j^*+\alpha_j\zeta_j)\|_Z\le 5N/f(N)$, from Lemma 4.5.

Let z be the normalized vector $\beta(\sum_{j=1}^N Bx_j^*+\alpha_j\zeta_j)$ where, by the above, $\beta\ge f(N)/(5N)$. Then $z\in\Lambda_Z(N,4)$ and $xz\ge yz/2\ge Bw/10$. This proves (7).

For (8) we notice that Lemma 3.4 now implies that $\Phi_X(Bw)-\langle Bw,\log x\rangle\le 10/e<4$.

Let us suppose that $n\in\mathcal{P}_1$ is fixed and large, say $f(n)>\exp(8K+4000)$, and let $\varepsilon=(\log f(n))^{-1}$. Let $M_1=p_{2n}$; we can construct an (M_1,ε)-sequence $\{w_{1j}\}_{j=1}^{M_1}$ with (M_1,ε)-average $w_1=M_1^{-1}\sum_{j=1}^{M_1}w_{1j}$ and ballast $\{\zeta_{1j}\}_{j=1}^{M_1}$. Let $x_1\in\mathbb{Q}_+\cap\mathcal{H}_Z(f;M_1)$ and $A_1\subset[a(w_1),b(w_1)]$ be such that the conclusions of Lemma 5.4 hold.

Next let $M_2=\sigma(Rx_1)$ and construct an (M_2,ε)-sequence $\{w_{2j}\}_{j=1}^{M_2}$ with associated (M_2,ε)-average w_2 and ballast $\{\zeta_{2j}\}_{j=1}^{M_2}$ so that $\zeta_{1M_1}<u_2$. Repeating this construction for n steps we obtain sequences $(w_{ij})_{j=1}^{M_i}$, $(\zeta_{ij})_{j=1}^{M_i}$ for $i=1,\dots,n$, $(w_i)_{i=1}^n$, $(M_i)_{i=1}^n$, $(A_i)_{i=1}^n$ and $(x_i)_{i=1}^n$ so that

(9) $(w_{ij})_{j=1}^{M_i}$ is an (M_i,ε)-sequence with associated (M_i,ε)-average w_i and ballast $\{\zeta_{ij}\}_{j=1}^{M_i}$ for $1\le i\le n$,

(10) $w_1<\zeta_{1M_1}<w_2<\zeta_{2M_2}<\dots<w_n<\zeta_{nM_n}$,

(11) $A_i\subset[a(w_i),b(w_i)]$ for $1\le i\le n$ and $\|A_iw_i\|_1>1-\varepsilon_i$,

(12) $\operatorname{supp}x_i\subset\operatorname{supp}w_i$, $x_i\in\mathcal{H}_Z(f;M_i)\cap\mathbb{Q}_+$, and so $\|x_i\|_X\le 1$,

(13) $\langle Bw_i,\log x_i\rangle>\Phi_X(Bw_i)-4$ whenever $B\subset A_i$,

(14) for any $B\subset A_i$ there exists $z\in\Lambda_Z(M_i,4)$ with $Bw_i\le 10x_iz$,

(15) $M_{i+1}=\sigma(Rx_1,\dots,Rx_i)$ for $1\le i\le n-1$.

We also have

(16) $(Rx_1,\dots,Rx_n)$ is a special sequence of length n in $X=Z^*$.

Let H_i be the union of the supports of the ballast at the ith step. Let $A=\bigcup_{i=1}^n A_i$ and then set $P=A\cap R$ and $Q=A\cap S$. We also define $u_i=2Rw_i$, $v_i=2Sw_i$ (so that $u_i,v_i\in D$) and then set $u=(1/n)(u_1+\dots+u_n)$, $v=(1/n)(v_1+\dots+v_n)$ and $w=(1/n)(w_1+\dots+w_n)$.

If we set $x=(f(n))^{-1/2}\sum_{i=1}^n Rx_i$ then $\|x\|_X\le 1$, since by (16), $\{Rx_1,\dots,Rx_n\}$ is a special sequence. Hence, using (13) above,

$$\Phi_X(Pw)\ge\langle Pw,\log x\rangle\ge\frac{1}{n}\sum_{i=1}^n\Phi_X(Pw_i)-\frac{1}{2}\log f(n)\|Pw\|_1-4.$$

Thus $(1/n)\Delta(Pu_1,\dots,Pu_n)\le(1/2)\log f(n)+8$. Now $(1/n)\sum_{i=1}^n\|Pu_i\|_1>1-2\varepsilon$ so that by Lemma 3.3, and the choice of ε,

(17) $$\frac{1}{n}\Delta(u_1,\dots,u_n)\le\frac{1}{2}\log f(n)+11.$$

On the other hand, by Lemma 5.4 we can find $z_i\in\Lambda_Z(M_i,4)$ supported on $((\operatorname{supp}w_i)\cap Q)\cup H_i$ so that $Qw_i\le 10x_iz_i$. At this point we can invoke Lemma 4.7. Let $E_i=[a(w_i),b(\zeta_{i,M_i})]$ and notice that Rx_i,z_i are both supported in E_i, but are disjoint. Since $f(n)\ge 1600$, $Rx_1,\dots,Rx_n$ is a special sequence and $z_i\in\Lambda_Z(M_j,4)$ where $M_1=p_{2n}$ and $M_{j+1}=\sigma(x_1,\dots,x_j)$ for $1\le j\le n-1$ we can conclude that $\|\sum_{j=1}^N z_j\|_Z\le 64nf(n)^{-1}$. At the same time, by the upper f-estimate on X, $\|\sum_{i=1}^n x_i\|_X\le f(n)$. Now we have

$$\frac{1}{640}Qw\le\Big(\frac{1}{f(n)}\sum_{i=1}^n x_i\Big)\Big(\frac{f(n)}{64n}\sum_{i=1}^n z_i\Big)$$

and we can apply Lemma 3.4 again to deduce that

$$\Phi_X(Qw)\le\sum_{i=1}^n\langle Qw,\log x_i\rangle-\log f(n)\|Qw\|_1+640,$$

and so, since $\|Qw\|_1>1/2-\varepsilon$,

$$\frac{1}{n}\Delta(Qw_1,\dots,Qw_n)\ge\log f(n)\|Qw\|_1-640\ge\frac{1}{2}\log f(n)-641.$$

Now recall that $Qv_i=2Qw_i$. Hence $(1/n)\Delta(Qv_1,\dots,Qv_n)\ge\log f(n)-1282$ and we can apply Lemma 3.3 to deduce that

(18) $$\frac{1}{n}\Delta(v_1,\dots,v_n)\ge\log f(n)-1283.$$

Notice that $u_i-v_i\in G$ for $1\le i\le n$. Now we have $|\Phi_X(u_i-v_i)|\le 2K$, for $1\le i\le n$, and $|\Phi_X(u-v)|\le 2K$. Hence $|\Phi_X(u_i)-\Phi_X(v_i)|\le 2K+8e^{-1}\le$

$2K+3$ for $1 \le i \le n$ and similarly $|\Phi_X(u) - \Phi_X(v)| \le 2K+3$. This implies that
$$\frac{1}{n}\Delta(v_1,\ldots,v_n) - \frac{1}{n}\Delta(u_1,\ldots,u_n) \le 4K+6.$$
Combining with (17) and (18) gives that $\log f(n) \le 8K + 2600$, which contradicts our initial choice of n and completes the proof of Theorem 5.1.

It is perhaps worth noting at this point that it is very simple to modify our example so that Theorem 1.1 holds with L of any specified dimension.

THEOREM 5.5. *For any $n \in \mathbb{N}$ there is a quasi-Banach space $Y^{(n)}$ with a subspace L of dimension n so that Y/L is isomorphic to ℓ_1 and if Y_0 is a closed infinite-dimensional subspace of $Y^{(n)}$ then $L \subset Y$.*

Proof. Let $A_k = \{nj+k\}_{j=0}^{\infty} \subset \mathbb{N}$, for $k = 1,\ldots,n$. Define $S_k : c_{00} \to c_{00}$ by $S_k u = \sum_{j=0}^{\infty} u(j) e_{nj+k}$. Define $\Phi : c_{00} \to \ell_\infty^n$ by $\Phi(u) = \{\Phi_X(S_k u)\}_{k=1}^n$. Then let $Y^{(n)}$ be the completion of $\ell_\infty^n \oplus c_{00}$ under the quasinorm
$$\|(\xi,u)\|_\Phi = \|\xi - \Phi(u)\|_\infty + \|u\|_1.$$
Let L be the space of all $(\xi,0)$ for $\xi \in \ell_\infty^n$. Clearly $Y^{(n)}/L$ is isomorphic to ℓ_1. Now suppose Y_0 is an infinite-dimensional subspace so that $Y_0 \cap L$ is a proper subspace of L. Then there is a non-trivial linear functional f on ℓ_∞^n so that $Y_0 \cap L \subset Z = f^{-1}(0)$. Suppose $f(\xi) = \sum_{k=1}^n \beta_k \xi_k$. It is easy to verify that Y/Z is isomorphic to the completion of $\mathbb{R} \oplus c_{00}$ under the quasinorm $\|(\alpha,u)\|_\Psi = |\alpha - \Psi(u)| + \|u\|_1$ where $\Psi(u) = \sum_{k=1}^n \beta_k \Phi_X(S_k u)$. However, there is a constant K depending only on $\beta_1,\ldots,\beta_n$ so that $|\Psi(u) - \Phi_X(\sum_{k=1}^n \beta_k S_k u)| \le K\|u\|_1$. It follows easily that Ψ is unbounded on every infinite-dimensional subspace of c_{00} and hence that $(Y_0+Z)/Z$ must contain L/Z, which is a contradiction to the fact that $Y_0 \cap L$ is contained in Z.

6. Some final remarks.

In this short final section we will present a proof of Theorem 1.2, which first appeared in [14], a reference which may not be readily available. Our proof here is slightly shorter. We begin with a lemma:

LEMMA 6.1. *Suppose X is a quasi-Banach space with a dense subspace V with* (HBEP). *Suppose $L = \{x \in X : x^*(x) = 0\ \forall x^* \in X^*\}$. Then:*

(1) *If $L = \{0\}$, so that X has a separating dual, then X is locally convex.*
(2) *If X contains a basic sequence then X is locally convex.*
(3) *If M is a closed subspace of L then X/M has a dense subspace with* (HBEP).

Proof. (1) (cf. [11]) Let $\| \ \|_c$ be the Banach envelope norm on X, i.e. $\|x\|_c = \sup\{|x^*(x)| : \|x^*\| \le 1\}$. If X is not locally convex we may choose $v_n \in V$ with $\|v_n\|_c \le 4^{-n}$ and $\|v_n\| = 1$. Pick any $x \in V$ and consider the sequence $w_n = v_n + 2^{-n}x$. Then (see Theorem 4.7 of [16]) there is a subsequence (w_{p_n}) which is a Markushevich basis for its closed linear span in X. Pick n_0 large enough so that $x \notin [w_{p_k} : k \ge n_0]$. Then by (HBEP) for V there is a linear functional $x^* \in X^*$ with $x^*(w_{p_k}) = 0$ for $k \ge n_0$ but $x^*(x) = 1$. However, $\lim_{n\to\infty} \|x - 2^n w_n\|_c = 0$ so that $x^*(x) = 0$, contrary to hypothesis.

(2) Pick any $u \in L$; we will show $u = 0$. Assume then that $u \ne 0$. Suppose $w \in V$ is non-zero, and u, w are linearly independent. Since X contains a basic sequence and V is dense in X we can apply standard perturbation arguments to suppose that we have a bounded basic sequence (x_n) with $x_n \in n(u+w) + V$, say $x_n = n(u+w) + v_n$ where $v_n \in V$. Then there exists n_0 so that $[u,w] \cap [x_n]_{n \ge n_0} = \{0\}$. Thus there is a bounded linear functional f on the span Y of u, w and $[x_n]_{n\ge n_0}$ with $f(u) = 1$, $f(w) = 0$ and $f(x_n) = 0$ for $n \ge n_0$. Since V has (HBEP) there is a bounded linear functional x^* on X with $x^*(v) = f(v)$ for $v \in V \cap Y$. Thus $x^*(w) = 0$ and $x^*(v_n) = -n$; also $x^*(u) = 0$ since $u \in L$. Hence $x^*(x_n) = -n$, contradicting the boundedness of x^*. Now since $L = \{0\}$ we can apply (1) to deduce that X is locally convex.

(3) Let $\pi : X \to X/M$ be the quotient map; we show $\pi(V)$ has (HBEP). Indeed, if $E \subset \pi(V)$ is a subspace and f is a continuous linear functional on E then we can find $x^* \in X^*$ so that $x^*(v) = f(\pi v)$ for $v \in \pi^{-1}E \cap V$. But then $x^*(x) = 0$ if $x \in M \subset L$ so that x^* factors to a linear functional on X/M.

THEOREM 6.2. *Suppose X is a decomposable quasi-Banach space. If X has a dense subspace V with* (HBEP) *then X is locally convex.*

Proof. Let P be a bounded projection on X so that both P and $Q = I - P$ have infinite rank. If L is defined as in the previous lemma then L is clearly invariant for P. From the hypotheses, X^* has infinite dimension and hence so has X/L. Therefore either $P(X)/P(L)$ or $Q(X)/Q(L)$ has infinite dimension. Suppose the former; then consider $X/P(L)$, which has a dense subspace with (HBEP) by Lemma 6.1(3). Then $P(X)/P(L)$ is isomorphic to a subspace of X/L which has separating dual; since it has infinite dimension, it contains a basic sequence. By Lemma 6.1(2) this implies that $X/P(L)$ is locally convex and hence that $Q(X)$ is locally convex. But now X itself must contain a basic sequence and Lemma 6.1(2) shows that X is locally convex.

Let us conclude by mentioning that in [14] we raised the question of whether every quasi-Banach space X with separating dual has a weakly closed subspace W and a bounded linear functional f on W which cannot be extended to X. We proved that this is equivalent to the following:

PROBLEM. *Suppose X is a quasi-Banach space with separating dual and suppose that every quotient X/E by an infinite-dimensional subspace E is locally convex. Is X locally convex?*

Of course our main example Y has every quotient Y/E by an infinite-dimensional subspace locally convex, but fails to have a separating dual.

References

[1] S. Banach, *Théorie des opérations linéaires*, reprint of the original 1932 edition, Chelsea, New York, 1978.

[2] J. Bastero, *ℓ_q-subspaces of stable p-Banach spaces*, Arch. Math. (Basel) 40 (1983), 538–544.

[3] L. Drewnowski, *On minimally subspace-comparable F-spaces*, J. Funct. Anal. 26 (1977), 315–332.

[4] —, *Quasi-complements in F-spaces*, Studia Math. 77 (1984), 373–391.

[5] P. L. Duren, B. W. Romberg and A. L. Shields, *Linear functionals on H^p-spaces when $0 < p < 1$*, J. Reine Angew. Math. 238 (1969), 32–60.

[6] T. A. Gillespie, *Factorisation in Banach function spaces*, Indag. Math. 43 (1981), 287–300.

[7] W. T. Gowers, *A solution to Banach's hyperplane problem*, Bull. London Math. Soc. 26 (1994), 532–540.

[8] —, *A new dichotomy for Banach spaces*, preprint.

[9] W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. 6 (1993), 851–874.

[10] J. Kąkol and P. Sorjonen, *Basic sequences and the Hahn–Banach extension property*, Acta Sci. Math. (Szeged) 59 (1994), 161–171.

[11] N. J. Kalton, *Basic sequences in F-spaces and their applications*, Proc. Edinburgh Math. Soc. 19 (1974), 151–167.

[12] —, *Compact and strictly singular operators on Orlicz spaces*, Israel J. Math. 26 (1977), 126–136.

[13] —, *The three space problem for locally bounded F-spaces*, Compositio Math. 37 (1978), 243–276.

[14] —, *The atomic space problem and related questions for F-spaces*, in: Proc. Orlicz Memorial Conf., Univ. of Mississippi, Oxford, Mississippi, 1991.

[15] —, *Differentials of complex interpolation processes for Köthe function spaces*, Trans. Amer. Math. Soc. 333 (1992), 479–529 .

[16] N. J. Kalton, N. T. Peck and J. W. Roberts, *An F-Space Sampler*, London Math. Soc. Lecture Note Ser. 89, Cambridge Univ. Press, Cambridge, 1984.

[17] N. J. Kalton and J. H. Shapiro, *Bases and basic sequences in F-spaces*, Studia Math. 56 (1976), 47–61.

[18] V. L. Klee, *Exotic topologies for linear spaces*, in: Proc. Sympos. on General Topology and its Relations to Modern Analysis and Algebra, Academic Press, 1962, 238–249.

[19] G. Ya. Lozanovskiĭ, *On some Banach lattices*, Siberian Math. J. 10 (1969), 419–430.

[20] E. Odell and T. Schlumprecht, *The distortion of Hilbert space*, Geom. Funct. Anal. 3 (1993), 201–217.

[21] E. Odell and T. Schlumprecht, *The distortion problem*, Acta Math. 173 (1994), 259–283.

[22] N. T. Peck, *Twisted sums and a problem of Klee*, Israel J. Math. 81 (1993), 357–368.

[23] N. T. Peck and H. Porta, *Linear topologies which are suprema of exotic topologies*, Studia Math. 47 (1973), 63–73.

[24] M. L. Reese, *Almost-atomic spaces*, Illinois J. Math. 36 (1992), 316–324.

[25] M. Ribe, *Necessary convexity conditions for the Hahn–Banach theorem in metrizable spaces*, Pacific J. Math. 44 (1973), 715–732.

[26] —, *Examples for the nonlocally convex three space problem*, Proc. Amer. Math. Soc. 73 (1979), 351–355.

[27] J. W. Roberts, *A nonlocally convex F-space with the Hahn–Banach approximation property*, in: Lecture Notes in Math. 604, Springer, 1977, 76–81.

[28] S. Rolewicz, *Metric Linear Spaces*, PWN, Warszawa, 1972.

[29] J. H. Shapiro, *Extension of linear functionals on F-spaces with bases*, Duke Math. J. 37 (1970), 639–645.

[30] S. C. Tam, *The basic sequence problem for quasi-normed spaces*, Arch. Math. (Basel) 62 (1994), 69–72.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MISSOURI
COLUMBIA, MISSOURI 65211
U.S.A.
E-mail: MATHNJK@MIZZOU1.MISSOURI.EDU

Received October 17, 1994
Revised version April 26, 1995 (3357)