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Weighted inequalities for monotone and concave functions

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Abstract. Characterizations of weight functions are given for which integral inequalities of monotone and concave functions are satisfied. The constants in these inequalities are sharp and in the case of concave functions, constitute weighted forms of Favard–Berwald inequalities on finite and infinite intervals. Related inequalities, some of Hardy type, are also given.

1. Introduction. In 1939 L. Berwald [8] proved, via a generalization of a mean value inequality of J. Favard [18], that if f is a non-negative concave continuous function on $[0,1]$ (1) and $0 < p \le q < \infty$, then

$$\|f\|_q \le (p+1)^{1/p}(q+1)^{-1/q}\|f\|_p, \tag{1.1}$$

where the constant $(p+1)^{1/p}(q+1)^{-1/q}$ is sharp. If $p = 1$ this is called *Favard's inequality*. This inequality may be interpreted as the converse of Hölder's inequality and in the limiting case with $q = 1$, $p \to 0$, it is a reverse Jensen inequality

$$\int_0^1 f(x)\,dx \le \frac{e}{2}\exp\Big(\int_0^1 \ln f(x)\,dx\Big),$$

where the constant $e/2$ is sharp.

Closely related to the Favard–Berwald inequality is an inequality of Grüss [21]. It asserts that the L^1-norms of certain functions f, g are dominated by the L^1-norm of the product fg. More generally, one obtains (cf. Barnes [4] with $r = 1$) the inequality

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(1) The interval of integration in Berwald's result is $[a,b]$ instead of $[0,1]$. This generalization may be achieved in this paper with minor modifications.

$$\|f\|_p\|g\|_q \le C\|fg\|_r$$

for certain indices p, q and r.

The object of this paper is to introduce weight functions u, v, w in these inequalities and provide conditions on the weights which are equivalent to corresponding weighted forms of the Favard–Berwald inequality, Grüss' inequality and the reverse Hölder inequality. These and corresponding variants are given both on finite and infinite intervals.

To prove these results, a number of auxilliary inequalities are required. One, due to Calderón–Scott [14], with special cases of earlier origin, is proved here with a sharp constant. These inequalities are of the form

$$\int_a^b f(x)\,dg(x) \le \Big(\int_a^b f(x)^\gamma\,d[g(x)^\gamma]\Big)^{1/\gamma}, \tag{1.2}$$

for $0<\gamma\le 1$ and f decreasing, g increasing as well as some other variants (cf. Th. 2.1). One of the consequences of this inequality is that it permits an easy proof of weight characterizations of Hardy type in the case $1\le p\le q$ from the special case of $p=q$. Also using inequality (1.2), we can give another proof of the imbeddings $\|f\|_{q,u}\le C\|f\|_{p,v}$ for $0\le f\downarrow$ and $\|f\|_{q,u}\le C\|f\|_{p,v}$ for $0\le f\uparrow$, which were proved earlier by Sawyer [39], Stepanov [44] and Heinig–Stepanov [24]. All these results are given in Section 2.

In Section 3, we study inequalities of the form

$$\|T_1 f\|_{q,u} \le C\|T_2 f\|_{p,v}, \tag{1.3}$$

where T_1, T_2 are the positive integral operators $T_i f(x) = \int_0^\infty k_i(x,t)f(t)\,dt$, $k_i(x,t)\ge 0$, $i=1,2$, $f\ge 0$ monotone and C a sharp constant. Some of these results are essentially known (cf. [2], [16], [24], [27], [33], [39], [42]–[44]) and in the main we prove them with sharp constants. The results are then applied in Sections 4 and 5 to obtain sharp weighted inequalities for concave functions, weighted Favard–Berwald inequalities as well as reverse Hölder inequalities. As special cases we obtain results of Barnard and Wells [3], whose work motivated this study.

All functions considered are assumed measurable and non-negative. We shall write $f\uparrow$, respectively $f\downarrow$ to mean that the function f is increasing $\equiv$ non-decreasing, respectively decreasing $\equiv$ non-increasing. For $f\downarrow$ we define $f^{-1}(t) = \inf\{s : f(s)\le t\}$, $\inf\emptyset = \infty$, and similarly for $f\uparrow$, $f^{-1}(t) = \inf\{s : f(s) > t\}$, $\inf\emptyset=\infty$.

Locally integrable non-negative *weight functions* on $(0,\infty)$ are denoted by u, v, w, and the conjugate index of $p\in(0,\infty)$ is denoted by $p' = p/(p-1)$, even if $0<p\le 1$, and similarly for q. We shall say $f\in L^p_w$, $p>0$, if $\|f\|_{p,w} = (\int_0^\infty |f(x)|^p w(x)\,dx)^{1/p} < \infty$. Finally, χ_E denotes the characteristic function of the set E. Inequalities (such as in Theorem 2.1) are interpreted to mean that if the right side is finite, so is the left, and the inequality holds.

2. Integral inequalities. In this section we prove inequalities frequently used in the sequel.

THEOREM 2.1. *Let $-\infty\le a<b\le\infty$ and $f\ge 0$ on (a,b) and g continuous on (a,b).*

(a) *Suppose $f\downarrow$ on (a,b) and $g\uparrow$ on (a,b) with $\lim_{x\to a^+} g(x) = 0$. Then for any $\gamma\in(0,1]$,*

$$\int_a^b f(x)\,dg(x) \le \Big(\int_a^b f(x)^\gamma\,d[g(x)^\gamma]\Big)^{1/\gamma}. \tag{2.1}$$

If $1\le\gamma<\infty$, the inequality in (2.1) is reversed.

(b) *Suppose $f\uparrow$ on (a,b) and $g\downarrow$ on (a,b) with $\lim_{x\to b^-} g(x) = 0$. Then for any $\gamma\in(0,1]$,*

$$\int_a^b f(x)\,d[-g(x)] \le \Big(\int_a^b f(x)^\gamma\,d[-g(x)^\gamma]\Big)^{1/\gamma}. \tag{2.2}$$

If $1\le\gamma<\infty$, the inequality in (2.2) is reversed.

We shall give two proofs of the theorem.

Proof 1. It suffices to prove the theorem when the integrals on the right of (2.1) and (2.2) are finite. Suppose $0<\gamma\le 1$.

(a) Since

$$\begin{aligned} f(x)\,dg(x) &= \frac{1}{\gamma}[f(x)^\gamma g(x)^\gamma]^{1/\gamma-1} f(x)^\gamma\,d(g(x)^\gamma) \\ &\le \frac{1}{\gamma}\Big[\int_a^x f(t)^\gamma\,d(g(t)^\gamma)\Big]^{1/\gamma-1} f(x)^\gamma\,d(g(x)^\gamma) \\ &= \frac{d}{dx}\Big[\int_a^x f(t)^\gamma\,d(g(t)^\gamma)\Big]^{1/\gamma}, \end{aligned}$$

integrating from a to b yields

$$\begin{aligned} \int_a^b f(x)\,dg(x) &\le \Big(\int_a^x f(t)^\gamma\,d(g(t)^\gamma)\Big)^{1/\gamma}\Big|_a^b \\ &= \Big(\int_a^b f(x)^\gamma\,d[g(x)^\gamma]\Big)^{1/\gamma}. \end{aligned}$$

(b) Also,

$$f(x)\,d[-g(x)] = \frac{1}{\gamma}[f(x)^\gamma g(x)^\gamma]^{1/\gamma-1} f(x)^\gamma\, d[-g(x)^\gamma]$$
$$\le \frac{1}{\gamma}\Big\{\int_x^b f(t)^\gamma d[-g(x)^\gamma]\Big\}^{1/\gamma-1} f(x)^\gamma d[-g(x)^\gamma]$$
$$= -\frac{d}{dx}\Big\{\int_x^b f(t)^\gamma d[-g(x)^\gamma]\Big\}^{1/\gamma},$$

and again integrating from a to b one obtains (2.2).

It is easily seen that inequalities in (2.1) and (2.2) are reversed if $1 \le \gamma < \infty$.

Proof 2. For $t \in (a,b)$, let

$$h(t) = \Big(\int_a^t f(x)\,dg(x)\Big)^\gamma - \int_a^t f(x)^\gamma\, d[g(x)^\gamma]$$

and

$$k(t) = \Big(\int_t^b f(x)\,d[-g(x)]\Big)^\gamma - \int_t^b f(x)^\gamma d[-g(x)^\gamma].$$

Then h and k are continuous and for $0 < \gamma \le 1$,

$$h'(t) = \gamma\Big(\int_a^t f(x)\,dg(x)\Big)^{\gamma-1} f(t)g'(t) - \gamma f(t)^\gamma g(t)^{\gamma-1} g'(t)$$
$$\le \gamma[f(t)g(t)]^{\gamma-1} f(t)g'(t) - \gamma f(t)^\gamma g(t)^{\gamma-1} g'(t) = 0 \quad \text{a.e.}$$

Therefore $h\downarrow$ on (a,b) and so $h(b) \le h(a) = 0$, which proves (2.1). Similarly with $k(t)$.

Remark 2.2. (i) If f and g are as in Theorem 2.1, then

$$F(\gamma) = \Big(\int_a^b f(x)^\gamma\, d[g(x)^\gamma]\Big)^{1/\gamma}, \quad \text{resp.} \quad G(\gamma) = \Big(\int_a^b f(x)^\gamma\, d[-g(x)^\gamma]\Big)^{1/\gamma}$$

are decreasing on $(0,\infty)$.

(ii) Inequality (2.2) with $a = 0$, $b = \infty$ and constant $1/\gamma$ on the right side was proved by Calderón–Scott [14, Lemma 6.1].

(iii) If $g(x) = x^{1/p}$, $0 < p \le 1$, $a = 0$ and $b = \infty$, then (2.1) takes the form

$$\int_0^\infty f(x)x^{1/p-1}\,dx \le p\Big(\int_0^\infty f(x)^p\,dx\Big)^{1/p}$$

for all $0 \le f\downarrow$, while $g(x) = x - a$ yields

$$\Big(\int_a^b f(x)\,dx\Big)^p \le p\int_a^b (x-a)^{p-1} f(x)^p\,dx$$

for all $0 \le f\downarrow$. The last inequality has been obtained by various authors (cf. Hardy–Littlewood–Pólya [22, p. 100], Lorentz [28, p. 39], Stein–Weiss [41, Th. 3.11i], Maz'ya [31, Lemma 2.2], Bergh–Burenkov–Persson [7, Lemma 2.1].

(iv) The special case of (2.1) when $g(x) = x^p$, $p \ge 1$, and f is positive increasing on $(0,b)$, $b < \infty$, has the form

$$\int_0^b f(b-x)\,dx^p \le \Big(\int_0^b f(b-x)^{1/p}dx\Big)^p = \Big(\int_0^b f(x)^{1/p}\,dx\Big)^p.$$

This inequality with some additional constant $C_p \in (1,2)$ was proved by García del Amo [20, Th. 5]. We can also write the above inequality in the form

$$p\int_0^b f(x)^p(b-x)^{p-1}\,dx \le \Big(\int_0^b f(x)\,dx\Big)^p,$$

where $p \ge 1$ and f is a positive increasing function. This inequality was proved by Bushell–Okrasiński [13] for natural p, and for any $p \ge 1$ quite recently by Walter–Weckesser [45].

(v) If $\gamma = \infty$, the reversed inequalities (2.1) and (2.2) are meaningless; however, one does have the following: If f and g are as in Theorem 2.1(a), then

$$\sup_{x\in(a,b)} f(x)g(x) \le \int_a^b f(x)\,dg(x).$$

In fact,

$$\sup_{x\in(a,b)} f(x)g(x) = \sup_{x\in(a,b)} f(x)\int_a^x dg(t)$$
$$\le \sup_{x\in(a,b)} \int_a^x f(t)\,dg(t) = \int_a^b f(x)\,dg(x).$$

Similarly, if f and g are as in Theorem 2.1(b) then

$$\sup_{x\in(a,b)} f(x)g(x) \le \int_a^b f(x)\,d[-g(x)].$$

The inequalities given in Theorem 2.1 may be applied in connection with Hardy's inequality. It is well known (cf. [36]) that the weighted Hardy inequality

$$(2.3)\qquad \Big(\int_0^\infty u(x)\Big(\int_0^x f(t)\,dt\Big)^q dx\Big)^{1/q} \le C\Big(\int_0^\infty v(x)f(x)^p\,dx\Big)^{1/p}$$

holds for all $f \ge 0$ if and only if for $1 \le p \le q < \infty$,

$$A_{p,q}(u,v) = \sup_{t>0}\Big(\int_t^\infty u(x)\,dx\Big)^{1/q}\Big(\int_0^t v(x)^{1-p'}\,dx\Big)^{1/p'} < \infty,$$

and for $0 < q < p$, $p > 1$,

$$B_{p,q}(u,v)$$
$$= \Big\{\int_0^\infty\Big[\Big(\int_t^\infty u(x)\,dx\Big)^{1/q}\Big(\int_0^t v(x)^{1-p'}\,dx\Big)^{1/q'}\Big]^r v(t)^{1-p'}\,dt\Big\}^{1/r} < \infty,$$

where $1/r = 1/q - 1/p$. Moreover, if $C = C_{p,q}(u,v)$ is the best constant for which (2.3) holds, then $A_{p,q}(u,v) \approx C_{p,q}(u,v)$, respectively, $B_{p,q}(u,v) \approx C_{p,q}(u,v)$.

Writing $A_{p,p} = A_p$ and $u_s(x) = u(x)(\int_x^\infty u(t)\,dt)^{s-1}$, $s > 0$, we obtain from Theorem 2.1 the following:

PROPOSITION 2.3. (a) *If* $1 \le p \le q < \infty$, *then* $A_{p,q}(u,v) = (p/q)^{1/p}A_p(u_{p/q},v)$ *and* $C_{p,q}(u,v) \le (p/q)^{1/p}C_p(u_{p/q},v)$.

(b) *If* $0 < q < p < \infty$ *and* $p > 1$, *then*

$$A_p(u,v) \le (q/p)^{1/q}(r/p')^{1/r}B_{p,q}(u_{q/p},v)$$

and $C_p(u,v) \le (p/q)^{1/q}C_{p,q}(u_{q/p},v)$.

Proof. (a) First, for all $t > 0$, we have

$$\Big(\int_t^\infty u_{p/q}(x)\,dx\Big)^{1/p}\Big(\int_0^t v(x)^{1-p'}\,dx\Big)^{1/p'}$$
$$= (q/p)^{1/p}\Big(\int_t^\infty d\Big[-\Big(\int_x^\infty u(s)\,ds\Big)^{p/q}\Big]\Big)^{1/p}\Big(\int_0^t v(x)^{1-p'}\,dx\Big)^{1/p'}$$
$$= (q/p)^{1/p}\Big(\int_t^\infty u(x)\,dx\Big)^{1/q}\Big(\int_0^t v(x)^{1-p'}\,dx\Big)^{1/p'},$$

which gives the equality.

Secondly, by Theorem 2.1(b) with $\gamma = p/q \le 1$ and f replaced by $(\int_0^x f(t)\,dt)^q$ and g replaced by $\int_x^\infty u(t)\,dt$, we have

$$\int_0^\infty u(x)\Big(\int_0^x f(t)\,dt\Big)^q dx = \int_0^\infty\Big(\int_0^x f(t)\,dt\Big)^q d\Big[-\int_x^\infty u(s)\,ds\Big]$$
$$\le \Big\{\int_0^\infty\Big(\int_0^x f(t)\,dt\Big)^p d\Big[-\Big(\int_x^\infty u(s)\,ds\Big)^{p/q}\Big]\Big\}^{q/p}$$
$$\le (p/q)^{q/p}\Big\{\int_0^\infty\Big(\int_0^x f(t)\,dt\Big)^p u_{p/q}(x)\,dx\Big\}^{q/p}$$
$$\le C\Big(\int_0^\infty v(x)f(x)^p\,dx\Big)^{q/p},$$

which means that $C_{p,q}(u,v) \le (p/q)^{1/p}C_p(u_{p/q},v)$.

(b) For any $\alpha > 0$,

$$B_{p,q}(u_{q/p},v)$$
$$= \Big\{\int_0^\infty\Big[\Big(\int_x^\infty u_{q/p}(t)\,dt\Big)^{1/q}\Big(\int_0^x v(t)^{1-p'}\,dt\Big)^{1/q'}\Big]^r v(x)^{1-p'}\,dx\Big\}^{1/r}$$
$$= \Big\{\int_0^\infty\Big[\Big(\frac{p}{q}\int_x^\infty \frac{d}{dt}\Big(-\int_t^\infty u(s)\,ds\Big)^{q/p}dt\Big)^{1/q}$$
$$\times\Big(\int_0^x v(t)^{1-p'}\,dt\Big)^{1/q'}\Big]^r v(x)^{1-p'}\,dx\Big\}^{1/r}$$
$$= \Big(\frac{p}{q}\Big)^{1/q}\Big\{\int_0^\infty\Big[\Big(\int_x^\infty u(s)\,ds\Big)^{1/p}\Big(\int_0^x v(t)^{1-p'}\,dt\Big)^{1/q'}\Big]^r v(x)^{1-p'}\,dx\Big\}^{1/r}$$
$$\ge \Big(\frac{p}{q}\Big)^{1/q}\Big\{\int_0^\alpha\Big[\Big(\int_x^\infty u(s)\,ds\Big)^{1/p}\Big(\int_0^x v(t)^{1-p'}\,dt\Big)^{1/q'}\Big]^r v(x)^{1-p'}\,dx\Big\}^{1/r}$$
$$\ge \Big(\frac{p}{q}\Big)^{1/q}\Big(\int_\alpha^\infty u(s)\,ds\Big)^{1/p}\Big\{\int_0^\alpha\Big(\int_0^x v(t)^{1-p'}\,dt\Big)^{r/q'} d\Big(\int_0^x v(t)^{1-p'}\,dt\Big)\Big\}^{1/r}$$
$$= \Big(\frac{p}{q}\Big)^{1/q}\Big(\int_\alpha^\infty u(s)\,ds\Big)^{1/p}\Big\{\int_0^\alpha d\Big[\Big(\int_0^x v(t)^{1-p'}\,dt\Big)^{1+r/q'}\Big]\frac{p'}{r}\Big\}^{1/r}$$
$$= \Big(\frac{p}{q}\Big)^{1/q}\Big(\frac{p'}{r}\Big)^{1/r}\Big(\int_\alpha^\infty u(s)\,ds\Big)^{1/p}\Big(\int_0^\alpha v(t)^{1-p'}\,dt\Big)^{(1+r/q')(1/r)}$$
$$= \Big(\frac{p}{q}\Big)^{1/q}\Big(\frac{p'}{r}\Big)^{1/r}\Big(\int_\alpha^\infty u(s)\,ds\Big)^{1/p}\Big(\int_0^\alpha v(t)^{1-p'}\,dt\Big)^{1/p'},$$

which gives

$$B_{p,q}(u_{q/p},v) \ge \left(\frac{p}{q}\right)^{1/q}\left(\frac{p'}{r}\right)^{1/r} A_p(u,v).$$

Similarly, using Theorem 2.1(b) with $\gamma = q/p \le 1$ and f replaced by $(\int_0^x f(t)\,dt)^p$, and g replaced by $\int_x^\infty u(t)\,dt$, we have

$$\int_0^\infty u(x)\Big(\int_0^x f(t)\,dt\Big)^p dx = \int_0^\infty \Big(\int_0^x f(t)\,dt\Big)^p d\Big[-\int_x^\infty u(s)\,ds\Big]$$

$$\le \Big\{\int_0^\infty \Big(\int_0^x f(t)\,dt\Big)^q d\Big[-\Big(\int_x^\infty u(s)\,ds\Big)^{q/p}\Big]\Big\}^{p/q}$$

$$\le (p/q)^{p/q}\Big\{\int_0^\infty \Big(\int_0^x f(t)\,dt\Big)^q u_{q/p}(x)\,dx\Big\}^{p/q} \le C\Big(\int_0^\infty v(x)f(x)^p\,dx\Big),$$

which means that $C_p(u,v) \le (p/q)^{1/q} C_{p,q}(u_{q/p},v)$.

Remark 2.4. (i) Artola, Talenti, Tomaselli and Muckenhoupt (cf. [32]) proved that Hardy's inequality (2.3) with $p = q \ge 1$ holds if and only if $A_p(u,v) < \infty$. Then Bradley [10], Kokilashvili and Maz'ya [31] extended this result to the case $1 \le p \le q$. The case $1 \le q < p$ was proved by Maz'ya [31] and Sawyer [38], and the case $0 < q < 1 < p$ by Sinnamon [40] (see also [38]). For more information we refer to [36]. Our Proposition 2.3(a) shows that if the Hardy inequality (2.3) is proved in the index range $p = q \ge 1$, then it holds for $1 \le p < q < \infty$. In fact, if $A_{p,q}(u,v) < \infty$, then by Proposition 2.3(a), $C_{p,q}(u,v) \le (p/q)^{1/p} C_p(u_{p/q},v) \approx A_p(u_{p/q},v) < \infty$. The best relationship between the constants $C_{p,q}(u,v)$ and $A_{p,q}(u,v)$ when $1 < p < q < \infty$ was found by Manakov [30].

(ii) Using Theorem 2.1(a) one obtains the same implications for the dual Hardy operator $\int_x^\infty f(t)\,dt$.

If, for $0 < p < \infty$ and for a weight function v, we define the Lorentz space $\Lambda_p(v)$ as the space generated by the quasi-norm

$$\|f\|_{\Lambda_p(v)} = \Big(\int_0^\infty f^*(x)^p v(x)\,dx\Big)^{1/p},$$

where f^* denotes the decreasing rearrangement of f, then the imbedding $\Lambda_p(v) \subset \Lambda_q(u)$ is equivalent to a corresponding weighted integral inequality for decreasing functions. Such results are due to Sawyer [39] and Stepanov [44]. Heinig and Stepanov [24, Th. 2.1(i)] proved a similar result for increasing functions. Theorem 2.1 may also be applied in a natural way in the proof of these results.

PROPOSITION 2.5. *Let* $0 < p \le q < \infty$.

(a) *The inequality*

$$\Big(\int_0^\infty u(x)f(x)^q\,dx\Big)^{1/q} \le C\Big(\int_0^\infty v(x)f(x)^p\,dx\Big)^{1/p} \tag{2.4}$$

holds for all $0 \le f\downarrow$ *if and only if*

$$\Big(\int_0^t u(x)\,dx\Big)^{1/q} \le C\Big(\int_0^t v(x)\,dx\Big)^{1/p} \quad \forall t > 0. \tag{2.5}$$

(b) *The inequality*

$$\Big(\int_0^\infty u(x)f(x)^q\,dx\Big)^{1/q} \le D\Big(\int_0^\infty v(x)f(x)^p\,dx\Big)^{1/p} \tag{2.6}$$

holds for all $0 \le f\uparrow$ *if and only if*

$$\Big(\int_t^\infty u(x)\,dx\Big)^{1/q} \le D\Big(\int_t^\infty v(x)\,dx\Big)^{1/p} \quad \forall t > 0. \tag{2.7}$$

Proof. (a) That (2.4) implies (2.5) follows on taking $f(x) = \chi_{[0,t]}(x)$, $t > 0$, in (2.4).

(2.5)⇒(2.4). It suffices to prove this implication for those functions f for which $\operatorname{supp} f = (0,N] \subset (0,\infty)$ and $\int_0^\infty v(x)f(x)^p\,dx < \infty$. Then standard limiting arguments give the result. If $\operatorname{supp} f = (0,N] \subset (0,\infty)$ and $\int_0^\infty v(x)f(x)^p\,dx < \infty$, then integration by parts, assumption (2.4), Theorem 2.1(b) with $\gamma = p/q$ and again integration by parts yield

$$\Big\{\int_0^\infty u(x)f(x)^q\,dx\Big\}^{p/q} = \Big\{\int_0^\infty f(x)^q\,d\Big[\int_0^x u(t)\,dt\Big]\Big\}^{p/q}$$

$$= \Big\{f(x)^q \int_0^x u(t)\,dt\Big|_0^\infty + \int_0^\infty \Big[\int_0^x u(t)\,dt\Big]\,d(-f(x)^q)\Big\}^{p/q}$$

$$\le \Big\{\int_0^\infty \Big[\int_0^x u(t)\,dt\Big]\,d(-f(x)^q)\Big\}^{p/q}$$

$$\le \Big\{\int_0^\infty C^q\Big[\int_0^x v(t)\,dt\Big]^{q/p} d(-f(x)^q)\Big\}^{p/q}$$

$$\le C^p \int_0^\infty \Big[\int_0^x v(t)\,dt\Big]\,d(-f(x)^p) = C^p \int_0^\infty v(x)f(x)^p\,dx.$$

(b) The necessity follows at once with $f(x) = \chi_{[t,\infty)}(x)$, $t > 0$.

Conversely, similarly to (a) we can assume that supp $f=[\delta,\infty)\subset(0,\infty)$ and $\int_0^\infty v(x)f(x)^p\,dx<\infty$. Using integration by parts, assumption (2.6), Theorem 2.1(a) with $\gamma=p/q$ and again integration by parts we obtain

$$\Big\{\int_0^\infty u(x)f(x)^q\,dx\Big\}^{p/q}=\Big\{-\int_0^\infty f(x)^q\,d\Big[\int_x^\infty u(t)\,dt\Big]\Big\}^{p/q}$$
$$=\Big\{-f(x)^q\int_x^\infty u(t)\,dt\Big|_0^\infty+\int_0^\infty\Big[\int_x^\infty u(t)\,dt\Big]\,d[f(x)^q]\Big\}^{p/q}$$
$$\le\Big\{\int_0^\infty\Big(\int_x^\infty u(t)\,dt\Big)\,d[f(x)^q]\Big\}^{p/q}$$
$$\le\Big\{\int_0^\infty D^q\Big(\int_x^\infty v(t)\,dt\Big)^{q/p}\,d[f(x)^q]\Big\}^{p/q}$$
$$\le D^p\int_0^\infty\Big(\int_x^\infty v(t)\,dt\Big)\,d[f(x)^p]=D^p\int_0^\infty v(x)f(x)^p\,dx,$$

and inequality (2.6) is proved.

3. Sharp weighted inequalities for monotone functions.

We now prove weighted inequalities of the form

$$\|T_1f\|_{q,u}\le C\|T_2f\|_{p,v},$$

where T_1 and T_2 are positive integral operators on monotone functions. These estimates are essential in proving the weighted Berwald–Favard inequalities given in the sequel. First we need the following known lemma (cf. Neugebauer [34, Lemmas 2.1, 4.2] and Carro–Soria [15, Th. 2.1]):

LEMMA 3.1. *Let $k\ge0$ be locally integrable on $(0,\infty)$ and $0<r<\infty$.*

(a) *If $0\le f\downarrow$ on $(0,\infty)$, then*

$$\int_0^\infty f(x)^rk(x)\,dx=r\int_0^\infty y^{r-1}\Big(\int_0^{f^{-1}(y)}k(x)\,dx\Big)\,dy.$$

(b) *If $0\le f\uparrow$ on $[0,a)$, $0<a\le\infty$, then*

$$\int_0^a f(x)^rk(x)\,dx=r\int_0^{f(a)}y^{r-1}\Big(\int_{f^{-1}(y)}^a k(x)\,dx\Big)\,dy.$$

(c) *If $0\le\varphi\downarrow$ on $(0,\infty)$, then*

$$\int_0^\infty\Big[\int_{\varphi(y)}^\infty k(x)\,dx\Big]\varphi(y)^r\,dy=\int_0^\infty\Big[\int_0^x\varphi^{-1}(s)\,d(s^r)-x^r\varphi^{-1}(x)\Big]k(x)\,dx.$$

Proof. (a) By the Fubini theorem

$$\int_0^\infty f(x)^rk(x)\,dx=r\int_0^\infty y^{r-1}\Big(\int_{\{x:f(x)>y\}}k(x)\,dx\Big)\,dy$$
$$=r\int_0^\infty y^{r-1}\Big(\int_0^{f^{-1}(y)}k(x)\,dx\Big)\,dy,$$

where the last equality follows since f is decreasing.

(b) In a similar way

$$\int_0^a f(x)^rk(x)\,dx=r\int_0^{f(a)}y^{r-1}\Big(\int_{\{x\in[0,a):f(x)>y\}}k(x)\,dx\Big)\,dy$$
$$=r\int_0^{f(a)}y^{r-1}\Big(\int_{f^{-1}(y)}^a k(x)\,dx\Big)\,dy.$$

(c) Interchanging the order of integration we have

$$\int_0^\infty\Big[\int_{\varphi(y)}^\infty k(x)\,dx\Big]\varphi(y)^r\,dy=\int_0^\infty\Big[\int_{\varphi^{-1}(x)}^\infty\varphi(y)^r\,dy\Big]k(x)\,dx.$$

The following equality is geometrically obvious (cf. O'Neil [35, p. 130] with $r=1$):

$$\int_0^{x^r}\varphi^{-1}(t^{1/r})\,dt=x^r\varphi^{-1}(x)+\int_{\varphi^{-1}(x)}^\infty\varphi(y)^r\,dy.$$

Applying these equalities together with $\int_0^{x^r}\varphi^{-1}(t^{1/r})\,dt=\int_0^x\varphi^{-1}(s)\,d(s^r)$ we obtain the proof of (c).

The proof of part (b) of Theorem 3.2 is taken from [16]. We repeat it here, with an emphasis on the constant which in our case plays an important role.

THEOREM 3.2. *Suppose $k_1(x,t)$ and $k_2(x,t)$ are two non-negative kernels.*

(a) *Let $0<p\le1\le q$. Then*

$$\Big(\int_0^\infty u(x)\Big[\int_0^\infty k_1(x,t)f(t)\,dt\Big]^q dx\Big)^{1/q}\le C\Big(\int_0^\infty v(x)\Big[\int_0^\infty k_2(x,t)f(t)\,dt\Big]^p dx\Big)^{1/p}\tag{3.1}$$

holds for all $0 \le f\downarrow$ *on* $[0,\infty)$ *if and only if*

$$(3.2)\qquad \Big(\int_0^\infty u(x)\Big[\int_0^\alpha k_1(x,t)\,dt\Big]^q dx\Big)^{1/q} \le C\Big(\int_0^\infty v(x)\Big[\int_0^\alpha k_2(x,t)\,dt\Big]^p dx\Big)^{1/p}$$

holds for all $\alpha > 0$.

(b) *Let* $0 < p \le 1$ *and* $p \le q$. *Then*

$$(3.3)\qquad \Big(\int_0^\infty u(x)\Big[\int_0^\infty k_1(x,t)f(t)\,dt\Big]^q dx\Big)^{1/q} \le C\Big(\int_0^\infty v(x)f(x)^p\,dx\Big)^{1/p}$$

holds for all $0 \le f\downarrow$ *on* $[0,\infty)$ *if and only if*

$$(3.4)\qquad \Big(\int_0^\infty u(x)\Big[\int_0^\alpha k_1(x,t)\,dt\Big]^q dx\Big)^{1/q} \le C\Big(\int_0^\alpha v(x)\,dx\Big)^{1/p}$$

holds for all $\alpha > 0$.

(c) *Let* $1 \le p \le q$. *Then*

$$(3.5)\qquad \Big(\int_0^\infty u(x)f(t)^q\,dx\Big)^{1/q} \le C\Big(\int_0^\infty v(x)\Big[\int_0^\infty k_2(x,t)f(t)\,dt\Big]^p dx\Big)^{1/p}$$

holds for all $0 \le f\downarrow$ *on* $[0,\infty)$ *if and only if*

$$(3.6)\qquad \Big(\int_0^\alpha u(x)\,dx\Big)^{1/q} \le C\Big(\int_0^\infty v(x)\Big[\int_0^\alpha k_2(x,t)\,dt\Big]^p dx\Big)^{1/p}$$

holds for all $\alpha > 0$.

Proof. The necessity parts of (a), (b) and (c) follow on taking $f(t) = \chi_{[0,\alpha]}(t)$, $\alpha > 0$ fixed.

(a) To prove sufficiency we apply Lemma 3.1(a) (with $r = 1$), Minkowski's inequality twice and (3.2) to obtain

$$\Big(\int_0^\infty u(x)\Big[\int_0^\infty k_1(x,t)f(t)\,dt\Big]^q dx\Big)^{1/q}$$

$$= \Big\{\int_0^\infty u(x)\Big[\int_0^\infty\Big(\int_0^{f^{-1}(y)} k_1(x,t)\,dt\Big)dy\Big]^q dx\Big\}^{1/q}$$

$$= \Big\|\int_0^\infty\Big(\int_0^{f^{-1}(y)} k_1(x,t)\,dt\Big)dy\Big\|_{q,u} \le \int_0^\infty\Big\|\int_0^{f^{-1}(y)} k_1(x,t)\,dt\Big\|_{q,u}dy$$

$$= \Big\{\int_0^\infty\Big[\int_0^\infty u(x)\Big(\int_0^{f^{-1}(y)} k_1(x,t)\,dt\Big)^q dx\Big]^{1/q}dy\Big\}$$

$$\le C\Big\{\int_0^\infty\Big[\int_0^\infty v(x)\Big(\int_0^{f^{-1}(y)} k_2(x,t)\,dt\Big)^p dx\Big]^{1/p}dy\Big\}$$

$$= C\Big\{\Big\|\int_0^\infty\Big(\int_0^{f^{-1}(y)} k_2(x,t)\,dt\Big)^p dx\Big\|_{1/p,v}\Big\}^{1/p}$$

$$\le C\Big\{\int_0^\infty\Big\|\Big(\int_0^{f^{-1}(y)} k_2(x,t)\,dt\Big)^p\Big\|_{1/p,v}dx\Big\}^{1/p}$$

$$= C\Big\{\int_0^\infty v(x)\Big[\int_0^\infty\Big(\int_0^{f^{-1}(y)} k_2(x,t)\,dt\Big)dy\Big]^p dx\Big\}^{1/p}$$

$$= C\Big(\int_0^\infty v(x)\Big[\int_0^\infty k_2(x,t)f(t)\,dt\Big]^p dx\Big)^{1/p}.$$

The last equality follows from Lemma 3.1(a) with $r = 1$.

(b) By Lemma 3.1(a) (with $r = 1$),

$$\Big(\int_0^\infty u(x)\Big(\int_0^\infty k_1(x,t)f(t)\,dt\Big)^q dx\Big)^{1/q} = \Big\{\int_0^\infty u(x)\Big[\int_0^\infty\Big(\int_0^{f^{-1}(y)} k_1(x,t)\,dt\Big)dy\Big]^q dx\Big\}^{1/q}.$$

Assume first that $q \le 1$. Since the innermost integral is a decreasing function of y, Theorem 2.1(a) applies with $\gamma = q$ (cf. Remark 2.2) so that the last integral is less than or equal to

$$\Big\{q\int_0^\infty u(x)\Big[\int_0^\infty y^{q-1}\Big(\int_0^{f^{-1}(y)} k_1(x,t)\,dt\Big)^q dy\Big]dx\Big\}^{1/q}$$

$$= \Big\{q\int_0^\infty y^{q-1}\Big[\int_0^\infty u(x)\Big(\int_0^{f^{-1}(y)} k_1(x,t)\,dt\Big)^q dx\Big]dy\Big\}^{1/q}$$

$$\le C\Big\{q\int_0^\infty y^{q-1}\Big(\int_0^{f^{-1}(y)} v(x)\,dx\Big)^{q/p}dy\Big\}^{1/q},$$

where the last inequality follows from (3.4). Since $p/q \le 1$ we apply again

Theorem 2.1(a) with $\gamma = p/q$ to see that the last expression is not larger than

$$C\Big\{p\int_0^\infty y^{p-1}\Big(\int_0^{f^{-1}(y)} v(x)\,dx\Big)\,dy\Big\}^{1/p} = C\Big(\int_0^\infty f(x)^p v(x)\,dx\Big)^{1/p}.$$

Here the equality follows again from Lemma 3.1(a).

If $q > 1$, let $g = f^q$. Then by Lemma 3.1(a),

$$\int_0^\infty k_1(x,t)f(t)\,dt = \int_0^\infty k_1(x,t)g(t)^{1/q}\,dt = \frac{1}{q}\int_0^\infty y^{1/q-1}\Big(\int_0^{g^{-1}(y)} k_1(x,t)\,dt\Big)\,dy.$$

Minkowski's inequality and (3.4) now shows that

$$\begin{aligned}
\Big(\int_0^\infty u(x)\Big(\int_0^\infty k_1(x,t)f(t)\,dt\Big)^q dx\Big)^{1/q}
&= \frac{1}{q}\Big\{\int_0^\infty u(x)\Big[\int_0^\infty y^{1/q-1}\Big(\int_0^{g^{-1}(y)} k_1(x,t)\,dt\Big)\,dy\Big]^q dx\Big\}^{1/q}\\
&= \frac{1}{q}\Big\|\int_0^\infty y^{1/q-1}\Big(\int_0^{g^{-1}(y)} k_1(x,t)\,dt\Big)\,dy\Big\|_{q,u}\\
&\le \frac{1}{q}\int_0^\infty y^{1/q-1}\Big\|\int_0^{g^{-1}(y)} k_1(x,t)\,dt\Big\|_{q,u}\,dy\\
&= \frac{1}{q}\int_0^\infty y^{1/q-1}\Big[\int_0^\infty u(x)\Big(\int_0^{g^{-1}(y)} k_1(x,t)\,dt\Big)^q dx\Big]^{1/q} dy\\
&\le \frac{C}{q}\int_0^\infty y^{1/q-1}\Big(\int_0^{g^{-1}(y)} v(x)\,dx\Big)^{1/p} dy.
\end{aligned}$$

But $p \le 1$, so again Theorem 2.1(a) with $\gamma = p$ applies and hence the last expression is not larger than

$$C\Big\{\frac{p}{q}\int_0^\infty y^{p/q-1}\Big(\int_0^{g^{-1}(y)} v(x)\,dx\Big)\,dy\Big\}^{1/p} = C\Big\{\int_0^\infty g(x)^{p/q}v(x)\,dx\Big\}^{1/p} = C\Big\{\int_0^\infty f(x)^p v(x)\,dx\Big\}^{1/p}.$$

(c) Let $g = f^q$. Then by Lemma 3.1(a),

$$\Big[\int_0^\infty f(x)^q u(x)\,dx\Big]^{p/q} = \Big[\int_0^\infty\Big(\int_0^{g^{-1}(y)} u(x)\,dx\Big)\,dy\Big]^{p/q}.$$

Theorem 2.1(a) with $\gamma = p/q \le 1$ and (3.6) show that this is not larger than

$$\begin{aligned}
&\frac{p}{q}\int_0^\infty y^{p/q-1}\Big(\int_0^{g^{-1}(y)} u(x)\,dx\Big)^{p/q} dy\\
&\le C^p\frac{p}{q}\int_0^\infty y^{p/q-1}\Big[\int_0^\infty\Big(\int_0^{g^{-1}(y)} k_2(x,t)\,dt\Big)^p v(x)\,dx\Big]\,dy\\
&= C^p\int_0^\infty\Big[\frac{p}{q}\int_0^\infty y^{p/q-1}\Big(\int_0^{g^{-1}(y)} k_2(x,t)\,dt\Big)^p dy\Big]v(x)\,dx = C^p\int_0^\infty A(x)v(x)\,dx
\end{aligned}$$

and again by Theorem 2.1(a) with $\gamma = 1/p \le 1$ we have

$$\begin{aligned}
A(x) &= \frac{p}{q}\int_0^\infty y^{p/q-1}\Big(\int_0^{g^{-1}(y)} k_2(x,t)\,dt\Big)^p dy\\
&= \int_0^\infty\Big(\int_0^{g^{-1}(y)} k_2(x,t)\,dt\Big)^p d(y^{p/q})\\
&\le \Big[\int_0^\infty\Big(\int_0^{g^{-1}(y)} k_2(x,t)\,dt\Big)\,d(y^{1/q})\Big]^p\\
&= \Big[\int_0^\infty \frac{1}{q}y^{1/q-1}\Big(\int_0^{g^{-1}(y)} k_2(x,t)\,dt\Big)\,dy\Big]^p.
\end{aligned}$$

Thus from the above and Lemma 3.1(a),

$$\begin{aligned}
\Big[\int_0^\infty f(x)^q u(x)\,dx\Big]^{p/q} &\le C^p\int_0^\infty\Big[\int_0^\infty \frac{1}{q}y^{1/q-1}\int_0^{g^{-1}(y)} k_2(x,t)\,dt\,dy\Big]^p v(x)\,dx\\
&= C^p\int_0^\infty\Big[\int_0^\infty g(t)^{1/q}k_2(x,t)\,dt\Big]^p v(x)\,dx\\
&= C^p\int_0^\infty\Big[\int_0^\infty f(t)k_2(x,t)\,dt\Big]^p v(x)\,dx.
\end{aligned}$$

This proves the theorem.

The next assertions for increasing functions on $[0,a)$ will be proved by similar arguments.

THEOREM 3.3. *Suppose $k_1(x,t)$ and $k_2(x,t)$ are two non-negative kernels.*

(a) *Let $0<p\le 1\le q$. Then*

$$\Big(\int_0^a u(x)\Big[\int_0^a k_1(x,t)f(t)\,dt\Big]^q dx\Big)^{1/q} \le D\Big(\int_0^a v(x)\Big[\int_0^a k_2(x,t)f(t)\,dt\Big]^p dx\Big)^{1/p} \tag{3.7}$$

holds for all $0\le f\uparrow$ on $[0,a)$, $0<a\le\infty$, if and only if

$$\Big(\int_0^a u(x)\Big[\int_\alpha^a k_1(x,t)\,dt\Big]^q dx\Big)^{1/q} \le D\Big(\int_0^a v(x)\Big[\int_\alpha^a k_2(x,t)\,dt\Big]^p dx\Big)^{1/p} \tag{3.8}$$

holds for all $\alpha\in(0,a)$.

(b) *Let $0<p\le 1$ and $p\le q$. Then*

$$\Big(\int_0^a u(x)\Big[\int_0^a k_1(x,t)f(t)\,dt\Big]^q dx\Big)^{1/q} \le D\Big(\int_0^a v(x)f(x)^p\,dx\Big)^{1/p} \tag{3.9}$$

holds for all $0\le f\uparrow$ on $[0,a)$, $0<a\le\infty$, if and only if

$$\Big(\int_0^a u(x)\Big[\int_\alpha^a k_1(x,t)\,dt\Big]^q dx\Big)^{1/q} \le D\Big(\int_\alpha^a v(x)\,dx\Big)^{1/p} \tag{3.10}$$

holds for all $\alpha\in(0,a)$.

(c) *Let $1\le p\le q$. Then*

$$\Big(\int_0^a u(x)f(x)^q\,dx\Big)^{1/q} \le D\Big(\int_0^a v(x)\Big[\int_0^a k_2(x,t)f(t)\,dt\Big]^p dx\Big)^{1/p} \tag{3.11}$$

holds for all $0\le f\uparrow$ on $[0,a)$, $0<a\le\infty$, if and only if

$$\Big(\int_\alpha^a u(x)\,dx\Big)^{1/q} \le D\Big(\int_0^a v(x)\Big[\int_\alpha^a k_2(x,t)\,dt\Big]^p dx\Big)^{1/p} \tag{3.12}$$

holds for all $\alpha\in(0,a)$.

Proof. The necessity parts of (a), (b) and (c) follow on taking $f(t)=\chi_{[\alpha,a)}(t)$, $\alpha>0$ fixed.

Sufficiency. (a) The proof is similar to that of Theorem 3.2(a), we need only use Lemma 3.1(b) twice (with $r=1$):

$$\Big\{\int_0^a u(x)\Big[\int_0^a k_i(x,t)f(t)\,dt\Big]^q dx\Big\}^{1/q} = \Big\{\int_0^a u(x)\Big[\int_0^{f(a)}\Big(\int_{f^{-1}(y)}^a k_i(x,t)\,dt\Big)dy\Big]^q dx\Big\}^{1/q}, \quad i=1,2.$$

(b) Similarly, by Lemma 3.1(b) (with $r=1$),

$$\Big\{\int_0^a u(x)\Big[\int_0^a k_1(x,t)f(t)\,dt\Big]^q dx\Big\}^{1/q} = \Big\{\int_0^a u(x)\Big[\int_0^{f(a)}\Big(\int_{f^{-1}(y)}^a k_1(x,t)\,dt\Big)dy\Big]^q dx\Big\}^{1/q}.$$

Assume again, first, that $q\le 1$. Since the inner integral is a decreasing function of y we may apply Lemma 2.1(a) with $\gamma=q$ so the last term is not larger than

$$\Big\{q\int_0^a u(x)\Big[\int_0^{f(a)} y^{q-1}\Big(\int_{f^{-1}(y)}^a k_1(x,t)\,dt\Big)^q dy\Big]dx\Big\}^{1/q} = \Big\{q\int_0^{f(a)} y^{q-1}\Big[\int_0^a u(x)\Big(\int_{f^{-1}(y)}^a k_1(x,t)\,dt\Big)^q dx\Big]dy\Big\}^{1/q}$$
$$\le D\Big\{q\int_0^{f(a)} y^{q-1}\Big[\int_{f^{-1}(y)}^a v(x)\,dx\Big]^{q/p}dy\Big\}^{1/q},$$

where the last inequality follows from (3.10). But since $p/q\le 1$ we apply Theorem 2.1(a) with $\gamma=p/q$ to see that the last expression is not larger than

$$D\Big\{p\int_0^{f(a)} y^{p-1}\Big[\int_{f^{-1}(y)}^a v(x)\,dx\Big]dy\Big\}^{1/p} = D\Big(\int_0^a f(x)^p v(x)\,dx\Big)^{1/p}.$$

If $q>1$, let $g=f^q$. Then by Lemma 3.1(b),

$$\int_0^a k_1(x,t)f(t)\,dt = \int_0^a k_1(x,t)g(t)^{1/q}\,dt = \frac{1}{q}\int_0^{g(a)} y^{1/q-1}\Big(\int_{g^{-1}(y)}^a k_1(x,t)\,dt\Big)dy.$$

By Minkowski's inequality and (3.10),

$$\Big\{\int_0^a u(x)\Big[\int_0^a k_1(x,t)f(t)\,dt\Big]^q dx\Big\}^{1/q}$$
$$=\frac{1}{q}\Big\{\int_0^a u(x)\Big[\int_0^{g(a)} y^{1/q-1}\Big(\int_{g^{-1}(y)}^a k_1(x,t)\,dt\Big)dy\Big]^q dx\Big\}^{1/q}$$
$$\le\frac{1}{q}\Big\{\int_0^{g(a)} y^{1/q-1}\Big[\int_0^a u(x)\Big(\int_{g^{-1}(y)}^a k_1(x,t)\,dt\Big)^q dx\Big]^{1/q} dy\Big\}$$
$$\le\frac{D}{q}\int_0^{g(a)} y^{1/q-1}\Big(\int_{g^{-1}(y)}^a v(x)\,dx\Big)^{1/p} dy.$$

But since $p\le 1$, Theorem 2.1 applies again and hence by Lemma 3.1(b) the last term is not larger than

$$D\Big\{\frac{p}{q}\int_0^{g(a)} y^{p/q-1}\Big(\int_{g^{-1}(y)}^a v(x)\,dx\Big)dy\Big\}^{1/p}$$
$$=D\Big(\int_0^a g(x)^{p/q}v(x)\,dx\Big)^{1/p}=D\Big(\int_0^a f(x)^p v(x)\,dx\Big)^{1/p}.$$

(c) The proof is similar to that of Theorem 3.2(c) and therefore omitted.

Remark 3.4. Theorems 3.2(b) and 3.3(b) were proved in a different way by Stepanov [43], Myasnikov–Persson–Stepanov [33] and, in the case of $0<p\le q\le 1$, by Lai [27]. Our method of proof is taken from the paper by Carro–Soria [15] and Lai [27]. Theorem 3.2(c) was also proved by Myasnikov–Persson–Stepanov [33] and Lai [27].

The choice of $k_1(x,t)=\chi_{[0,x]}(t)$ and $k_1(x,t)=\chi_{[x,a]}(t)$, in Theorem 3.2(b), respectively Theorem 3.3(b), gives

COROLLARY 3.5. *Let $0<p\le 1$ and $p\le q$.*

(a) *The inequality*

$$\Big(\int_0^\infty u(x)\Big(\int_0^x f(t)\,dt\Big)^q dx\Big)^{1/q}\le C\Big(\int_0^\infty f(x)^p v(x)\,dx\Big)^{1/p}$$

holds for all $0\le f\downarrow$ if and only if for every $\alpha>0$,

$$\Big(\int_0^\infty u(x)(\min\{x,\alpha\})^q\,dx\Big)^{1/q}\le C\Big(\int_0^\alpha v(x)\,dx\Big)^{1/p}.$$

(b) *For $a<\infty$, the inequality*

$$\Big(\int_0^a u(x)\Big(\int_x^a f(t)\,dt\Big)^q dx\Big)^{1/q}\le D\Big(\int_0^a f(x)^p v(x)\,dx\Big)^{1/p}$$

holds for all $0\le f\uparrow$ if and only if for every $\alpha\in(0,a)$,

$$\Big(\int_0^a u(x)(\min\{a-x,a-\alpha\})^q\,dx\Big)^{1/q}\le D\Big(\int_\alpha^a v(x)\,dx\Big)^{1/p}.$$

Remark 3.6. If $p=q\ge 1$ and $u=v$, then the result corresponding to Corollary 3.5(a) was proved by Neugebauer [34, Th. 2.2].

In the case when both the integral operators are equal and of the form $\int_0^x f(t)\,dt$, Theorem 3.2(a) extends to the range $p=q\ge 1$.

THEOREM 3.7. *If either $0<p\le 1\le q<\infty$ or $1\le p=q<\infty$, then*

$$(3.13)\qquad \Big(\int_0^\infty u(x)\Big(\int_0^x f(t)\,dt\Big)^q dx\Big)^{1/q}\le C\Big(\int_0^\infty v(x)\Big(\int_0^x f(t)\,dt\Big)^p dx\Big)^{1/p}$$

holds for all $0\le f\downarrow$ if and only if

$$(3.14)\qquad \Big(\int_0^\infty u(x)(\min\{x,\alpha\})^q\,dx\Big)^{1/q}\le C\Big(\int_0^\infty v(x)(\min\{x,\alpha\})^p\,dx\Big)^{1/p}$$

holds for every $\alpha>0$.

Proof. If $0<p\le 1\le q$, then the proof follows immediately from Theorem 3.2(a) by taking $k_1(x,t)=k_2(x,t)=\chi_{[0,x]}(t)$.

If $1\le p=q$, then (3.14) with $\alpha=\varphi(y)$ means

$$\int_0^\infty u(x)(\min\{x,\varphi(y)\})^p\,dx\le C^p\int_0^\infty v(x)(\min\{x,\varphi(y)\})^p\,dx.$$

Integrating from 0 to ∞ with respect to y and using Lemma 3.1(a) and (c), we obtain

$$\int_0^\infty\Big(\int_0^\infty u(x)(\min\{x,\varphi(y)\})^p\,dx\Big)dy$$
$$=\int_0^\infty\Big[\int_0^{\varphi(y)} u(x)x^p\,dx+\varphi(y)^p\int_{\varphi(y)}^\infty u(x)\,dx\Big]dy$$
$$=\int_0^\infty \varphi^{-1}(x)u(x)x^p\,dx+\int_0^\infty\Big[\int_0^x \varphi^{-1}(s)\,d(s^p)-x^p\varphi^{-1}(x)\Big]u(x)\,dx$$
$$=\int_0^\infty\Big[\int_0^x \varphi^{-1}(s)\,d(s^p)\Big]u(x)\,dx.$$

Similarly for the expression with weight v we obtain from (3.14),

$$\int_0^\infty \Big[\int_0^x \varphi^{-1}(s)\,d(s^p)\Big] u(x)\,dx \le C^p \int_0^\infty \Big[\int_0^x \varphi^{-1}(s)\,d(s^p)\Big] v(x)\,dx.$$

Taking $\varphi^{-1}(s) = f(s)(s^{-1}\int_0^s f(t)\,dt)^{p-1}$ we have $\int_0^x \varphi^{-1}(s)\,d(s^p) = (\int_0^x f(t)\,dt)^p$ and so

$$\int_0^\infty \Big[\int_0^x f(t)\,dt\Big]^p u(x)\,dx \le C^p \int_0^\infty \Big[\int_0^x f(t)\,dt\Big]^p v(x)\,dx.$$

Remark 3.8. The same result with a different constant in the range $0 < p \le q < \infty$ and $q \ge 1$ was proved by Stepanov [43, Th. 3.3].

4. Sharp weighted inequalities for concave functions. Consider the Green kernel

$$K(x,t) = \begin{cases} x(1-t) & \text{if } 0 \le x \le t \le 1,\\ t(1-x) & \text{if } 0 \le t \le x \le 1.\end{cases} \tag{4.1}$$

Then the function

$$f(x) = \int_0^1 K(x,t)g(t)\,dt, \tag{4.2}$$

where g ranges over all non-negative functions in $L^1[0,1]$, constitute a dense subset of non-negative concave functions on $[0,1]$ (cf. [3], [5]). This fact is used in the proof of the following theorem:

THEOREM 4.1. *Suppose either $0 < p \le 1 \le q$ or $1 \le p \le q$. Then*

$$\Big(\int_0^1 u(x)f(x)^q\,dx\Big)^{1/q} \le C\Big(\int_0^1 v(x)f(x)^p\,dx\Big)^{1/p} \tag{4.3}$$

holds for any concave function $f \ge 0$ on $[0,1]$ if and only if

$$\Big(\int_0^1 u(x)K(x,\alpha)^q\,dx\Big)^{1/q} \le C\Big(\int_0^1 v(x)K(x,\alpha)^p\,dx\Big)^{1/p} \tag{4.4}$$

holds for all $\alpha \in (0,1)$, where $K(x,\alpha) = \min\{x,\alpha\}\min\{1-x, 1-\alpha\}$.

Proof. The necessity follows at once on taking $f(x) = K(x,\alpha)$, $\alpha \in (0,1)$ fixed, in (4.3). To prove sufficiency in the case of $0 < p \le 1 \le q$, it suffices to prove (4.3) for those functions f having representation (4.2) and then standard limiting arguments give the general result. We may also assume that the left side of (4.3) is finite on taking first a suitable dense subset. This restriction again can be removed by limiting procedures.

Assuming this, we use Hölder's inequality, (4.4) and Minkowski's inequality to obtain

$$\begin{aligned}
\|f\|_{q,u}^q &= \int_0^1 f(x)^{q-1}u(x)\Big[\int_0^1 K(x,t)g(t)\,dt\Big]dx\\
&= \int_0^1 g(t)\Big[\int_0^1 K(x,t)u(x)f(x)^{q-1}\,dx\Big]dt\\
&\le \int_0^1 g(t)\Big[\int_0^1 K(x,t)^q u(x)\,dx\Big]^{1/q}\Big[\int_0^1 f(x)^q u(x)\,dx\Big]^{1/q'}dt\\
&\le C\int_0^1 g(t)\Big[\int_0^1 K(x,t)^p v(x)\,dx\Big]^{1/p}dt\,\|f\|_{q,u}^{q/q'}\\
&= C\Big(\Big\|\int_0^1 K(x,t)^p v(x)\,dx\Big\|_{1/p,g}\Big)^{1/p}\|f\|_{q,u}^{q-1}\\
&\le C\Big(\int_0^1 \|K(x,t)^p v(x)\|_{1/p,g}\,dx\Big)^{1/p}\|f\|_{q,u}^{q-1}\\
&= C\Big(\int_0^1\Big(\int_0^1 K(x,t)v(x)^{1/p}g(t)\,dt\Big)^p dx\Big)^{1/p}\|f\|_{q,u}^{q-1}\\
&= C\Big(\int_0^1 f(x)^p v(x)\,dx\Big)^{1/p}\|f\|_{q,u}^{q-1} = C\|f\|_{p,v}\|f\|_{q,u}^{q-1}
\end{aligned}$$

and the result follows on division by $\|f\|_{q,u}^{q-1}$.

Sufficiency in the case $1 \le p \le q$. For $p \ge 0$ let the differential operator on $C^2(0,1)$ be given by

$$\begin{aligned}
L_p[y] &= (1-x)^p\frac{d}{dx}\Big\{x^{p+1}(1-x)^{1-p}\frac{d}{dx}(x^{-p}y)\Big\}\\
&= x(1-x)y'' + (p-1)(2x-1)y' + p(1-p)y
\end{aligned}$$

and consider the Radon-concave functions of order p, $R_p = \overline{Q}_p$ (cf. [17], [37]), i.e. the closure in the topology of locally uniform convergence on $(0,1)$ of

$$Q_p = \{f \in C^2[0,1] : -L_p[f] \ge 0,\ f(0) = f(1) = 0\}.$$

For every continuous function $g : [0,1] \to \mathbb{R}$ the boundary value problem

$$L_p[f] = g, \quad f(0) = f(1) = 0,$$

has a solution

$$f(x)=\int_0^1 G_p(x,t)g(t)\,dt,$$

where

$$G_p(x,t)=\frac{1}{p}\begin{cases} x^p t^{-p} & \text{if } 0<x<t<1,\\ (1-x)^p(1-t)^{-p} & \text{if } 0<t\le x<1.\end{cases}$$

Our result now follows from the following two facts (cf. also [17] and [37]):

1. If $1\le p\le q$, (4.4) holds and $f\in R_p$, then

$$\Big(\int_0^1 f(x)^{q/p}u(x)\,dx\Big)^{p/q}\le C\int_0^1 f(x)v(x)\,dx.$$

In fact (and similarly to the above),

$$\begin{aligned}\|f\|_{q/p,u}^{q/p}&=\int_0^1 f(x)^{q/p-1}u(x)\Big[\int_0^1 G_p(x,t)g(t)\,dt\Big]dx\\
&=\int_0^1 g(t)\Big[\int_0^1 G_p(x,t)f(x)^{q/p-1}u(x)\,dx\Big]dt\\
&\le\int_0^1 g(t)\Big[\int_0^1 G_p(x,t)^{q/p}u(x)\,dx\Big]^{p/q}\Big[\int_0^1 f(x)^{q/p}u(x)\,dx\Big]^{1-p/q}dt\\
&=\frac{1}{p}\int_0^1 g(t)\Big[\int_0^1 qG_q(x,t)u(x)\,dx\Big]^{p/q}dt\,\|f\|_{q/p,u}^{q/p-1}\\
&=\frac{1}{p}\int_0^1 g(t)t^{-p}(1-t)^{-p}\Big[\int_0^1 K(x,t)^q u(x)\,dx\Big]^{p/q}dt\,\|f\|_{q/p,u}^{q/p-1}\\
&\le\frac{1}{p}C^p\int_0^1 g(t)t^{-p}(1-t)^{-p}\Big[\int_0^1 K(x,t)^p v(x)\,dx\Big]dt\,\|f\|_{q/p,u}^{q/p-1}\\
&=C^p\int_0^1 g(t)\Big[\int_0^1 G_p(x,t)v(x)\,dx\Big]dt\,\|f\|_{q/p,u}^{q/p-1}\\
&=C^p\int_0^1\Big[\int_0^1 G_p(x,t)g(t)\,dt\Big]v(x)\,dx\,\|f\|_{q/p,u}^{q/p-1}\\
&=C^p\int_0^1 f(x)v(x)\,dx\,\|f\|_{q/p,u}^{q/p-1}.\end{aligned}$$

2. If $p\ge 1$ and $f\in R_1$, i.e. f is a positive concave function on $[0,1]$, then $f^p\in R_p$. It is enough to prove that if $y\in Q_1$ (i.e. $y''\le 0$ and $y(0)=y(1)=0$), then $y^p\in Q_p$. Since

$$\begin{aligned}L_p[y^p]&=x(1-x)p[(p-1)y^{p-2}(y')^2+y^{p-1}y'']\\
&\quad+(p-1)(2x-1)py^{p-1}y'+p(1-p)y^p\\
&=(p-1)y^{p-1}L_p[y]+p(p-1)y^{p-2}(xy'-y)[(1-x)y'+y]\end{aligned}$$

and $xy'(x)-y(x)=\int_0^x ty''(t)\,dt\le 0$, $(1-x)y'(x)+y(x)=\int_x^1(t-1)y''(t)\,dt\ge 0$ it follows that $L_p[y^p]\le 0$ and so $y^p\in Q_p$.

Remark 4.2. (i) Theorem 4.1 in the case $p=1$ was also proved by Maligranda–Pečarić–Persson [29, Th. 3]. In the case $0<p\le 1\le q$, Theorem 4.1 is still true for all functions which have representation (4.2) with the non-negative kernel $K(x,t)$. In particular, if

$$K(x,t)=\begin{cases}1 & \text{if } 0\le t\le x\le 1,\\ 0 & \text{if } 0\le x<t\le 1,\end{cases}$$

then $f(x)=\int_0^x g(t)\,dt$ is increasing on $[0,1]$. Similarly, if

$$K(x,t)=\begin{cases}0 & \text{if } 0\le t\le x\le 1,\\ 1 & \text{if } 0\le x<t\le 1,\end{cases}$$

then $f(x)=\int_x^1 g(t)\,dt$ is decreasing on $[0,1]$. Therefore Theorem 4.1 also gives the proof of Proposition 2.5 but only in the index range $0<p\le 1\le q$.

(ii) If either $0<p\le 1\le q$ or $1\le p\le q$ and $u(x)=v(x)=1$, then

$$\begin{aligned}h(t)&=\Big[\int_0^1 K(x,t)^q\,dx\Big]^{1/q}\Big[\int_0^1 K(x,t)^p\,dx\Big]^{-1/p}\\
&=\Big[\int_0^t x^q(1-t)^q\,dx+\int_t^1 t^q(1-x)^q\,dx\Big]^{1/q}\\
&\quad\times\Big[\int_0^t x^p(1-t)^p\,dx+\int_t^1 t^p(1-x)^p\,dx\Big]^{-1/p}\\
&=(p+1)^{1/p}(q+1)^{-1/q},\end{aligned}$$

which is the quotient, i.e. the constant C of (4.4), and hence Theorem 4.1 gives the Favard–Berwald inequality $\|f\|_q\le(p+1)^{1/p}(q+1)^{-1/q}\|f\|_p$ for any positive concave function f on $[0,1]$. Berwald [8] proved this inequality in the full range $0<p\le q$. In particular, if $q=1$ then $\|f\|_1\le\frac{1}{2}(p+1)^{1/p}\|f\|_p$, $0<p\le 1$. As $p\to 0$ this inequality becomes the reverse Jensen inequality given in the introduction.

(iii) If $u(x) = v(x) = x^\alpha$, $\alpha \ge 0$ and $p = 1$ in Theorem 4.1, then the function

$$h(t) = \Big[\int_0^1 K(x,t)^q x^\alpha\,dx\Big]^{1/q} \Big/ \Big[\int_0^1 K(x,t)x^\alpha\,dx\Big] = \frac{\Big[(1-t)^q \frac{t^{\alpha+1}}{q+\alpha+1} + \int_t^1 (1-x)^q x^\alpha\,dx\Big]^{1/q}}{\frac{(1-t)t^{\alpha+1}}{\alpha+2} + \frac{1-t^{\alpha+1}}{\alpha+1} - \frac{1-t^{\alpha+2}}{\alpha+2}}$$

can be shown to be decreasing on $(0,1)$. Hence $h(t) \le h(0)$ and therefore the constant of (4.3) is

$$(\alpha+1)(\alpha+2)\Big(\int_0^1 (1-x)^q x^\alpha\,dx\Big)^{1/q} = (\alpha+1)(\alpha+2)B(q+1,\alpha+1)^{1/q},$$

B being the Beta function. This is a result of Barnard–Wells [3, Th. 1].

We can apply Corollary 3.5 and Theorem 3.7 to prove Favard–Berwald inequalities for increasing and decreasing concave functions. Note that if $0 \le f$ is concave on $[0,\infty)$ then f is necessarily increasing.

Remark 4.3. (i) From Theorem 3.7 it follows at once that if either $0 < p \le q$ or $1 \le p = q < \infty$, then

$$\|f\|_{q,u} \le C\|f\|_{p,v}$$

holds for any concave increasing function f on $[0,\infty)$ if and only if (3.14) holds for all $\alpha > 0$.

(ii) If $1 = p \le q < \infty$, then it follows from Corollary 3.5 and Theorem 2.1(a) that

$$\Big(\int_0^1 u(x)f(x)^q\,dx\Big)^{1/q} \le D\int_0^1 v(x)f(x)\,dx$$

holds for any concave decreasing function $f \ge 0$ on $[0,1]$ if and only if

$$\Big(\int_0^1 u(x)[\min(1-x,1-\alpha)]^q\,dx\Big)^{1/q} \le D\int_0^1 v(x)\min(1-x,1-\alpha)\,dx$$

is satisfied for all $\alpha \in (0,1)$.

We conclude this section with three examples.

EXAMPLE 4.4. If $0 \le f$ is concave increasing on $[0,1]$, then by Remark 4.3(i) with $p = 1$, $q = 2$ and $u(x) = v(x) = x^\alpha$, $\alpha > 0$, we obtain

$$\Big(\int_0^1 f(x)^2 x^\alpha\,dx\Big)^{1/2} \le \frac{\alpha+2}{\sqrt{\alpha+3}}\int_0^1 f(x)x^\alpha\,dx. \tag{4.5}$$

The constant $(\alpha+2)/\sqrt{\alpha+3}$ is best possible, as may be seen on taking $f(x) = x$ in (4.5).

EXAMPLE 4.5. If $0 \le f$ is concave on $[0,\infty)$, then for $t > 0$,

$$\Big(\int_0^\infty e^{-tx}f(x)^q\,dx\Big)^{1/q} \le t^{1-1/q}\Gamma(q+1)^{1/q}\int_0^\infty e^{-tx}f(x)\,dx, \quad 1 < q < \infty. \tag{4.6}$$

The constant $t^{1-1/q}\Gamma(q+1)^{1/q}$ is best possible, as may be seen on taking $f(x) = x$ in (4.6).

In order to obtain the constant in (4.6) it clearly suffices to take $t = 1$. By Remark 4.3(i) with $p = 1$, we must show that

$$\sup_{\alpha>0}\Big\{\Big(\int_0^\infty e^{-x}[\min(x,\alpha)]^q\,dx\Big)^{1/q}\Big/\int_0^\infty e^{-x}\min(x,\alpha)\,dx\Big\} = \sup_{\alpha>0}\Big\{\Big(\int_0^\alpha e^{-x}x^q\,dx + \alpha^q e^{-\alpha}\Big)^{1/q}\Big/(1-e^{-\alpha})\Big\} = \Gamma(q+1)^{1/q}.$$

But if $h(\alpha) = (\int_0^\alpha e^{-x}x^q\,dx + \alpha^q e^{-\alpha})/(1-e^{-\alpha})^q$, then h is increasing on $(0,\infty)$ since

$$h'(\alpha) = (1-e^{-\alpha})^{-q}\Big\{q\alpha^{q-1}e^{-\alpha} + qe^{-\alpha}(1-e^{-\alpha})^{q-1}\Big[\int_0^\alpha e^{-x}x^q\,dx + \alpha^q e^{-\alpha}\Big]\Big\} > 0,$$

and the above supremum equals $\lim_{\alpha\to\infty} h(\alpha)^{1/q} = (\int_0^\infty e^{-x}x^q\,dx)^{1/q} = \Gamma(q+1)^{1/q}$.

EXAMPLE 4.6. If $0 \le f$ is concave on $[0,\infty)$, then

$$\Big(\int_0^\infty e^{-x^2}f(x)^q\,dx\Big)^{1/q} \le 2^{1-1/q}\Gamma\Big(\frac{q+1}{2}\Big)^{1/q}\int_0^\infty e^{-x^2}f(x)\,dx, \quad 1 < q < \infty, \tag{4.7}$$

and the constant $2^{1-1/q}\Gamma((q+1)/2)^{1/q}$ is sharp, as may be seen on taking $f(x)=x$ in (4.7).

Again by Remark 4.3(i) we only need to show that

$$\sup_{\alpha>0}\Big\{\Big(\int_0^\infty e^{-x^2}[\min(x,\alpha)]^q\,dx\Big)^{1/q}\Big/\int_0^\infty e^{-x^2}\min(x,\alpha)\,dx\Big\}$$
$$=\sup_{\alpha>0}\Big\{\Big(\int_0^\alpha e^{-x^2}x^q\,dx+\alpha^q\int_\alpha^\infty e^{-x^2}\,dx\Big)^{1/q}$$
$$\times\Big(\int_0^\alpha e^{-x^2}x\,dx+\alpha\int_\alpha^\infty e^{-x^2}\,dx\Big)^{-1}\Big\}=2^{1-1/q}\Gamma\Big(\frac{q+1}{2}\Big)^{1/q}.$$

For that it suffices to show that

$$k(\alpha)=\Big(\int_0^\alpha e^{-x^2}x^q\,dx+\alpha^q\int_\alpha^\infty e^{-x^2}\,dx\Big)\Big/\Big(\int_0^\alpha e^{-x^2}x\,dx+\alpha\int_\alpha^\infty e^{-x^2}\,dx\Big)^q$$

is an increasing function on $(0,\infty)$, for then the above supremum is

$$\lim_{\alpha\to\infty}k(\alpha)^{1/q}=\Big(\int_0^\infty e^{-x^2}x^q\,dx\Big)^{1/q}\Big/\int_0^\infty e^{-x^2}x\,dx=2^{1-1/q}\Gamma\Big(\frac{q+1}{2}\Big)^{1/q}.$$

But

$$k'(\alpha)=q\Big(\int_0^\alpha e^{-x^2}x\,dx+\alpha\int_\alpha^\infty e^{-x^2}\,dx\Big)^{-q-1}\Big(\int_\alpha^\infty e^{-x^2}\,dx\Big)$$
$$\times\Big[\alpha^{q-1}\int_0^\alpha e^{-x^2}x\,dx-\int_0^\alpha e^{-x^2}x^q\,dx\Big]>0$$

and hence the function k is increasing as required.

5. Weighted reverse Hölder inequalities. Closely related to the Favard–Berwald inequality is Grüss' inequality [21] (see also [23]). It asserts that the L^1-norms of f and g are dominated by the L^1-norm of the product fg, where f,g are from some class of functions. This inequality together with Favard's gives a reverse Hölder inequality (cf. [3], [4], [6], [9], [26]).

We now discuss weighted versions of Grüss' inequality and also give weight characterizations for which such inequalities hold.

In our first result we assume $0\le w\in L^1(I)$, where $I\subset\mathbb{R}$ is an interval, and we let $\mathcal{P}$ be the class of positive measurable functions on I.

THEOREM 5.1. (a) *If* $\|f\|_{2,w}\le C_2\|f\|_{1,w}$ *for any* $f\in\mathcal{P}$, *then*

$$C_2^*\|f\|_{1,w}\|g\|_{1,w}\le\|fg\|_{1,w}\quad\text{for any } f,g\in\mathcal{P},\tag{5.1}$$

where $C_2^*=2/w(I)-C_2^2$ *and* $w(I)=\int_I w(x)\,dx$.

(b) *If* $\|f\|_{r,w}\le C_r\|f\|_{1,w}$ *for every* $r>1$ *and* $f\in\mathcal{P}$, *then*

$$C_{p,q}^*\|f\|_{p,w}\|g\|_{q,w}\le\|fg\|_{1,w}\tag{5.2}$$

for $1\le p,q<\infty$, $f,g\in\mathcal{P}$ *and* $C_{p,q}^*=C_2^*/(C_pC_q)$.

Proof. (a) Let $\|f\|_{1,w}=\|g\|_{1,w}=1$. Then

$$\|f+g\|_{2,w}^2=\|f\|_{2,w}^2+2\|fg\|_{1,w}+\|g\|_{2,w}^2$$
$$\le C_2^2\|f\|_{1,w}^2+2\|fg\|_{1,w}+C_2^2\|g\|_{1,w}^2=2C_2^2+2\|fg\|_{1,w}.$$

On the other hand, Hölder's inequality shows that

$$2=\|f+g\|_{1,w}\le\|f+g\|_{2,w}w(I)^{1/2},$$

so that

$$2\|fg\|_{1,w}\ge\|f+g\|_{2,w}^2-2C_2^2\ge\frac{4}{w(I)}-2C_2^2,$$

and hence

$$\|fg\|_{1,w}\ge\Big(\frac{2}{w(I)}-C_2^2\Big)\|f\|_{1,w}\|g\|_{1,w}.$$

(b) The proof follows immediately from (a) and the assumption.

For concave functions the constants given in Theorem 5.1 are not always sharp. We illustrate this fact by examples given next.

EXAMPLE 5.2. (a) Let $\mathcal{P}=\{f\ge0: f \text{ is concave on } [0,1]\}$. If $w(x)=x^\alpha$, $\alpha\ge0$, then Remark 4.2(iii) shows that the constant C_r equals $(\alpha+1)(\alpha+2)\times B(r+1,\alpha+1)^{1/r}$, where B is the Beta function. Since we have $C_2=[2(\alpha+1)(\alpha+2)/(\alpha+3)]^{1/2}$, it follows that $C_2^*=2(\alpha+1)/(\alpha+3)$ and

$$C_{p,q}^*=2[(\alpha+1)(\alpha+2)^2(\alpha+3)]^{-1}B(p+1,\alpha+1)^{-1/p}B(q+1,\alpha+1)^{-1/q}.$$

In particular, $C_{2,2}^*=(\alpha+2)^{-1}$ but it is known that the best constant is $\{(\alpha+1)/[2(\alpha+3)]\}^{1/2}$ (cf. Barnard–Wells [3]). If $\alpha=0$ the constant $C_{p,q}^*=\frac{1}{6}(p+1)^{1/p}(q+1)^{1/q}$ is sharp (cf. Barnes [4]).

(b) If $\mathcal{P}=\{f\ge0: f \text{ is concave on } [0,\infty)\}$ and $w(x)=e^{-x^2}$, then Example 4.6 shows that $C_r=2^{1-1/r}\Gamma((r+1)/2)^{1/r}$. Hence we have $C_2^*=(4-\pi)\pi^{-1/2}>0$ and

$$C_{p,q}^*=(4-\pi)\pi^{-1/2}2^{1/p+1/q-2}\Gamma\Big(\frac{p+1}{2}\Big)^{-1/p}\Gamma\Big(\frac{q+1}{2}\Big)^{-1/q}.$$

To obtain sharp constants in inequality (5.2) for a specific class of functions, say concave functions, we require properties of these functions which are not as general as those of $\mathcal{P}$.

THEOREM 5.3. *Suppose f, g are non-negative concave functions on $[0,1]$ and $0 < r \le 1 \le p,\ q < \infty$. If u, v and w are weight functions on $[0,1]$, then*

$$(5.3)\qquad \Big(\int_0^1 u(x)f(x)^p\,dx\Big)^{1/p}\Big(\int_0^1 v(x)g(x)^q\,dx\Big)^{1/q} \le C\Big(\int_0^1 w(x)[f(x)g(x)]^r\,dx\Big)^{1/r}$$

holds if and only if

$$(5.4)\qquad C = \sup_{s,t\in(0,1)} \frac{[\int_0^1 K(x,t)^p u(x)\,dx]^{1/p}[\int_0^1 K(x,s)^q v(x)\,dx]^{1/q}}{[\int_0^1 [K(x,t)K(x,s)]^r w(x)\,dx]^{1/r}} < \infty,$$

where

$$K(x,y) = \begin{cases} x(1-y) & \text{if } 0\le x\le y\le 1,\\ y(1-x) & \text{if } 0\le y\le x\le 1.\end{cases}$$

Proof. *Necessity.* Substitute $f(x) = K(x,t)$ and $g(x) = K(x,s)$, where $s,t\in(0,1)$ are arbitrary, into (5.3). Then the necessity follows.

Sufficiency. As noted earlier, it suffices to prove (5.3) if f and g have the representation

$$f(x) = \int_0^1 K(x,t)f_1(t)\,dt, \qquad g(x) = \int_0^1 K(x,s)g_1(s)\,ds,$$

where f_1, g_1 are some non-negative functions in $L^1[0,1]$. Applying Minkowski's inequality twice and (5.3) one obtains

$$\begin{aligned}
\|f\|_{p,u}\|g\|_{q,v} &= \Big[\int_0^1 u(x)\Big(\int_0^1 K(x,t)f_1(t)\,dt\Big)^p dx\Big]^{1/p} \times \Big[\int_0^1 v(x)\Big(\int_0^1 K(x,s)g_1(s)\,ds\Big)^q dx\Big]^{1/q}\\
&\le \int_0^1 f_1(t)\Big[\int_0^1 K(x,t)^p u(x)\,dx\Big]^{1/p} dt \int_0^1 g_1(s)\Big[\int_0^1 K(x,s)^q v(x)\,dx\Big]^{1/q} ds\\
&= \int_0^1\int_0^1 f_1(t)g_1(s)\Big[\int_0^1 K(x,t)^p u(x)\,dx\Big]^{1/p} dt\Big[\int_0^1 K(x,s)^q v(x)\,dx\Big]^{1/q} ds\,dt\\
&\le C\int_0^1\int_0^1 f_1(t)g_1(s)\Big\{\int_0^1 [K(x,t)K(x,s)]^r w(x)\,dx\Big\}^{1/r} ds\,dt\\
&\le C\Big\{\int_0^1 w(x)\Big[\int_0^1\int_0^1 f_1(t)g_1(s)K(x,t)K(x,s)\,ds\,dt\Big]^r dx\Big\}^{1/r}\\
&= C\Big\{\int_0^1 w(x)\Big[\int_0^1 f_1(t)K(x,t)\,dt\Big]^r\Big[\int_0^1 g_1(s)K(x,s)\,ds\Big]^r\Big\}^{1/r}\\
&= C\|fg\|_{r,w}.
\end{aligned}$$

Remark 5.4. (i) The method of proof of Theorem 5.3 with the Green function $K(x,y)$ is usually called the Bellman or Bellman–Weinberger method. Bellman proved such a result in the case of $u = v = w = 1$, $r = 1$ and $1/p + 1/q = 1$ (cf. [5], [6], [3], [26], [37], [46], [47]). This method was also used earlier by Bückner [11] who proved the case $u = v = w$, $r = 1$ and $p = q = 2$.

(ii) Computations of the supremum (5.4) or (5.6) are often quite tedious. If $u(x) = v(x) = w(x) = 1$ in Theorem 5.3, then the supremum over $0 < s \le t < 1$ of

$$\begin{aligned}
H(s,t) &= \Big(\int_0^1 K(x,t)^p\,dx\Big)^{1/p} \times \Big(\int_0^1 K(x,s)^q\,dx\Big)^{1/q}\Big/\Big(\int_0^1 [K(x,t)K(x,s)]^r\,dx\Big)^{1/r}\\
&= (p+1)^{-1/p}(q+1)^{-1/q}t(1-s)\Big[\frac{(1-s)^r s^{r+1}}{2r+1} + \int_s^t x^r(1-x)^r\,dx + \frac{t^r(1-t)^{r+1}}{2r+1}\Big]^{-1/r}
\end{aligned}$$

is not easy to compute (cf. [3], [6], [25], [37], [46] and [47]).

(iii) If f and g are non-negative concave increasing functions on $[0,a)$, $0 < a \le \infty$, then the result corresponding to Theorem 5.3 is the following: If $0 < r \le 1 \le p,\ q < \infty$, then

$$(5.5)\qquad \Big(\int_0^a u(x)f(x)^p\,dx\Big)^{1/p}\Big(\int_0^a v(x)g(x)^q\,dx\Big)^{1/q} \le D\Big(\int_0^a w(x)[f(x)g(x)]^r\,dx\Big)^{1/r}$$

holds if and only if

$$(5.6)\qquad D = \sup_{s,t\in(0,a)} \frac{[\int_0^a K(x,t)^p u(x)\,dx]^{1/p}[\int_0^a K(x,s)^q v(x)\,dx]^{1/q}}{[\int_0^a [K(x,t)K(x,s)]^r w(x)\,dx]^{1/r}} < \infty,$$

where

$$K(x,y)=\begin{cases} x & \text{if } 0\le x\le y<a,\\ y & \text{if } 0\le y\le x<a.\end{cases}$$

Finally, if for $p,q\ge 1$ and $\alpha\ge 0$, $H_{p,q,\alpha}$ denotes the best constant H in the reverse Hölder inequality

$$H\Big(\int_0^1 f(x)^p x^\alpha\,dx\Big)^{1/p}\Big(\int_0^1 g(x)^q x^\alpha\,dx\Big)^{1/q}\le\int_0^1 f(x)g(x)x^\alpha\,dx,$$

f,g concave positive functions on $[0,1]$, then the known results are the following:

$$H_{2,2,0}=1/2 \quad \text{(Frank–Pick [19])},$$

$$H_{1,1,0}=2/3 \quad \text{(Grüss [21])},$$

$$H_{p,p',0}=\frac{1}{6}(p+1)^{1/p}(p'+1)^{1/p'} \quad \text{(Bellman [6])},$$

$$H_{p,q,0}=\frac{1}{6}(p+1)^{1/p}(q+1)^{1/q} \quad \text{(Barnes [4])},$$

$$H_{2,2,\alpha}=\sqrt{\frac{\alpha+1}{2(\alpha+2)}} \quad \text{(Barnard–Wells [3])},$$

$$H_{p,p',\alpha}=\min\{V_\alpha(p),V_\alpha(p'),W_\alpha(p)\}/[(\alpha+2)(\alpha+3)] \quad \text{(Wang–Chen [47])},$$

where

$$V_\alpha(p)=(p'+1+\alpha)^{1/p'}B(p+1,\alpha+1)^{-1/p},$$

$$W_\alpha(p)=2/[(\alpha+1)B(p+1,\alpha+1)^{1/p}B(p'+1,\alpha+1)^{1/p'}].$$

Also if $H^+_{p,q,\alpha}$ is the corresponding constant for increasing concave functions f,g on $[0,1]$, then

$$H^+_{2,2,0}=\sqrt{3}/2 \quad \text{(Kraft, Bückner [11])}$$

and

$$H^+_{p,q,0}=\min\left\{\tfrac{1}{2}(p+1)^{1/p},\tfrac{1}{2}(q+1)^{1/q},\tfrac{1}{3}(p+1)^{1/p}(q+1)^{1/q}\right\} \quad \text{(Petschke [37])}.$$

Other results may also be obtained from Remark 5.4(iii); in particular, we get

$$H^+_{2,2,\alpha}=\frac{\sqrt{(\alpha+1)(\alpha+3)}}{\alpha+2}.$$

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