## Contents of Volume 116, Number 2





The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

Detailed information for authors is given on the inside back cover. Manuscripts and correspondence concerning editorial work should be addressed to

STUDIA MATHEMATICA

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-293997

Correspondence concerning subscription, exchange and back numbers should be addressed to

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-293997



Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by

PRINTED IN POLAND

ISSN 0039-3223



# The local versions of $H^p(\mathbb{R}^n)$ spaces at the origin

by

SHANZHEN LU and DACHUN YANG (Beijing)

**Abstract.** Let $0 < p \le 1 < q < \infty$ and $\alpha = n(1/p - 1/q)$. We introduce some new Hardy spaces $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ which are the local versions of $H^p(\mathbb{R}^n)$ spaces at the origin. Characterizations of these spaces in terms of atomic and molecular decompositions are established, together with their $\varphi$-transform characterizations in M. Frazier and B. Jawerth's sense. We also prove an interpolation theorem for operators on $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and discuss the $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$-boundedness of Calderón–Zygmund operators. Similar results can also be obtained for the non-homogeneous Hardy spaces $HK_q^{\alpha,p}(\mathbb{R}^n)$.

**0. Introduction.** The Herz spaces turn out to be very useful in the study of the sharpness of multiplier theorems on $H^p$ spaces (see [1]). The purpose of this paper is mainly twofold. First, we shall give a decomposition characterization of Herz spaces in terms of blocks, where the definition of a block is a modification of the original one due to M. H. Taibleson and G. Weiss [12]. Next, by means of the grand maximal functions of C. Fefferman and E. M. Stein [3], we define some Hardy spaces associated with the Herz spaces and establish their characterizations in terms of atomic and molecular decompositions. The last fact shows that the Hardy spaces associated with the Herz spaces are just the local versions of the standard Hardy spaces $H^p(\mathbb{R}^n)$ $(0 < p \le 1)$ at the origin.

In §1 we introduce the definition of a central $(p,q)$-block and formulate a characterization of the homogeneous Herz spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ in terms of decompositions into central $(p,q)$-blocks, where $0 < p \le 1 < q < \infty$ and $\alpha = n(1/p - 1/q)$.

In §2, using the grand maximal function $G(f)$, we define the Hardy spaces $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ associated with $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ as follows:

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) : G(f) \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)\}.$$

In addition, we establish their characterizations in terms of decompositions into central $(p,q)$-atoms and central $(p,q,s,\varepsilon)$-molecules. Here, the notions

1991 *Mathematics Subject Classification*: 42B30, 42C15, 42B20.
The research was supported by the NNSF of China.

of central $(p,q)$-atoms and central $(p,q,s,\varepsilon)$-molecules are modifications of the standard $(p,q)$-atoms and $(p,q,s,\varepsilon)$-molecules.

In §3, using the atomic and molecular theory of the spaces $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ established in §2, we give their $\varphi$-transform characterizations in M. Frazier and B. Jawerth's sense (see [5] and [6]).

In the last section, §4, as other applications of §2, we prove an interpolation theorem for operators on $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$. We also study the boundedness of Calderón–Zygmund operators on $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$. More interesting applications will be discussed in another paper.

It should be pointed out that we have similar results for non-homogeneous Herz spaces $K_q^{\alpha,p}(\mathbb{R}^n)$.

**1. Herz spaces.** Let $0<p\le 1<q<\infty$ and $\alpha=n(1/p-1/q)$.

DEFINITION 1.1. (a) The *homogeneous Herz space* is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n)=\{f\in L^q_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\}):\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}<\infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}:=\Big\{\sum_{k=-\infty}^{\infty}\Big(\int_{A_k}|f(x)|^q\,dx\Big)^{p/q}2^{k\alpha p}\Big\}^{1/p}$$

and $A_k:=\{x\in\mathbb{R}^n:2^{k-1}<|x|\le 2^k\}$.

(b) The *non-homogeneous Herz space* is defined by

$$K_q^{\alpha,p}(\mathbb{R}^n)=L^q(\mathbb{R}^n)\cap\dot{K}_q^{\alpha,p}(\mathbb{R}^n).$$

Moreover,

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}:=\|f\|_{L^q(\mathbb{R}^n)}+\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)},$$

where $\|f\|_{L^q(\mathbb{R}^n)}:=(\int_{\mathbb{R}^n}|f(x)|^q\,dx)^{1/q}$.

DEFINITION 1.2. A function $b(x)$ on $\mathbb{R}^n$ is called a *central* $(p,q)$-*block* if it satisfies

(i) $\operatorname{supp} b\subset B(0,r)$, $r>0$,
(ii) $\|b\|_{L^q(\mathbb{R}^n)}\le|B(0,r)|^{1/q-1/p}$,

where $B(0,r):=\{x\in\mathbb{R}^n:|x|<r\}$.

DEFINITION 1.2′. A function $b(x)$ on $\mathbb{R}^n$ is called a *central* $(p,q)$-*block of restricted type* if it satisfies

(i) $\operatorname{supp} b\subset B(0,r)$, $r\ge 1$,
(ii) $\|b\|_{L^q(\mathbb{R}^n)}\le|B(0,r)|^{1/q-1/p}$.

Note that the definition of a $(1,q)$-block was first introduced by M. H. Taibleson and G. Weiss in the study of a.e. convergence of Fourier series (see [12]).



We now can formulate a decomposition theorem for Herz spaces in terms of central $(p,q)$-blocks.

THEOREM 1.1. *Let $0<p\le 1<q<\infty$ and $\alpha=n(1/p-1/q)$. Then the following two statements are equivalent.*

(1) $f\in\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.
(2) *$f(x)$ can be represented as*

$$f(x)=\sum_{k=1}^{\infty}\lambda_k b_k(x),$$

*where each $b_k$ is a central $(p,q)$-block and $\sum_k|\lambda_k|^p<\infty$. Moreover, $\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}\sim\inf(\sum_k|\lambda_k|^p)^{1/p}$, where the infimum is taken over all block decompositions of $f$.*

For non-homogeneous Herz spaces, we have a similar result.

THEOREM 1.1′. *Let $0<p\le 1<q<\infty$ and $\alpha=n(1/p-1/q)$. Then the following two statements are equivalent.*

(1) $f\in K_q^{\alpha,p}(\mathbb{R}^n)$.
(2) *$f(x)$ can be represented as*

$$f(x)=\sum_{k=1}^{\infty}\lambda_k b_k(x),$$

*where each $b_k$ is a central $(p,q)$-block of restricted type and $\sum_k|\lambda_k|^p<\infty$. Moreover, $\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}\sim\inf(\sum_k|\lambda_k|^p)^{1/p}$, where the infimum is taken over all block decompositions of $f$.*

We only prove Theorem 1.1. The proof of Theorem 1.1′ is similar.

Proof of Theorem 1.1. Suppose $f\in\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$; without loss of generality, we can assume $f(0)=0$. For $k\in\mathbb{Z}$, set $A_k=\{x\in\mathbb{R}^n:2^{k-1}<|x|\le 2^k\}$ and $\chi_k(x)=\chi_{A_k}(x)$, where $\chi_{A_k}(x)$ is the indicator function of $A_k$. Write

$$\begin{aligned}f(x)&=\sum_{k\in\mathbb{Z}}f(x)\chi_k(x)\\&=\sum_{k\in\mathbb{Z}}|B(0,2^k)|^{1/p-1/q}\|f\chi_k\|_{L^q(\mathbb{R}^n)}\frac{f(x)\chi_k(x)}{|B(0,2^k)|^{1/p-1/q}\|f\chi_k\|_{L^q(\mathbb{R}^n)}}\\&=:\sum_{k\in\mathbb{Z}}\lambda_k b_k(x),\end{aligned}$$

where $\lambda_k=|B(0,2^k)|^{1/p-1/q}\|f\chi_k\|_{L^q(\mathbb{R}^n)}$ and

$$b_k(x)=\frac{f(x)\chi_k(x)}{|B(0,2^k)|^{1/p-1/q}\|f\chi_k\|_{L^q(\mathbb{R}^n)}}.$$

It is easy to see that $\operatorname{supp} b_k \subset B(0,2^k)$ and $\|b_k\|_{L^q(\mathbb{R}^n)} \le |B(0,2^k)|^{1/q-1/p}$. Thus, each $b_k(x)$ is a central $(p,q)$-block, and

$$\Big(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\Big)^{1/p} = c\Big(\sum_{k=-\infty}^{\infty}2^{kn(1/p-1/q)p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)}\Big)^{1/p} = c\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} < \infty.$$

Conversely, suppose $f(x)=\sum_{k=1}^{\infty}\lambda_k b_k(x)$, where each $b_k(x)$ is a central $(p,q)$-block and $\sum_{k=1}^{\infty}|\lambda_k|^p<\infty$. We want to prove $f\in \dot K_q^{\alpha,p}(\mathbb{R}^n)$. Suppose $\operatorname{supp} b_k \subset B(0,r)$ and $2^{N_k} < r \le 2^{N_k+1}$ for some $N_k\in\mathbb{Z}$. For $j\in\mathbb{Z}$, we have

$$\|f\chi_j\|_{L^q(\mathbb{R}^n)} = \Big\|\Big(\sum_{k\in\mathbb{Z}}\lambda_k b_k\Big)\chi_j\Big\|_{L^q(\mathbb{R}^n)} \le \sum_{k\in\mathbb{Z}}|\lambda_k|\cdot\|b_k\chi_j\|_{L^q(\mathbb{R}^n)}.$$

Therefore,

$$\begin{aligned}
\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} &= \Big\{\sum_{j=-\infty}^{\infty}2^{jn(1/p-1/q)p}\|f\chi_j\|^p_{L^q(\mathbb{R}^n)}\Big\}^{1/p}\\
&\le \Big\{\sum_{j=-\infty}^{\infty}2^{jn(1-p/q)}\Big(\sum_{k=-\infty}^{\infty}|\lambda_k|\cdot\|b_k\chi_j\|_{L^q(\mathbb{R}^n)}\Big)^p\Big\}^{1/p}\\
&\le \Big\{\sum_{j=-\infty}^{\infty}2^{jn(1-p/q)}\Big(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\|b_k\chi_j\|^p_{L^q(\mathbb{R}^n)}\Big)\Big\}^{1/p}\\
&= \Big\{\sum_{k=-\infty}^{\infty}|\lambda_k|^p\Big(\sum_{j=-\infty}^{N_k+1}2^{jn(1-p/q)}\|b_k\chi_j\|^p_{L^q(\mathbb{R}^n)}\Big)\Big\}^{1/p}\\
&\le c\Big\{\sum_{k=-\infty}^{\infty}|\lambda_k|^p\|b_k\|^p_{L^q(\mathbb{R}^n)}2^{N_k n(1-p/q)}\Big\}^{1/p}\\
&\le c\Big\{\sum_{k=-\infty}^{\infty}|\lambda_k|^p r^{n(1/q-1/p)p}r^{n(1-p/q)}\Big\}^{1/p}\\
&= c\Big\{\sum_{k=-\infty}^{\infty}|\lambda_k|^p\Big\}^{1/p}<\infty,
\end{aligned}$$

that is, $f\in \dot K_q^{\alpha,p}(\mathbb{R}^n)$. This finishes the proof of Theorem 1.1.

**2. Hardy space associated with Herz space.** As in [3], let $G(f)(x)$ be the grand maximal function of $f(x)$ defined by

$$G(f)(x)=\sup_{\phi\in\mathcal{A}_N}|\phi^*_\nabla(f)(x)|,$$



where $\mathcal{A}_N=\{\phi\in\mathcal{S}(\mathbb{R}^n):\sup_{|\alpha|,|\beta|\le N}|x^\alpha D^\beta\phi(x)|\le 1\}$ and $N>n+1$, $\phi_t(x)=t^{-n}\phi(x/t)$ for $t>0$ and $\phi^*_\nabla(f)(x)=\sup_{|x-y|<t}|(f*\phi_t)(y)|$.

DEFINITION 2.1. Let $0<p\le 1<q<\infty$ and $\alpha=n(1/p-1/q)$. The *Hardy space* $H\dot K_q^{\alpha,p}(\mathbb{R}^n)$ *associated with* $\dot K_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$H\dot K_q^{\alpha,p}(\mathbb{R}^n)=\{f\in\mathcal{S}'(\mathbb{R}^n):G(f)\in\dot K_q^{\alpha,p}(\mathbb{R}^n)\}.$$

Moreover, we define $\|f\|_{H\dot K_q^{\alpha,p}(\mathbb{R}^n)}=\|G(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}$.

For the non-homogeneous case, we have

DEFINITION 2.1′. Let $0<p\le 1<q<\infty$ and $\alpha=n(1/p-1/q)$. The *Hardy space* $HK_q^{\alpha,p}(\mathbb{R}^n)$ *associated with* $K_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$HK_q^{\alpha,p}(\mathbb{R}^n)=\{f\in\mathcal{S}'(\mathbb{R}^n):G(f)\in K_q^{\alpha,p}(\mathbb{R}^n)\}.$$

Moreover, we define $\|f\|_{HK_q^{\alpha,p}(\mathbb{R}^n)}=\|G(f)\|_{K_q^{\alpha,p}(\mathbb{R}^n)}$.

Remark 2.1. Let $1<q<\infty$. When $p=1$ and $\alpha=n(1-1/q)$, the spaces $HK_q^{\alpha,p}(\mathbb{R}^n)$ and $H\dot K_q^{\alpha,p}(\mathbb{R}^n)$ are respectively the space $HA^q(\mathbb{R}^n)$ introduced by Chen–Lau in [2] and García-Cuerva in [7] and the space $HK_q(\mathbb{R}^n)$ introduced by the authors in [10] (see also [9]). Moreover, in [9] the authors show that

$$HK_q^{\alpha,1}(\mathbb{R}^n)=H\dot K_q^{\alpha,1}(\mathbb{R}^n)\cap L^q(\mathbb{R}^n).$$

Now, we turn to the atomic decomposition of the space $H\dot K_q^{\alpha,p}(\mathbb{R}^n)$. The following definition is a modification of the definition of the standard $(p,q)$-atom.

DEFINITION 2.2. Let $0<p\le 1<q<\infty$ and fix a non-negative integer $s\ge[n(1/p-1)]$. A function $a(x)$ on $\mathbb{R}^n$ is called a *central* $(p,q)$-*atom* if it satisfies

(i) $\operatorname{supp} a\subset B(0,r)$, $r>0$,
(ii) $\|a\|_{L^q(\mathbb{R}^n)}\le|B(0,r)|^{1/q-1/p}$,
(iii) $\int_{\mathbb{R}^n}a(x)x^\alpha\,dx=0$, $|\alpha|\le s$.

DEFINITION 2.2′. Let $p,q,s$ be as in Definition 2.2. A function $a(x)$ on $\mathbb{R}^n$ is called a *central* $(p,q)$-*atom of restricted type* if it satisfies

(i) $\operatorname{supp} a\subset B(0,r)$, $r\ge 1$,
(ii) $\|a\|_{L^q(\mathbb{R}^n)}\le|B(0,r)|^{1/q-1/p}$,
(iii) $\int_{\mathbb{R}^n}a(x)x^\alpha\,dx=0$, $|\alpha|\le s$.

Now, we can state our atomic decomposition theorems for Hardy spaces.

THEOREM 2.1. *Let $0<p\le 1<q<\infty$ and $\alpha=n(1/p-1/q)$. Then $f\in H\dot K_q^{\alpha,p}(\mathbb{R}^n)$ if and only if $f(x)$ can be represented as $f(x)=$*

*$\sum_{k=1}^{\infty}\lambda_k a_k(x)$, in distributional sense, where each $a_k$ is a central $(p,q)$-atom, and $\sum_{k=1}^{\infty}|\lambda_k|^p<\infty$. Moreover,*

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}\sim\inf\Big(\sum_{k=1}^{\infty}|\lambda_k|^p\Big)^{1/p},$$

*where the infimum is taken over all atomic decompositions of $f$.*

Proof. We first verify the necessity. Suppose $a(x)$ is a central $(p,q)$-atom. It is enough to verify that $\|G(a)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}\le c$, where $c$ is independent of $a$. Let $\operatorname{supp}a\subset B(0,r)$ and $2^{k_0}<r\le 2^{k_0+1}$ for some $k_0\in\mathbb{Z}$. Write

$$\begin{aligned}\|G(a)\|^p_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}&=\sum_{k=-\infty}^{\infty}2^{kn(1/p-1/q)p}\|G(a)\chi_k\|^p_{L^q(\mathbb{R}^n)}\\&=\sum_{k=-\infty}^{k_0+3}\ldots+\sum_{k=k_0+4}^{\infty}\ldots=:I_1+I_2.\end{aligned}$$

For $I_1$, we have

$$I_1\le c\|a\|^p_{L^q(\mathbb{R}^n)}\sum_{k=-\infty}^{k_0+3}2^{kn(1-p/q)}\le cr^{n(1/q-1/p)p}2^{k_0n(1-p/q)}=c<\infty.$$

In order to estimate $I_2$, we need a pointwise estimate of $G(a)(x)$ on $A_k$ for $k\ge k_0+4$. Suppose $\phi\in\mathcal{A}_N$ and

$$\frac{n}{n+m+1}<p\le\frac{n}{n+m}\quad\text{ for some } m\in\mathbb{N}\cup\{0\}.$$

Let $P_m$ be the $m$th order Taylor expansion of $\phi$. Let $x\in A_k$ and $|x-y|<t$. Then we have

$$\begin{aligned}|(a*\phi_t)(y)|&=t^{-n}\Big|\int_{\mathbb{R}^n}a(z)\Big(\phi\Big(\frac{y-z}{t}\Big)-P_m\Big(\frac{y}{t}\Big)\Big)\,dz\Big|\\&\le t^n\int_{\mathbb{R}^n}|a(z)|\Big|\frac{z}{t}\Big|^{m+1}\frac{c}{(1+|y-\theta z|/t)^{n+m+1}}\,dz\\&\le c\int_{\mathbb{R}^n}|a(z)|\cdot|z|^{m+1}\frac{1}{(t+|y-\theta z|)^{n+m+1}}\,dz,\qquad\theta\in(0,1).\end{aligned}$$

Since $x\in A_k$ for $k\ge k_0+4$, we have $|x|\ge 2\cdot 2^{k_0+1}$. From $|x-y|<t$ and $|z|<2^{k_0+1}$, we deduce that

$$t+|y-\theta z|\ge|x-y|+|y-\theta z|\ge|x|-|z|\ge|x|/2.$$

Thus,

$$|(a*\phi_t)(y)|\le\frac{c}{|x|^{n+m+1}}r^{m+1}\int_{\mathbb{R}^n}|a(z)|\,dz\le c\frac{2^{k_0(m+n+1-n/p)}}{|x|^{n+m+1}}.$$



Therefore, we have

$$G(a)(x)\le c\frac{2^{k_0(m+n+1-n/p)}}{|x|^{n+m+1}}.$$

From this, we deduce that

$$I_2\le c2^{k_0(m+n+1-n/p)p}\sum_{k=k_0+4}^{\infty}2^{kn(1-p/q)}\frac{2^{knp/q}}{2^{k(m+n+1)p}}=c<\infty,$$

where $c$ is independent of $a$. This finishes the proof of the necessity.

Now, we turn to the proof of the sufficiency. Let $f\in\mathcal{S}'(\mathbb{R}^n)$; we consider its regularization. Suppose $\varphi\in C^\infty(\mathbb{R}^n)$, $\varphi\ge 0$, $\int_{\mathbb{R}^n}\varphi(x)\,dx=1$ and $\operatorname{supp}\varphi\subset\{x\in\mathbb{R}^n:|x|\le 1\}$. Set $\varphi_{(i)}(x)=2^{in}\varphi(2^ix)$ and $f^{(i)}(x):=f*\varphi_{(i)}(x)$ for $i\in\mathbb{N}$ and $f\in\mathcal{S}'(\mathbb{R}^n)$. Then the $C^\infty$-function $f^{(i)}(x)$ is said to be the *regularization* of $f\in\mathcal{S}'(\mathbb{R}^n)$ realized by $\varphi_{(i)}(x)$. It is well known that $\lim_{i\to\infty}f^{(i)}=f$ in distributional sense. In addition, let $\psi$ be a radial smooth function with $\operatorname{supp}\psi\subset\{x:1/2-\varepsilon\le|x|\le 1+\varepsilon\}$, $0<\varepsilon<1/2$, and $\psi(x)=1$ if $1/2\le|x|\le 1$. Define $\psi_k(x)=\psi(2^{-k}x)$, $k\in\mathbb{Z}$. It is easy to see that

$$\operatorname{supp}\psi_k\subset\widetilde{A}_k=\{x:2^{k+1}-2^k\varepsilon\le|x|\le 2^k+2^k\varepsilon\}$$

and $\psi_k(x)=1$ if $x\in A_k=\{x:2^{k-1}<|x|\le 2^k\}$. Evidently, $1\le\sum_{k\in\mathbb{Z}}\psi_k(x)\le 2$ for each $x\ne 0$. Let

$$\Phi_k(x)=\begin{cases}\psi_k(x)/(\sum_{j\in\mathbb{Z}}\psi_j(x)), & x\ne 0,\\ 0, & x=0.\end{cases}$$

Then $\sum_k\Phi_k(x)=1$, $x\ne 0$. We denote by $\mathcal{P}_m$ the class of all real polynomials of degree $m$. Let $P_k^{(i)}(x)=P_{\widetilde{A}_k}(f^{(i)}\Phi_k)(x)\chi_{\widetilde{A}_k}(x)\in\mathcal{P}_m$ be the unique polynomial satisfying

$$\int_{\mathbb{R}^n}\{f^{(i)}(x)\Phi_k(x)-P_k^{(i)}(x)\}x^\alpha\,dx=0,\qquad|\alpha|\le[n(1/p-1)]=:m.$$

Write

$$f^{(i)}(x)=\sum_k[f^{(i)}(x)\Phi_k(x)-P_k^{(i)}(x)]+\sum_kP_k^{(i)}(x)=:\Sigma_1^{(i)}+\Sigma_2^{(i)}.$$

For $\Sigma_1^{(i)}$, set $g_k^{(i)}(x)=f^{(i)}(x)\Phi_k(x)-P_k^{(i)}(x)$ and

$$a_k^{(i)}(x)=\frac{g_k^{(i)}(x)}{c_1|B(0,2^{k+1})|^{1/p-1/q}\sum_{j=k-1}^{k+1}\|G(f)\chi_j\|_{L^q(\mathbb{R}^n)}},$$

where $c_1$ is a constant to be determined later. Then $\operatorname{supp}a_k^{(i)}\subset\widetilde{A}_k\subset$

$B(0,2^{k+1})$, and

$$\Sigma_1^{(i)} := \sum_{k=-\infty}^{\infty} \lambda_k a_k^{(i)}(x),$$

where $\lambda_k = c2^{kn(1/p-1/q)} \sum_{j=k-1}^{k+1} \|G(f)\chi_j\|_{L^q(\mathbb{R}^n)}$.

Next, we estimate $\|g_k^{(i)}\|_{L^q(\mathbb{R}^n)}$. To do this, let $\{\varphi_l^k : |l| \le m\}$ be the orthogonal polynomials restricted to $\widetilde{A}_k$ with respect to the weight $1/|\widetilde{A}_k|$, which are obtained from $\{x^\alpha : |\alpha| \le m\}$ by Gram–Schmidt's method, that is,

$$\langle \varphi_\nu^k, \varphi_\mu^k\rangle = \frac{1}{|\widetilde{A}_k|} \int_{\widetilde{A}_k} \varphi_\nu^k(x)\varphi_\mu^k(x)\,dx = \delta_{\nu\mu}.$$

It is easy to see that for $x \in \widetilde{A}_k$, we have

$$P_k^{(i)}(x) = \sum_{|l|\le m} \langle f^{(i)}\Phi_k, \varphi_l^k\rangle \varphi_l^k(x).$$

On the other hand, from $\frac{1}{|\widetilde{A}_k|}\int_{\widetilde{A}_k} \varphi_\nu^k(x)\varphi_\mu^k(x)\,dx = \delta_{\nu\mu}$ we deduce that

$$\frac{1}{|\widetilde{A}_1|} \int_{\widetilde{A}_1} \varphi_\nu^k(2^{k-1}y)\varphi_\mu^k(2^{k-1}y)\,dy = \delta_{\nu\mu}.$$

That is, $\varphi_\nu^k(2^{k-1}y) = \varphi_\nu^1(y)$. In other words, $\varphi_\nu^k(x) = \varphi_\nu^1(2^{-(k-1)}x)$ for $x \in \widetilde{A}_k$. Thus, $|\varphi_\nu^k(x)| \le c$, and for $x \in \widetilde{A}_k$,

$$|P_k^{(i)}(x)| \le \frac{c}{|\widetilde{A}_k|} \int_{\widetilde{A}_k} |f^{(i)}(x)\Phi_k(x)|\,dx.$$

Therefore,

$$\begin{aligned}
\|g_k^{(i)}\|_{L^q(\mathbb{R}^n)} &\le \|f^{(i)}\Phi_k\|_{L^q(\mathbb{R}^n)} + \Big(\int_{\widetilde{A}_k} |P_k^{(i)}(x)|^q\,dx\Big)^{1/q}\\
&\le \|f^{(i)}\Phi_k\|_{L^q(\mathbb{R}^n)} + \frac{c}{|\widetilde{A}_k|}\Big(\int_{\widetilde{A}_k} |f^{(i)}\Phi_k|\,dx\Big)|\widetilde{A}_k|^{1/q}\\
&\le c\|f^{(i)}\Phi_k\|_{L^q(\mathbb{R}^n)} \le c_1 \sum_{j=k-1}^{k+1} \|G(f)\chi_j\|_{L^q(\mathbb{R}^n)}.
\end{aligned}$$

From this, we obtain



$$\|a_k^{(i)}\|_{L^q(\mathbb{R}^n)} \le \frac{1}{|B(0,2^{k+1})|^{1/p-1/q}}.$$

Thus, $a_k^{(i)}$ is a central $(p,q)$-atom supported on $B(0,2^{k+1})$. Moreover,

$$\begin{aligned}
\Big(\sum_{k=-\infty}^{\infty} |\lambda_k|^p\Big)^{1/p} &= c\Big\{\sum_{k=-\infty}^{\infty} 2^{kn(1-p/q)}\Big(\sum_{j=k-1}^{k+1} \|G(f)\chi_j\|_{L^q(\mathbb{R}^n)}\Big)^p\Big\}^{1/p}\\
&\le c\Big\{\sum_{k=-\infty}^{\infty} 2^{kn(1-p/q)}\|G(f)\chi_j\|_{L^q(\mathbb{R}^n)}^p\Big\}^{1/p}\\
&= c\|G(f)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)},
\end{aligned}$$

where $c$ is independent of $i$ and $f$.

Now, we decompose $\Sigma_2^{(i)}$. Let $\{\psi_l^k : |l| \le m\}$ be the dual basis of $\{x^\alpha : |\alpha| \le m\}$ restricted to $\widetilde{A}_k$ with respect to the weight $1/|\widetilde{A}_k|$, that is,

$$\langle \psi_l^k, x^\alpha\rangle := \frac{1}{|\widetilde{A}_k|}\int_{\widetilde{A}_k} x^\alpha \psi_l^k(x)\,dx = \delta_{\alpha l}.$$

We can prove that if $\varphi_l^k(x) = \sum_{|\nu|\le m} \beta_{l\nu}^k x^\nu$, then $\psi_l^k(x) = \sum_{|\nu|\le m} \beta_{\nu l}^k \varphi_\nu^k(x)$. In fact, if we let $\psi_l^k(x) = \sum_{|\nu|\le m} c_{\nu l}^k \varphi_\nu^k(x)$, then

$$c_{\nu l}^k = \langle \psi_l^k, \varphi_\nu^k\rangle = \Big\langle \psi_l^k, \sum_{|\alpha|\le m} \beta_{\nu\alpha}^k x^\alpha\Big\rangle = \sum_{|\alpha|\le m} \beta_{\nu\alpha}^k \langle \psi_l^k, x^\alpha\rangle = \beta_{\nu l}^k.$$

Thus, for $x \in \widetilde{A}_k$,

$$\begin{aligned}
P_k^{(i)}(x) &= \sum_{|\nu|\le m} \langle f^{(i)}\Phi_k, \varphi_\nu^k\rangle\varphi_\nu^k(x) = \sum_{|\nu|\le m}\Big\langle f^{(i)}\Phi_k, \sum_{|l|\le m} \beta_{\nu l}^k x^l\Big\rangle\varphi_\nu^k(x)\\
&= \sum_{|l|\le m} \langle f^{(i)}\Phi_k, x^l\rangle \sum_{|\nu|\le m} \beta_{\nu l}^k\varphi_\nu^k(x) = \sum_{|l|\le m}\langle f^{(i)}\Phi_k, x^l\rangle\psi_l^k(x).
\end{aligned}$$

Next, we want to prove that $|\psi_l^k(x)| \le c2^{-k|l|}$ for $x \in \widetilde{A}_k$. In fact, if we set $E = \{x : 1/2 - \varepsilon \le |x| \le 1 + \varepsilon\}$ and $\{e_l : |l| \le m\}$ is the dual basis of $\{x^\alpha : |\alpha| \le m\}$ restricted to $E$ with respect to the weight $1/|E|$, then, from the equality

$$\delta_{l\alpha} = \frac{1}{|\widetilde{A}_k|}\int_{\widetilde{A}_k} \psi_l^k(x)x^\alpha\,dx = \frac{1}{|E|}\int_E 2^{k|\alpha|}\psi_l^k(2^k y)y^\alpha\,dy,$$

we deduce that $e_l(y) = 2^{k|l|}\psi_l^k(2^k y)$. In other words, if $x \in \widetilde{A}_k$, then $\psi_l^k(x) = 2^{-k|l|}e_l(x/2^k)$. Thus, if $x \in \widetilde{A}_k$, then $|\psi_l^k(x)| \le c2^{-k|l|}$. Therefore,

$$\begin{aligned}
\Sigma_2^{(i)} &= \sum_k P_k^{(i)}(x) = \sum_k \sum_{|l|\le m} \langle f^{(i)}\Phi_k, x^l\rangle \psi_l^k(x)\chi_{\widetilde{A}_k}(x)\\
&= \sum_{|l|\le m}\sum_k \Big(\int_{\mathbb{R}^n} f^{(i)}(x)\Phi_k(x)x^l\,dx\Big)\frac{\psi_l^k(x)\chi_{\widetilde{A}_k}(x)}{|\widetilde{A}_k|}\\
&= \sum_{|l|\le m}\sum_k \Big(\sum_{j=-\infty}^{k}\int_{\mathbb{R}^n} f^{(i)}(x)\Phi_j(x)x^l\,dx\Big)\\
&\quad\times\Big(\frac{\psi_l^k(x)\chi_{\widetilde{A}_k}(x)}{|\widetilde{A}_k|}-\frac{\psi_l^{k+1}(x)\chi_{\widetilde{A}_{k+1}}(x)}{|\widetilde{A}_{k+1}|}\Big)\\
&=: \sum_l\sum_k h_{k,l}^{(i)}(x).
\end{aligned}$$

From

$$\int_{\mathbb{R}^n}\Big(\sum_{j=-\infty}^{k}\Phi_j(x)\Big)x^l\,dx = \sum_{j=-\infty}^{k}\int_{\widetilde{A}_j}\Phi_j(x)x^l\,dx \sim c2^{kn}2^{k|l|},$$

we deduce that

$$\Big|\int_{\mathbb{R}^n} f^{(i)}(x)\Big(\sum_{j=-\infty}^{k}\Phi_j(x)\Big)x^l\,dx\Big| \le c2^{kn+k|l|}G(f)(x)\chi_{B(0,2^{k+2})}(x).$$

In addition,

$$\Big|\frac{\psi_l^k(x)\chi_{\widetilde{A}_k}(x)}{|\widetilde{A}_k|}-\frac{\psi_l^{k+1}(x)\chi_{\widetilde{A}_{k+1}}(x)}{|\widetilde{A}_{k+1}|}\Big| \le c2^{-k|l|-kn}\sum_{j=k-1}^{k+2}\chi_j(x).$$

Thus,

$$\begin{aligned}
\sum_k P_k^{(i)}(x) &= \sum_{|l|\le m}\sum_k h_{k,l}^{(i)}\\
&= \sum_{|l|\le m}\sum_k\Big\{c_2\sum_{j=k-1}^{k+2}\|G(f)\chi_k\|_{L^q(\mathbb{R}^n)}\Big\}|B(0,2^{k+2})|^{1/p-1/q}\\
&\quad\times\frac{h_{k,l}^{(i)}}{|B(0,2^{k+2})|^{1/p-1/q}\{c_2\sum_{j=k-1}^{k+2}\|G(f)\chi_k\|_{L^q(\mathbb{R}^n)}\}}\\
&=: \sum_{|l|\le m}\sum_k \alpha_{k,l}a_{k,l}^{(i)}(x),
\end{aligned}$$

where $c_2$ is a constant to be determined later.

Note that $\|h_{k,l}^{(i)}\|_{L^q(\mathbb{R}^n)} \le c_2\sum_{j=k-1}^{k+1}\|G(f)\chi_k\|_{L^q(\mathbb{R}^n)}$. It is easy to see that $a_{k,l}^{(i)}$ is a central $(p,q)$-atom with support in $\widetilde{A}_k\cup\widetilde{A}_{k+1}\subset B(0,2^{k+2})$ and $\alpha_{k,l}=c_2|B(0,2^{k+1})|^{1/p-1/q}\sum_{j=k-1}^{k+2}\|G(f)\chi_k\|_{L^q(\mathbb{R}^n)}$, where $c_2$ is a constant independent of $i,f,k$ and $l$. Moreover,



$$\begin{aligned}
\Big(\sum_{k,l}|\alpha_{k,l}|^p\Big)^{1/p} &\le c\Big\{\sum_{k=-\infty}^{\infty}2^{kn(1/p-1/q)p}\Big(\sum_{j=k-1}^{k+1}\|G(f)\chi_k\|_{L^q(\mathbb{R}^n)}\Big)^p\Big\}^{1/p}\\
&\le c\Big\{\sum_{k=-\infty}^{\infty}2^{kn(1-p/q)}\|G(f)\chi_k\|_{L^q(\mathbb{R}^n)}^p\Big\}^{1/p}\\
&= c\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}<\infty.
\end{aligned}$$

So far, we have proved that

$$f^{(i)}(x)=\sum_{l=-\infty}^{\infty}\lambda_l a_l^{(i)}(x),$$

where $a_l^{(i)}(x)$ is a central $(p,q)$-atom supported on $\widetilde{A}_l\cup\widetilde{A}_{l+1}\subset B(0,2^{l+2})$, $\lambda_l$ is independent of $i$ and

$$\Big(\sum_l|\lambda_l|^p\Big)^{1/p}\le c\|G(f)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}<\infty,$$

where $c$ is independent of $i$ and $f$.

Since

$$\sup_{i\in\mathbb{N}}\|a_0^{(i)}\|_{L^q(\mathbb{R}^n)}\le|B(0,2^2)|^{1/q-1/p},$$

the Banach–Alaoglu theorem implies that there exists a subsequence $\{a_0^{(i_{n_0})}\}$ of $\{a_0^{(i)}\}$ converging in the weak* topology of $L^q(\mathbb{R}^n)$ to some $a_0\in L^q(\mathbb{R}^n)$. It is easy to verify that $a_0$ is a central $(p,q)$-atom supported on $B(0,2^2)$. Next, since

$$\sup_{i_{n_0}\in\mathbb{N}}\|a_1^{(i_{n_0})}\|_{L^q(\mathbb{R}^n)}\le|B(0,2^3)|^{1/q-1/p},$$

another application of the Banach–Alaoglu theorem yields a subsequence $\{a_1^{(i_{n_1})}\}$ of $\{a_1^{(i_{n_0})}\}$ which converges weak* in $L^q(\mathbb{R}^n)$ to a central $(p,q)$-atom $a_1$ with support in $B(0,2^3)$. Furthermore,

$$\sup_{i_{n_1}\in\mathbb{N}}\|a_{-1}^{(i_{n_1})}\|_{L^q(\mathbb{R}^n)}\le|B(0,2)|^{1/q-1/p}.$$

Similarly, by the Banach–Alaoglu theorem, we get a subsequence $\{a_{-1}^{(i_{n_{-1}})}\}$ of $\{a_{-1}^{(i_{n_1})}\}$ which converges weak* in $L^q(\mathbb{R}^n)$ to some $a_{-1}\in L^q(\mathbb{R}^n)$, and $a_{-1}$ is a central $(p,q)$-atom supported on $B(0,2)$. Repeating the above procedure for each $l\in\mathbb{Z}$, we can find a subsequence $\{a_l^{(i_{n_l})}\}$ of $\{a_l^{(i)}\}$ converging weak* in $L^q(\mathbb{R}^n)$ to some $a_l \in L^q(\mathbb{R}^n)$ which is a central $(p,q)$-atom supported on $B(0,2^{l+2})$. Using the usual diagonal method we get a subsequence $\{i_\nu\}$ of natural numbers such that for each $l \in \mathbb{Z}$, $\lim_{\nu\to\infty} a_l^{(i_\nu)} = a_l$ in the weak* topology of $L^q(\mathbb{R}^n)$ and therefore in $\mathcal{S}'(\mathbb{R}^n)$.

Next, we shall prove that

$$f = \sum_{l=-\infty}^{\infty} \lambda_l a_l \tag{2.1}$$

holds in $\mathcal{S}'(\mathbb{R}^n)$. To do this, take any $\phi \in \mathcal{S}(\mathbb{R}^n)$; noting that $\operatorname{supp} a_l^{(i_\nu)} \subset \widetilde{A}_l \cup \widetilde{A}_{l+1} \subset A_{l-1} \cup A_l \cup A_{l+1} \cup A_{l+2}$, we have

$$\begin{aligned}
\langle f,\phi\rangle &= \lim_{\nu\to\infty} \langle f^{(i_\nu)},\phi\rangle = \lim_{\nu\to\infty} \int_{\mathbb{R}^n} \sum_{l=-\infty}^{\infty} \lambda_l a_l^{(i_\nu)}(x)\phi(x)\,dx \\
&= \lim_{\nu\to\infty} \sum_{k=-\infty}^{\infty} \int_{A_k} \sum_{l=-\infty}^{\infty} \lambda_l a_l^{(i_\nu)}(x)\phi(x)\,dx \\
&= \lim_{\nu\to\infty} \sum_{k=-\infty}^{\infty} \int_{A_k} \sum_{l=k-2}^{k+1} \lambda_l a_l^{(i_\nu)}(x)\phi(x)\,dx \\
&= \lim_{\nu\to\infty} \sum_{k=-\infty}^{\infty} \sum_{l=k-2}^{k+1} \int_{A_k} \lambda_l a_l^{(i_\nu)}(x)\phi(x)\,dx.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\lim_{\substack{k_1\to\infty\\ k_2\to\infty}} \int_{\mathbb{R}^n} \sum_{l=-k_2}^{k_1} \lambda_l a_l^{(i_\nu)}(x)\phi(x)\,dx \\
&\qquad= \lim_{\substack{k_1\to\infty\\ k_2\to\infty}} \sum_{l=-k_2}^{k_1} \int_{\mathbb{R}^n} \lambda_l a_l^{(i_\nu)}(x)\phi(x)\,dx \\
&\qquad= \lim_{\substack{k_1\to\infty\\ k_2\to\infty}} \sum_{l=-k_2}^{k_1} \sum_{k=l-1}^{l+2} \int_{A_k} \lambda_l a_l^{(i_\nu)}(x)\phi(x)\,dx \\
&\qquad= \lim_{\substack{k_1\to\infty\\ k_2\to\infty}} \sum_{k=-k_2-1}^{k_1+2} \sum_{l=k-2}^{k+1} \int_{A_k} \lambda_l a_l^{(i_\nu)}(x)\phi(x)\,dx \\
&\qquad= \sum_{k=-\infty}^{\infty} \sum_{l=k-2}^{k+1} \int_{A_k} \lambda_l a_l^{(i_\nu)}(x)\phi(x)\,dx.
\end{aligned}$$



Thus,

$$\langle f,\phi\rangle = \lim_{\nu\to\infty} \sum_{l=-\infty}^{\infty} \lambda_l \int_{\mathbb{R}^n} a_l^{(i_\nu)}(x)\phi(x)\,dx.$$

Recall that $m = [n(1/p-1)]$. If $l \le 0$, then we have

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n} a_l^{(i_\nu)}(x)\phi(x)\,dx\Big| &= \Big|\int_{\mathbb{R}^n} a_l^{(i_\nu)}(x)\Big(\phi(x) - \sum_{|\beta|\le m} \frac{D^\beta\phi(0)}{|\beta|!}x^\beta\Big)\,dx\Big| \\
&\le c\int_{\mathbb{R}^n} |a_l^{(i_\nu)}(x)|\cdot|x|^{m+1}\,dx \le c2^{l(m+1)}\int_{\mathbb{R}^n} |a_l^{(i_\nu)}(x)|\,dx \\
&\le c2^{l(m+1)}\|a_l^{(i_\nu)}\|_{L^q(\mathbb{R}^n)} 2^{ln(1-1/q)} \\
&\le c2^{l(m+1-n(1/p-1))} \le c < \infty,
\end{aligned}$$

where $c$ is independent of $l$. If $l > 0$, then,

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n} a_l^{(i_\nu)}(x)\phi(x)\,dx\Big| &= \Big|\int_{\mathbb{R}^n} a_l^{(i_\nu)}\Big(\phi(x) - \sum_{|\beta|\le m-1} \frac{D^\beta\phi(0)}{|\beta|!}x^\beta\Big)\,dx\Big| \\
&\le c\int_{\mathbb{R}^n} |a_l^{(i_\nu)}(x)|\cdot|x|^{m}\,dx \le c2^{lm}\int_{\mathbb{R}^n} |a_l^{(i_\nu)}(x)|\,dx \\
&\le c2^{l[m-n(1/p-1)]} \le c < \infty,
\end{aligned}$$

where $c$ is independent of $l$. Therefore,

$$|\lambda_l|\Big|\int_{\mathbb{R}^n} a_l^{(i_\nu)}(x)\phi(x)\,dx\Big| \le c|\lambda_l|.$$

Note that

$$\sum_{l=-\infty}^{\infty} |\lambda_l| \le \Big(\sum_{l=-\infty}^{\infty} |\lambda_l|^p\Big)^{1/p} \le c\|G(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} < \infty.$$

By Lebesgue's dominated convergence theorem, we get

$$\langle f,\phi\rangle = \sum_{l=-\infty}^{\infty} \lim_{\nu\to\infty} \lambda_l \int_{\mathbb{R}^n} a_l^{(i_\nu)}(x)\phi(x)\,dx = \sum_{l=-\infty}^{\infty} \lambda_l \int_{\mathbb{R}^n} a_l(x)\phi(x)\,dx.$$

This means (2.1) holds in distributional sense, and the proof of Theorem 2.1 is finished.

Similarly, we have

THEOREM 2.1′. *Let $0 < p \le 1 < q < \infty$ and $\alpha = n(1/p - 1/q)$. Then $f \in HK_q^{\alpha,p}(\mathbb{R}^n)$ if and only if $f(x)$ can be represented as $f(x) = \sum_{k=1}^{\infty} \lambda_k a_k(x)$ in distributional sense, where each $a_k$ is a central $(p,q)$-atom of restricted* *type, and* $\sum_{k=1}^{\infty}|\lambda_k|^p<\infty$. *Moreover,*

$$\|f\|_{HK_q^{\alpha,p}(\mathbb{R}^n)}\sim\inf\Big\{\Big(\sum_{k=1}^{\infty}|\lambda_k|^p\Big)^{1/p}\Big\},$$

*where the infimum is taken over all atomic decompositions of* $f$.

Remark 2.2. Let $0<p\le 1<q<\infty$ and $\alpha=n(1/p-1/q)$. Suppose $\varphi\in\mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}\varphi(x)\,dx=1$ or $\varphi$ is the standard Poisson kernel. Recall that $\varphi_t(x)=t^{-n}\varphi(x/t)$ for $t>0$. Then for $f\in\mathcal{S}'(\mathbb{R}^n)$, the following four statements are equivalent.

(1) $f\in H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.
(2) $\varphi_+^*(f)(x)=\sup_{t>0}|(f*\varphi_t)(x)|\in\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.
(3) $\varphi_\nabla^*(f)(x)=\sup_{|x-y|<t}|(f*\varphi_t)(y)|\in\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.
(4) For $N\ge 1$, $\varphi_{\nabla,N}^*(f)(x)=\sup_{|x-y|<Nt}|(f*\varphi_t)(y)|\in\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.

Similar conclusions are true for the spaces $HK_q^{\alpha,p}(\mathbb{R}^n)$. They can be proved by a method similar to the proof of Theorem 2.1 (see [7]).

As in the theory of $H^p(\mathbb{R}^n)$, we can establish molecular decompositions of $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.

Definition 2.3. Let $0<p\le 1<q<\infty$, fix a non-negative integer $s\ge[n(1/p-1)]$, and let $\varepsilon>\max\{s/n,1/p-1\}$, $a=1-1/p+\varepsilon$ and $b=1-1/q+\varepsilon$. A function $M\in L^q(\mathbb{R}^n)$ is called a *central* $(p,q,s,\varepsilon)$-*molecule* if it satisfies

(1) $\mathcal{R}_q(M):=\|M\|_{L^q(\mathbb{R}^n)}^{a/b}\||x|^{nb}M(x)\|_{L^q(\mathbb{R}^n)}^{1-a/b}<\infty$,
(2) $\int_{\mathbb{R}^n}M(x)x^\alpha\,dx=0$, $|\alpha|\le s$.

As in [11], we can prove

Theorem 2.2. *Let* $p,q,s,\varepsilon$ *be as Definition* 2.3. *Then* $f\in H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ *if and only if* $f(x)$ *can be represented as*

$$f(x)\stackrel{\mathcal{S}'}{=}\sum_{k=1}^{\infty}\lambda_kM_k(x),$$

*where each* $M_k$ *is a central* $(p,q,s,\varepsilon)$-*molecule, and* $\sum_{k=1}^{\infty}|\lambda_k|^p<\infty$. *Moreover,*

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}\sim\inf\Big\{\Big(\sum_{k=1}^{\infty}|\lambda_k|^p\Big)^{1/p}\Big\},$$

*where the infimum is taken over all molecular decompositions of* $f$.

For the space $HK_q^{\alpha,p}(\mathbb{R}^n)$, we have a similar molecule decomposition.



Definition 2.3′. Let $0<p\le 1<q<\infty$, fix a non-negative integer $s\ge[n(1/p-1)]$, and let $\varepsilon>\max\{s/n,1/p-1\}$, $a=1-1/p+\varepsilon$ and $b=1-1/q+\varepsilon$. A function $M\in L^q(\mathbb{R}^n)$ is called a *central* $(p,q,s,\varepsilon)$-*molecule of restricted type* if it satisfies

(1) $\|M\|_{L^q(\mathbb{R}^n)}\le 1$,
(2) $\mathcal{R}_q(M):=\|M\|_{L^q(\mathbb{R}^n)}^{a/b}\||x|^{nb}M(x)\|_{L^q(\mathbb{R}^n)}^{1-a/b}<\infty$,
(3) $\int_{\mathbb{R}^n}M(x)x^\alpha\,dx=0$, $|\alpha|\le s$.

Theorem 2.2 is still true if we replace the space $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and the central $(p,q,s,\varepsilon)$-molecules respectively by $HK_q^{\alpha,p}(\mathbb{R}^n)$ and the central $(p,q,s,\varepsilon)$-molecules of restricted type.

**3. The $\varphi$-transform characterizations of $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.** To establish the $\varphi$-transform characterizations of $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, we need to introduce the definition of a variant of a tent space. Let $\nu\in\mathbb{Z}$ and $K\in\mathbb{Z}^n$. Define $Q_{\nu K}=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n:2^\nu x-K\in[0,1)^n\}$ and $\mathcal{D}=\{Q_{\nu K}:\nu\in\mathbb{Z},K\in\mathbb{Z}^n\}$. For a complex numerical series $\beta=\{\beta(Q)\}_{Q\in\mathcal{D}}$, we define

$$s(\beta)(x)=\Big(\sum_{x\in Q\in\mathcal{D}}|\beta(Q)|^2|Q|^{-1}\Big)^{1/2}\quad\text{and}\quad\operatorname{supp}\beta=\bigcup_{\{Q\in\mathcal{D}:\beta(Q)\ne0\}}Q.$$

Definition 3.1. Let $0<p\le 1<q<\infty$, $\alpha=n(1/p-1/q)$ and $\beta=\{\beta(Q)\}_{Q\in\mathcal{D}}$. Then the *tent space associated with* $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$T\dot{K}_q^{\alpha,p}(\mathbb{R}^n)=\{\beta:s(\beta)\in\dot{K}_q^{\alpha,p}(\mathbb{R}^n)\}.$$

In addition, we define $\|\beta\|_{T\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}=\|s(\beta)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$.

Similarly, we can define the space $TK_q^{\alpha,p}(\mathbb{R}^n)$ by replacing $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ by $K_q^{\alpha,p}(\mathbb{R}^n)$ in Definition 3.1.

Let $\varphi,\psi\in\mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp}\widehat{\varphi}\cup\operatorname{supp}\widehat{\psi}\subset\{\xi\in\mathbb{R}^n:1/2\le|\xi|\le 2\}$, $|\widehat{\varphi}(\xi)|,|\widehat{\psi}(\xi)|\ge c>0$ if $3/5\le|\xi|\le 5/3$, and $\sum_{\nu\in\mathbb{Z}}\overline{\widehat{\varphi}(2^\nu\xi)}\widehat{\psi}(2^\nu\xi)=1$ $(\xi\ne0)$. Further, write $\varphi_\nu(x)=2^{\nu n}\varphi(2^\nu x)$, $\psi_\nu(x)=2^{\nu n}\psi(2^\nu x)$, $\varphi_Q(x)=|Q|^{-1/2}\varphi(2^\nu x-K)=|Q|^{1/2}\varphi_\nu(x-x_Q)$ and $\psi_Q(x)=|Q|^{-1/2}\psi(2^\nu x-K)=|Q|^{1/2}\psi_\nu(x-x_Q)$, where $\nu\in\mathbb{Z}$, $K\in\mathbb{Z}^n$ and $x_Q=2^{-\nu}K$. It was proved in [4] that for $f\in\mathcal{S}'(\mathbb{R}^n)$,

$$f(x)=\sum_Q\langle f,\varphi_Q\rangle\psi_Q(x).$$

Now, we can state our result in this section.

THEOREM 3.1. *Let* $0 < p \le 1 < q < \infty$, $\alpha = n(1/p - 1/q)$, $\varphi$, $\psi$ *be as above*, $f \in \mathcal{S}'(\mathbb{R}^n)$, *and* $f(x) = \sum_Q \langle f, \varphi_Q\rangle \psi_Q$. *Then the following four statements are equivalent.*

(1) $f \in H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.

(2) *There is a constant* $c_0 \in (0,1]$ *such that for any* $Q \in \mathcal{D}$, *there exists a dyadic cube* $R(Q) \subset Q$ *with* $R(Q) \in \mathcal{D}$ *and* $|R(Q)| \ge c_0|Q|$ *satisfying*

$$W(f)(x) := \Big(\sum_{Q\in\mathcal{D}} |\langle f, \varphi_Q\rangle|^2 |Q|^{-1} \chi_{R(Q)}(x)\Big)^{1/2} \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n).$$

(3) $$S(f)(x) := \Big(\sum_{Q\in\mathcal{D}} |\langle f, \varphi_Q\rangle|^2 |Q|^{-1} \chi_Q(x)\Big)^{1/2} \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n).$$

(4) $$G(f)(x) := \Big\{\sum_{\nu\in\mathbb{Z}} \Big(\sum_{Q\in\mathcal{D}} |\langle f, \varphi_Q\rangle| \cdot |(\widetilde{\varphi}_\nu * \psi_Q)(x)|\Big)^2\Big\}^{1/2} \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n),$$

*where* $\widetilde{\varphi}_\nu(x) = \overline{\varphi_\nu(-x)}$ *for each* $\nu \in \mathbb{Z}$.

*Moreover, the relevant norms are equivalent.*

Similarly, Theorem 3.1 holds if we replace $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ by $HK_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$ respectively.

In order to prove Theorem 3.1, we must first establish a central $(p,q)$-atom-sequence decomposition characterization of the space $T\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.

DEFINITION 3.2. Let $0 < p \le 1 < q < \infty$ and $\alpha = n(1/p - 1/q)$. If there exists a cube $R$ with center at the origin such that $R \supset \operatorname{supp}\beta$ and

$$\Big\|\Big(\sum_{Q\in\mathcal{D}} |\beta(Q)|^2 |Q|^{-1} \chi_Q(x)\Big)^{1/2}\Big\|_q \le |R|^{1/q-1/p},$$

then $\beta = \{\beta(Q)\}_{Q\in\mathcal{D}}$ is said to be a *central* $(p,q)$-*atom-sequence*, and the smallest cube $R$ as above is called the *base* of $\beta$.

THEOREM 3.2. *Let* $0 < p \le 1 < q < \infty$ *and* $\alpha = n(1/p - 1/q)$. *The following three statements are equivalent.*

(i) $\beta \in T\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.

(ii) *There is a constant* $c_0 \in (0,1]$ *such that for any* $Q \in \mathcal{D}$, *there exists a dyadic cube* $R(Q) \subset Q$ *with* $R(Q) \in \mathcal{D}$ *and* $|R(Q)| \ge c_0|Q|$ *satisfying*

$$\sigma(x) = \Big(\sum_{Q\in\mathcal{D}} |\beta(Q)|^2 |Q|^{-1} \chi_{R(Q)}(x)\Big)^{1/2} \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n).$$

(iii) *There are a constant* $c_1 \ge 1$, *a sequence* $\{\beta_j\}_{j=-\infty}^{\infty}$ *of central* $(p,q)$-*atom-sequences and a sequence* $\{\lambda_j\}_{j=-\infty}^{\infty}$ *of numbers such that*

$$\operatorname{supp}\beta_j \subset c_1 Q_j, \quad \beta = \sum_{j\in\mathbb{Z}} \lambda_j \beta_j \quad \text{and} \quad \Big(\sum_{j\in\mathbb{Z}} |\lambda_j|^p\Big)^{1/p} < \infty,$$

*where* $Q_j = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| \le 2^j,\ i = 1, \ldots, n\}$, $j \in \mathbb{Z}$.

*In addition, the norms* $\|\beta\|_{T\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$, $\|\sigma\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$ *and* $\inf\{(\sum |\lambda_j|^p)^{1/p}\}$ *are mutually equivalent, where the infimum is taken over all central* $(p,q)$-*atom-sequence decompositions of* $\beta$.

Proof. Note that $\sigma(x) \le s(\beta)(x)$. From this, (i) obviously implies (ii).

Now we prove (ii)$\Rightarrow$(iii). For $k \in \mathbb{Z}$, define $\Gamma_k = \{Q \in \mathcal{D} : R(Q) \subset Q_k,\ R(Q) \cap A_k \ne \emptyset\}$. Evidently, $\{\Gamma_k\}_{k=-\infty}^{\infty}$ is a disjoint division of $\mathcal{D}$. From $|R(Q)| \ge c_0|Q|$ and the fact that $R(Q)$ and $Q$ are dyadic cubes, we easily deduce that there is a constant $c_1$ independent of $k$ such that $Q \subset c_1 Q_k$ for each $Q \in \Gamma_k$. Set

$$\lambda_k = c_2 2^{kn(1/p-1/q)} \Big\|\Big(\sum_{Q\in\mathcal{D}} |\beta(Q)|^2 |Q|^{-1} \chi_{R(Q)}(x)\Big)^{1/2} \chi_{A_k}(x)\Big\|_{L^q(\mathbb{R}^n)},$$

where $c_2$ is a constant to be determined. Define $\beta_k = \{\beta_k(Q)\}_{Q\in\mathcal{D}}$ by

$$\beta_k(Q) = \begin{cases} \lambda_k^{-1}\beta(Q), & Q \in \Gamma_k, \\ 0, & \text{otherwise.} \end{cases}$$

Then, obviously, $\beta = \sum_{k=-\infty}^{\infty} \lambda_k \beta_k$, $\operatorname{supp}\beta_k \subset c_1 Q_k$, and $(\sum_{j=-\infty}^{\infty} |\lambda_j|^p)^{1/p} = c_2\|\sigma\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$. In the following, we need to estimate

$$\Big\|\Big(\sum_{Q\in\Gamma_k} |\beta_k(Q)|^2 |Q|^{-1} \chi_Q(x)\Big)^{1/2}\Big\|_{L^q(\mathbb{R}^n)}.$$

We first point out that if $Q \in \Gamma_k$ and $R(Q) \not\subset A_k$, then $R(Q) \in \{Q_{-k,K} : K \in \{-1,0\}^n\}$. Let $\{R_1, \ldots, R_{i_0}\}$ ($i_0 \le 2^n$) denote all the different $R(Q)$'s. Obviously, $\{R_i\}_{i=1}^{i_0}$ are mutually disjoint. We have

$$\begin{aligned}\Big\|\Big(\sum_{Q\in\Gamma_k} |\beta(Q)|^2 |Q|^{-1} \chi_{R(Q)}(x)\Big)^{1/2}\Big\|_{L^q(\mathbb{R}^n)}^q &= \int_{A_k} \Big(\sum_{Q\in\Gamma_k} |\beta(Q)|^2 |Q|^{-1} \chi_{R(Q)}(x)\Big)^{q/2} dx \\ &\quad + \int_{Q_{k-1}} \Big(\sum_{Q\in\Gamma_k} |\beta(Q)|^2 |Q|^{-1} \chi_{R(Q)}(x)\Big)^{q/2} dx.\end{aligned}$$

The second term is

$$\int_{Q_{k-1}} \Big\{ \sum_{i=1}^{i_0} \Big( \sum_{R(Q)=R_i} |\beta(Q)|^2 |Q|^{-1} \Big) \chi_{R_i}(x) \Big\}^{q/2} dx$$
$$= \sum_{i=1}^{i_0} \Big( \sum_{R(Q)=R_i} |\beta(Q)|^2 |Q|^{-1} \Big)^{q/2} |R_i \cap Q_{k-1}|$$
$$= c_3 \sum_{i=1}^{i_0} \Big( \sum_{R(Q)=R_i} |\beta(Q)|^2 |Q|^{-1} \Big)^{q/2} |R_i \cap A_k|$$
$$\le c_3 \int_{A_k} \Big( \sum_{Q\in\Gamma_k} |\beta(Q)|^2 |Q|^{-1} \chi_{R(Q)}(x) \Big)^{q/2} dx.$$

Therefore,

$$\Big\| \Big( \sum_{Q\in\Gamma_k} |\beta(Q)|^2 |Q|^{-1} \chi_{R(Q)}(x) \Big)^{1/2} \Big\|_{L^q(\mathbb{R}^n)}^q$$
$$\le (1+c_3) \int_{A_k} \Big( \sum_{Q\in\Gamma_k} |\beta(Q)|^2 |Q|^{-1} \chi_{R(Q)}(x) \Big)^{q/2} dx,$$

where $c_3$ is a geometric constant. Note that $|R(Q)| \sim |Q|$; using the proposition in [6], we know that there exists a geometric constant $c_4 > 0$ such that

$$\Big\| \Big( \sum_{Q\in\Gamma_k} |\beta(Q)|^2 |Q|^{-1} \chi_Q(x) \Big)^{1/2} \Big\|_{L^q(\mathbb{R}^n)}$$
$$= \lambda_k^{-1} \Big\| \Big( \sum_{Q\in\Gamma_k} |\beta(Q)|^2 |Q|^{-1} \chi_Q(x) \Big)^{1/2} \Big\|_{L^q(\mathbb{R}^n)}$$
$$\le c_4 \lambda_k^{-1} \Big\| \Big( \sum_{Q\in\Gamma_k} |\beta(Q)|^2 |Q|^{-1} \chi_{R(Q)}(x) \Big)^{1/2} \Big\|_{L^q(\mathbb{R}^n)}$$
$$\le c_4 (1+c_3)^{1/q} \lambda_k^{-1} \Big\| \Big( \sum_{Q\in\Gamma_k} |\beta(Q)|^2 |Q|^{-1} \chi_{R(Q)}(x) \Big)^{1/2} \chi_{A_k}(x) \Big\|_{L^q(\mathbb{R}^n)}$$
$$\le |c_1 Q_k|^{1/q-1/p},$$

where we take $c_2 = c_1^{1/p-1/q} c_4 (1+c_3)^{1/q}$. Thus, $\beta_k = \{\beta_k(Q)\}_{Q\in\mathcal{D}}$ is a central $(p,q)$-atom-sequence. This proves that (ii) implies (iii).

We still need to prove (iii) implies (i). Without loss of generality, we can suppose $c_1 = 1$. That is, $\beta_j$ is a central $(p,q)$-atom-sequence with base $Q_j$. We want to prove that $\|s(\beta_j)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \le c < \infty$ with the constant $c$



independent of $\beta_j$. Obviously, $\operatorname{supp} s(\beta_j) \subset Q_j$, and therefore, we have

$$\|s(\beta_j)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} = \Big\{ \sum_{k=-\infty}^{j} 2^{kn(1/p-1/q)p} \Big( \int_{A_k} |s(\beta_j)(x)|^q dx \Big)^{p/q} \Big\}^{1/p}$$
$$\le \|s(\beta_j)\|_{L^q(\mathbb{R}^n)} \Big\{ \sum_{k=-\infty}^{j} 2^{kn(1-p/q)} \Big\}^{1/p}$$
$$\le \frac{1}{|Q_j|^{1/p-1/q}} \cdot 2^{jn(1/p-1/q)} \le c < \infty.$$

Moreover, by the generalized Minkowski inequality, we get

$$s(\beta)(x) = \Big( \sum_{Q\in\mathcal{D}} \Big| \sum_{j=-\infty}^{\infty} \lambda_j \beta_j(Q) \Big|^2 |Q|^{-1} \chi_Q(x) \Big)^{1/2}$$
$$\le \sum_{j=-\infty}^{\infty} |\lambda_j| \Big( \sum_{Q\in\mathcal{D}} |\beta_j(Q)|^2 |Q|^{-1} \chi_Q(x) \Big)^{1/2} = \sum_{j=-\infty}^{\infty} |\lambda_j| s(\beta_j)(x).$$

From this, we deduce that

$$\|s(\beta)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \le \Big\{ \sum_{j=-\infty}^{\infty} |\lambda_j|^p \|s(\beta_j)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}^p \Big\}^{1/p}$$
$$\le c \Big\{ \sum_{j=-\infty}^{\infty} |\lambda_j|^p \Big\}^{1/p} < \infty.$$

That is, (i) holds, and we have proved Theorem 3.2.

Similarly, we have the definition of a central $(p,q)$-atom-sequence of restricted type:

DEFINITION 3.2′. Let $0 < p \le 1 < q < \infty$ and $\alpha = n(1/p - 1/q)$. If there exists a cube $R$ with side length $\ge 2$ and center at the origin such that $R \supset \operatorname{supp} \beta$ and

$$\Big\| \Big( \sum_{Q\in\mathcal{D}} |\beta(Q)|^2 |Q|^{-1} \chi_Q(x) \Big)^{1/2} \Big\|_{L^q(\mathbb{R}^n)} \le |R|^{1/q-1/p},$$

then $\beta = \{\beta(Q)\}_{Q\in\mathcal{D}}$ is said to be a *central* $(p,q)$-*atom-sequence of restricted type*, and the smallest cube $R$ as above is called the *base* of $\beta$.

The conclusion of Theorem 3.2 also holds if we replace the spaces $T\dot K_q^{\alpha,p}(\mathbb{R}^n)$ and $\dot K_q^{\alpha,p}(\mathbb{R}^n)$ and central $(p,q)$-atom-sequences by $TK_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$ and central $(p,q)$-atom-sequences of restricted type.

Now, we turn to the proof of Theorem 3.1.

Proof of Theorem 3.1. The equivalence of (2) and (3) is proved in Theorem 3.2. (1)$\Rightarrow$(3) is essentially Proposition 2.1 of [9]. There, we only gave the proof for $p = 1$, $1 < q < \infty$ and $\alpha = n(1 - 1/q)$. But, it is parallel to generalize Proposition 2.1 of [9] to the case of $0 < p \le 1 < q < \infty$ and $\alpha = n(1/p - 1/q)$. We only need to note that if $0 < p < 1$, then the atom has vanishing high order moments.

We still need to prove (3)$\Rightarrow$(1), (4)$\Rightarrow$(2) and (3)$\Rightarrow$(4). We first show (3) implies (1). Suppose (3) holds and let $s(Q) = \langle f, \varphi_Q \rangle$ and $s = \{s(Q)\}_{Q\in\mathcal{D}}$. Using Theorem 3.2, we know that there exist a constant $c_1 > 0$, a sequence $\{s_j\}_{j=-\infty}^{\infty}$ of central atom-sequences and a sequence $\{\lambda_j\}_{j=-\infty}^{\infty}$ of numbers such that $\operatorname{supp} s_j \subset c_1 Q_j$, $s = \sum_{j\in\mathbb{Z}} \lambda_j s_j$ and $\|S(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \sim (\sum_{j=-\infty}^{\infty} |\lambda_j|^p)^{1/p}$. Without loss of generality, we can assume $c_1 = 1$. Set

$$\mathcal{D}_i = \{Q \in \mathcal{D} : Q \subset Q_i,\ Q \not\subset Q_{i-1}\}.$$

By the lemma of [4], we have

$$f(x) = \sum_Q s(Q)\psi_Q(x) = \sum_{i=-\infty}^{\infty} \lambda_i \Big( \sum_{Q\in\mathcal{D}_i} s_i(Q)\psi_Q(x) \Big).$$

Write $s_i(x) = \sum_{Q\in\mathcal{D}_i} s_i(Q)\psi_Q(x)$. By Theorem 2.2, to prove $f \in H\dot K_q^{\alpha,p}(\mathbb{R}^n)$, we only need to verify that $s_i(x)$ is a central $(p,q,s,\varepsilon)$-molecule, where $s \ge [n(1/p - 1)]$ and $\varepsilon > \max\{s/n, 1/p - 1\}$.

Obviously, we only need to prove that

$$\mathcal{R}_q(s_i) = \|s_i\|_{L^q(\mathbb{R}^n)}^{a/b} \|s_i(x)|x|^{nb}\|_{L^q(\mathbb{R}^n)}^{1-a/b} \le c < \infty,$$

where $a = 1 - 1/p + \varepsilon$, $b = 1 - 1/q + \varepsilon$ and $c$ is independent of $s_i$. Set $\mu = nb$. We first estimate $\|s_i(x)|x|^\mu\|_{L^q(\mathbb{R}^n)}$. In fact,

$$\|s_i(x)|x|^\mu\|_{L^q(\mathbb{R}^n)} = \Big( \int_{3Q_i} |s_i(x)|^q |x|^{\mu q}\,dx \Big)^{1/q} + \Big( \int_{\mathbb{R}^n\setminus 3Q_i} |s_i(x)|^q |x|^{\mu q}\,dx \Big)^{1/q} =: I_1 + I_2.$$

For $I_1$, by the results of [5] or [6], we have

$$\begin{aligned} I_1 &\le c|Q_i|^{\mu/n} \|s_i(x)\|_{L^q(\mathbb{R}^n)} \\ &\le c|Q_i|^{\mu/n} \Big\| \Big( \sum_{Q\in\mathcal{D}} |s_i(Q)|^2 |Q|^{-1} \chi_Q(x) \Big)^{1/2} \Big\|_{L^q(\mathbb{R}^n)} \\ &\le c|Q_i|^{b+1/q-1/p} = c|Q_i|^a. \end{aligned}$$

On the other hand, since $\psi \in \mathcal{S}(\mathbb{R}^n)$, we have $|\psi_Q(x)| \le c2^{\nu n/2}/(1 + |2^\nu x - K|)^L$, where the side length $l(Q)$ of $Q$ is $2^{-\nu}$. In the following, we take $L > n/q + \mu$. If $x \notin 3Q_i$ and $Q \in \mathcal{D}_i$, there exists a geometric constant $c > 0$ such that $|2^\nu x - K| = 2^\nu |x - 2^{-\nu}K| \ge c2^\nu |x|$. Therefore, $|\psi_Q(x)| \le c2^{\nu n/2}/(2^{\nu L}|x|^L)$. Thus,



$$\begin{aligned} I_2 &\le \Big( \int_{\mathbb{R}^n\setminus 3Q_i} \Big| \sum_{\nu=-i}^{\infty} \sum_{l(Q)=2^{-\nu}} s_i(Q)\psi_Q(x) \Big|^q |x|^{\mu q}\,dx \Big)^{1/q} \\ &\le \sum_{\nu=-i}^{\infty} \Big( \int_{\mathbb{R}^n\setminus 3Q_i} \Big( \sum_{l(Q)=2^{-\nu}} |s_i(Q)||\psi_Q(x)| \Big)^q |x|^{\mu q}\,dx \Big)^{1/q} \\ &\le c \sum_{\nu=-i}^{\infty} \Big( \sum_{l(Q)=2^{-\nu}} |s_i(Q)|^q \Big)^{1/q} 2^{\nu(n/2-L)+i(\mu-L+n/q)+(1+i+\nu)n/q'}, \end{aligned}$$

where $1/q + 1/q' = 1$. Since $s_i$ is a central $(p,q)$-atom-sequence, it follows that

$$\Big( \sum_{l(Q)=2^{-\nu}} |s_i(Q)|^q \Big)^{1/q} \le c2^{\nu n(1/q-1/2)} |Q_i|^{1/q-1/p}.$$

Therefore,

$$I_2 \le c|Q_i|^{1/q-1/p+\mu/n-L/n+1} \sum_{\nu=-i}^{\infty} 2^{\nu(n-L)} = c|Q_i|^{\mu/n+1/q-1/p} = c|Q_i|^a.$$

To sum up, we have proved that

$$\|s_i(x)|x|^\mu\|_{L^q(\mathbb{R}^n)} \le c|Q_i|^a.$$

Thus

$$\begin{aligned} \mathcal{R}_q(s_i) &= \|s_i\|_{L^q(\mathbb{R}^n)}^{a/b} \|s_i(x)|x|^\mu\|_{L^q(\mathbb{R}^n)}^{1-a/b} \\ &\le c \Big\| \Big( \sum_{Q\in\mathcal{D}} |s_i(Q)|^2 |Q|^{-1} \chi_Q(x) \Big)^{1/2} \Big\|_{L^q(\mathbb{R}^n)}^{a/b} |Q_i|^{a(1-a/b)} \\ &\le c|Q_i|^{(1/q-1/p)a/b + a(1-a/b)} = c < \infty, \end{aligned}$$

where $c$ is independent of $s_i$. This finishes the proof of (3)$\Rightarrow$(1).

Now we prove (4)$\Rightarrow$(2). Set $s(Q) = \langle f, \varphi_Q \rangle$. Obviously, we have

$$\begin{aligned} G(f)(x) &\ge \Big\{ \sum_{\nu\in\mathbb{Z}} \sum_Q |s(Q)|^2 |(\widetilde\varphi_\nu * \psi_Q)(x)|^2 \Big\}^{1/2} \\ &= \Big\{ \sum_Q |s(Q)|^2 \Big( \sum_{\nu\in\mathbb{Z}} |(\widetilde\varphi_\nu * \psi_Q)(x)|^2 \Big) \Big\}^{1/2}. \end{aligned}$$

Fix $Q \in \mathcal{D}$. Let $l(Q) = 2^{-\mu}$ be the side length of $Q$. Then

$$(\widetilde{\varphi}_\nu * \psi_Q)(x) = \frac{2^{\nu n}}{|Q|^{1/2}} \int_{\mathbb{R}^n} \overline{\varphi}(-2^\nu y)\psi(2^\mu x - K - 2^\mu y)\,dy$$
$$= |Q|^{1/2} \int_{\mathbb{R}^n} \overline{\widehat{\varphi}}\Big(\frac{\xi}{2^\nu}\Big)\widehat{\psi}\Big(\frac{\xi}{2^\mu}\Big) e^{-i(2^\mu x - K)\xi/2^\mu}\,d\xi.$$

Therefore,

$$\sum_{\nu\in\mathbb{Z}} |(\widetilde{\varphi}_\nu * \psi_Q)(x)|^2 = \sum_{\nu=\mu-1}^{\mu+1} |Q| \Big| \int_{\mathbb{R}^n} \overline{\widehat{\varphi}}\Big(\frac{\xi}{2^\nu}\Big)\widehat{\psi}\Big(\frac{\xi}{2^\mu}\Big) e^{-i(2^\mu x - K)\xi/2^\mu}\,d\xi\Big|^2$$
$$= |Q|^{-1} \sum_{\nu=\mu-1}^{\mu+1} \Big| \int_{\mathbb{R}^n} \overline{\widehat{\varphi}}\Big(\frac{2^\mu}{2^\nu}\xi\Big)\widehat{\psi}(\xi) e^{-i(2^\mu x - K)\xi}\,d\xi\Big|^2.$$

Without loss of generality, we can suppose that

$$\sum_{\nu=-1}^{1} \Big| \int_{\mathbb{R}^n} \overline{\widehat{\varphi}}\Big(\frac{\xi}{2^\nu}\Big)\widehat{\psi}(\xi)\,d\xi\Big|^2 \geq c_1 > 0. \tag{3.1}$$

Otherwise, $\int_{\mathbb{R}^n} \overline{\widehat{\varphi}}(\xi)\widehat{\psi}(\xi)\,d\xi=0$. Then $\int_{\mathbb{R}^n} \overline{\widehat{\varphi}}(2^\nu\xi)\widehat{\psi}(2^\nu\xi)\,d\xi=0$ for each $\nu\in\mathbb{Z}$. Therefore $\sum_{\nu\in\mathbb{Z}}\int_{\mathbb{R}^n} \overline{\widehat{\varphi}}(2^\nu\xi)\widehat{\psi}(2^\nu\xi)\,d\xi=0$. That is, $\int_{\mathbb{R}^n}\sum_{\nu\in\mathbb{Z}} \overline{\widehat{\varphi}}(2^\nu\xi)\widehat{\psi}(2^\nu\xi)\,d\xi = 0$. But $\sum_{\nu\in\mathbb{Z}} \overline{\widehat{\varphi}}(2^\nu\xi)\widehat{\psi}(2^\nu\xi)\,d\xi = 1$ for $\xi\neq 0$. This is a contradiction.

From (3.1) and the continuity of $\sum_{\nu=-1}^{1} |\int_{\mathbb{R}^n} e^{-ix\xi}\overline{\widehat{\varphi}}(\xi/2^\nu)\widehat{\psi}(\xi)\,d\xi|^2$ at $x=0$, we deduce that there exist a constant $c_0\in(0,1]$ and $R_0\in\mathcal{D}$ with $R_0\subset[0,1)^n$ such that $|R_0|\geq c_0>0$ and

$$\sum_{\nu=-1}^{1} \Big| \int_{\mathbb{R}^n} e^{-ix\xi}\overline{\widehat{\varphi}}\Big(\frac{\xi}{2^\nu}\Big)\widehat{\psi}(\xi)\,d\xi\Big|^2 \geq c_1\chi_{R_0}(x).$$

Therefore, if for $\mu\in\mathbb{Z}$ and $K\in\mathbb{Z}^n$, we set $Q=\{x: 2^\mu x - K\in[0,1)^n\}$ and $R(Q)=\{x: 2^\mu x - K\in R_0\}$, then $|R(Q)|\geq c_0|Q|$ and

$$\sum_{\nu=\mu-1}^{\mu+1} \Big| \int_{\mathbb{R}^n} \overline{\widehat{\varphi}}\Big(\frac{\xi}{2^{\nu-\mu}}\Big)\widehat{\psi}(\xi)e^{-i(2^\mu x-K)\xi}\,d\xi\Big|^2 \geq c_1\chi_{R_0}(2^\mu x-K) = c_1\chi_{R(Q)}(x).$$

Thus,

$$G(f)(x) \geq c_1\Big\{\sum_{Q\in\mathcal{D}} |s(Q)|^2|Q|^{-1}\chi_{R(Q)}(x)\Big\}^{1/2} = c_1 W(f)(x).$$

That is, (4)⇒(2).

Next, suppose (3) holds. We want to prove that $G(f)\in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$. Set $s(Q)=\langle f,\varphi_Q\rangle$. Using the same notations as in the proof of (3)⇒(1), in



order to prove (3)⇒(4), we only need to show that

$$\|G(s_i)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \Big\|\Big\{\sum_{\nu\in\mathbb{Z}}\Big(\sum_{Q\in\mathcal{D}_i}|s_i(Q)|\cdot|(\widetilde{\varphi}_\nu*\psi_Q)(x)|\Big)^2\Big\}^{1/2}\Big\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$$
$$\leq c < \infty,$$

where $c$ is independent of $i$. Write

$$\|G(s_i)\|^q_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \sum_{k=-\infty}^{\infty} 2^{kn(1/p-1/q)p}\|G(s_i)\chi_{A_k}\|^p_{L^q(\mathbb{R}^n)}$$
$$= \sum_{k=-\infty}^{i+3}\ldots + \sum_{k=i+4}^{\infty}\ldots =: I_1 + I_2.$$

We first estimate $I_1$. Note that each $\psi_Q$ is a smooth molecule for $Q$ up to a constant which is independent of $Q$ (see [5] for the definition). By the proof of Theorem II B in [5], we get

$$I_1 \leq c\|G(s_i)\|^p_{L^q(\mathbb{R}^n)}\sum_{k=-\infty}^{i+3} 2^{kn(1-p/q)}$$
$$\leq c\Big\|\Big\{\sum_{\nu\in\mathbb{Z}}\Big(\sum_{Q\in\mathcal{D}}|s_i(Q)(\widetilde{\varphi}_\nu*\psi_Q)(x)|\Big)^2\Big\}^{1/2}\Big\|^p_{L^q(\mathbb{R}^n)} 2^{in(1-p/q)}$$
$$\leq c\Big\|\Big(\sum_{Q\in\mathcal{D}_i}|s(Q)|^2|Q|^{-1}\chi_Q\Big)^{1/2}\Big\|^p_{L^q(\mathbb{R}^n)} 2^{in(1-p/q)}$$
$$\leq \frac{c}{|Q_i|^{(1/p-1/q)p}}2^{in(1-p/q)} = c < \infty.$$

For $I_2$, we first have the following fact: if $\nu\geq -i-1$, $Q\subset Q_i$, and the side length $l(Q)$ of $Q$ is equal to $2^{-\nu-1}$, $2^{-\nu}$ or $2^{-\nu+1}$, then

$$|(\widetilde{\varphi}_\nu*\psi_Q)(x)| \leq c\frac{2^{\nu n/2}}{2^{\nu L}|x|^L} \tag{3.2}$$

holds for $x\in A_k$, $k\geq i+4$ and $L>n/p$.

We only prove (3.2) for $l(Q)=2^{-\nu}$. The other cases are similar. Note that $\varphi,\psi\in\mathcal{S}(\mathbb{R}^n)$. Therefore, if $l(Q)=2^{-\nu}$, then

$$|\widetilde{\varphi}_\nu(x)| \leq c_L\frac{2^{\nu n}}{(1+2^\nu|x|)^L},\quad |\psi_Q(x)| \leq c_L\frac{2^{\nu n/2}}{(1+|2^\nu x - K|)^L}.$$

When $x\in A_k$, $k\geq i+4$ and $Q\subset Q_i$, there is a geometric constant $c>0$ such that $|2^\nu x - K| = 2^\nu|x-2^{-\nu}K| \geq c2^\nu|x|$. On the other hand, we have

$$|(\widetilde{\varphi}_\nu * \psi_Q)(x)| = \Big|\int_{\mathbb{R}^n} \widetilde{\varphi}_\nu(x-\xi)\psi_Q(\xi)\,d\xi\Big|$$
$$\le c\int_{\mathbb{R}^n} \frac{2^{\nu n/2}}{(1+|2^\nu \xi - K|)^L}\cdot\frac{2^{\nu n}}{(1+|2^\nu x - 2^\nu \xi|)^L}\,d\xi$$
$$\le c2^{\nu n/2}\int_{\mathbb{R}^n} \frac{d\xi}{(1+|\xi-K|)^L(1+|2^\nu x-\xi|)^L}.$$

If $|2^\nu x - K| > 2|\xi - K|$, then
$$1+|2^\nu x-\xi| \ge 1+|2^\nu x - K|/2 \ge c2^\nu|x|.$$
If $|2^\nu x - K| \le 2|\xi - K|$, then
$$1+|\xi-K| \ge |2^\nu x - K|/2 \ge c2^\nu |x|.$$
Therefore, if we choose $L > n/p$, then for $x \in A_k$, $k \ge i+4$, we have
$$|(\widetilde{\varphi}_\nu * \psi_Q)(x)| \le c\frac{2^{\nu n/2}}{2^{\nu L}|x|^L}\int_{|\xi-K|<|2^\nu x-K|/2}\frac{d\xi}{(1+|\xi-K|)^L}$$
$$+ c\frac{2^{\nu n/2}}{2^{\nu L}|x|^L}\int_{|\xi-K|\ge|2^\nu x-K|/2}\frac{d\xi}{(1+|2^\nu x-\xi|)^L}$$
$$\le \frac{c2^{\nu n/2}}{2^{\nu L}|x|^L},$$
that is, (3.2) holds. Using (3.2), we deduce
$$I_2 \le \sum_{k=i+4}^{\infty} 2^{kn(1-p/q)}$$
$$\times\Big\{\int_{A_k}\Big(\sum_{\nu=-i-1}^{\infty}\Big[\sum_{\mu=-\nu-1}^{-\nu+1}\sum_{l(Q)=2^\mu}|s_i(Q)|\cdot|(\widetilde{\varphi}_\nu*\psi_Q)(x)|\Big]^2\Big)^{q/2}dx\Big\}^{p/q}$$
$$\le c\sum_{k=i+4}^{\infty} 2^{kn(1-p/q)}$$
$$\times\Big\{\int_{A_k}\Big(\sum_{\nu=-i-1}^{\infty}\sum_{\mu=-\nu-1}^{-\nu+1}\sum_{l(Q)=2^\mu}|s_i(Q)|\cdot|(\widetilde{\varphi}_\nu*\psi_Q)(x)|\Big)^q dx\Big\}^{p/q}$$
$$\le c\sum_{k=i+4}^{\infty} 2^{kn(1-p/q)}$$
$$\times\sum_{\nu=-i-1}^{\infty}\sum_{\mu=-\nu-1}^{-\nu+1}\Big(\sum_{l(Q)=2^\mu}|s_i(Q)|^q\Big)^{p/q}2^{\nu(n/2-L)p-k(L-n/q)p+(i+\nu)np/q'},$$



where $1/q+1/q'=1$. Because $s_i = \{s_i(Q)\}_{Q\in\mathcal{D}}$ is a central $(p,q)$-atom-sequence with support in $Q_i$, we have
$$\sum_{\mu=-\nu-1}^{-\nu+1}\Big(\sum_{l(Q)=2^\mu}|s_i(Q)|^q\Big)^{p/q} \le c2^{-\nu n(1/2-1/q)p}|Q_i|^{(1/q-1/p)p}.$$
Therefore,
$$I_2 \le c\sum_{k=i+4}^{\infty}\frac{1}{2^{k(L-n)p}}\cdot\frac{1}{|Q_i|^{1-p}}\sum_{\nu=-i-1}^{\infty}2^{\nu(n-L)p} \le c < \infty.$$
That is, $\|G(s_i)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \le c < \infty$, where $c$ is independent of $i$. Thus we have proved (3)⇒(4), which finishes the proof of Theorem 3.1.

**4. Some applications.** Using Theorem 2.1, we can prove an interpolation theorem.

THEOREM 4.1. *Let $0 < p_1 < p_2 \le 1 \le q < \infty$ ($p_2 \ne q$), $0 < r_1 < r_2 \le 1$, and $p_i \le r_i$, $\alpha_i = n(1/p_i - 1/q)$, $\beta_i = n(1/r_i - 1/q)$, $i = 1,2$. If a sublinear operator $T$ maps $H\dot{K}_q^{\alpha_i,p_i}(\mathbb{R}^n)$ to $H\dot{K}_q^{\beta_i,r_i}(\mathbb{R}^n)$ for $i=1,2$, then $T: H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) \to H\dot{K}_q^{\beta,r}(\mathbb{R}^n)$, where*
$$\frac{1}{p} = \frac{t}{p_1}+\frac{1-t}{p_2},\quad \frac{1}{r} = \frac{t}{r_1}+\frac{1-t}{r_2},\quad 0<t<1,$$
$$\alpha = n\Big(\frac{1}{p}-\frac{1}{q}\Big),\quad \beta = n\Big(\frac{1}{r}-\frac{1}{q}\Big).$$

Proof. By Theorem 2.1 and the definition of $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, it suffices to show that if $f$ is a central $(p,q)$-atom, then
$$\|G(Tf)\|_{\dot{K}_q^{\beta,r}(\mathbb{R}^n)} \le c < \infty,$$
where $c$ is independent of $f$ and $G(Tf)$ is the grand maximal function of $Tf$.

Set
$$\theta_0 = \frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{r_1}-\frac{1}{r_2}}.$$
Suppose $\operatorname{supp} f \subset B(0,R)$, $R>0$ and $2^{(l_1-1)n} < |B(0,R)|^{\theta_0} \le 2^{l_1 n}$ for some $l_1 \in \mathbb{Z}$. Moreover, write
$$\|G(Tf)\|^r_{\dot{K}_q^{\beta,r}(\mathbb{R}^n)} = \sum_{l=-\infty}^{\infty}2^{ln(1-r/q)}\|G(Tf)\chi_{A_l}\|^r_{L^q(\mathbb{R}^n)}$$
$$= \sum_{l=-\infty}^{l_1}\ldots+\sum_{l=l_1+1}^{\infty}\ldots =: I_1+I_2.$$

For $I_1$, note that $r < r_2$; from Hölder's inequality, it follows that

$$\begin{aligned} I_1 &\le \Big(\sum_{l=-\infty}^{l_1} 2^{ln}\Big)^{1-r/r_2}\Big(\sum_{l=-\infty}^{l_1} 2^{ln(1-r_2/q)}\|G(Tf)\chi_{A_l}\|^{r_2}_{L^q(\mathbb{R}^n)}\Big)^{r/r_2} \\ &\le c2^{l_1n(1-r/r_2)}\|Tf\|^r_{H\dot K_q^{\beta_2,r_2}(\mathbb{R}^n)} \le c2^{l_1n(1-r/r_2)}\|f\|^r_{H\dot K_q^{\alpha_2,p_2}(\mathbb{R}^n)} \\ &\le c|B(0,R)|^{\theta_0(1-r/r_2)+r(1/p_2-1/p)}. \end{aligned}$$

For $I_2$, since $r_1 < r$, we have

$$\begin{aligned} I_2 &\le 2^{l_1n(1-r/r_1)}\sum_{l=l_1+1}^{\infty} 2^{ln(1-r_1/q)r/r_1}\|G(Tf)\chi_{A_l}\|^{r_1\cdot r/r_1}_{L^q(\mathbb{R}^n)} \\ &\le 2^{l_1n(1-r/r_1)}\|Tf\|^r_{H\dot K_q^{\beta_1,r_1}(\mathbb{R}^n)} \le c2^{l_1n(1-r/r_1)}\|f\|^r_{H\dot K_q^{\alpha_1,p_1}(\mathbb{R}^n)} \\ &\le c|B(0,R)|^{\theta_0(1-r/r_1)+r(1/p_1-1/p)}. \end{aligned}$$

Since $\frac{1}{p} = \frac{t}{p_1} + \frac{1-t}{p_2}$ and $\frac{1}{r} = \frac{t}{r_1} + \frac{1-t}{r_2}$, we have

$$\theta_0 = \frac{\frac{1}{p_1}-\frac{1}{p}}{\frac{1}{r_1}-\frac{1}{r}} = \frac{\frac{1}{p_2}-\frac{1}{p}}{\frac{1}{r_2}-\frac{1}{r}}.$$

Therefore,

$$\theta_0\Big(1-\frac{r}{r_2}\Big) + r\Big(\frac{1}{p_2}-\frac{1}{p}\Big) = \theta_0\Big(1-\frac{r}{r_1}\Big) + r\Big(\frac{1}{p_1}-\frac{1}{p}\Big) = 0.$$

Thus,

$$\|G(Tf)\|^r_{\dot K_q^{\beta,r}(\mathbb{R}^n)} \le c < \infty,$$

where $c$ is independent of $f$. This finishes the proof of Theorem 4.1.

Let us turn to the boundedness of operators on Herz spaces. It was shown in [8] that the Hilbert transform is not a bounded operator on $\dot K_q^{\alpha,p}(\mathbb{R}^n)$, where $\alpha = n(1/p-1/q)$. However, the space $H\dot K_q^{\alpha,p}(\mathbb{R}^n)$ can be applied to yield a substitute result (see [8] for the details). More generally, let us consider the boundedness of Calderón–Zygmund operators on $H\dot K_q^{\alpha,p}(\mathbb{R}^n)$.

Definition 4.1. Suppose $T$ is a linear operator which is continuous from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$, and there are a kernel $K(x,y)$ defined for $x \ne y$ in $\mathbb{R}^n$ and constants $c > 0$ and $0 < \delta \le 1$ such that

$$|K(x,y)-K(x,0)| \le \frac{c|y|^\delta}{|x-y|^{n+\delta}} \quad \text{if } 2|y| < |x|, \tag{4.1}$$

and for $f, g \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$,

$$\langle Tf, g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)f(y)g(x)\,dy\,dx. \tag{4.2}$$



If, moreover, $T$ can be extended to a bounded operator in $L^q(\mathbb{R}^n)$ for some $q \in (1,\infty)$, then $T$ is called a *$\delta$-Calderón–Zygmund operator* in $L^q(\mathbb{R}^n)$.

Using the atom-molecule theory of $H\dot K_q^{\alpha,p}(\mathbb{R}^n)$ from §2, we can prove the following theorem.

Theorem 4.2. *Let $1 < q < \infty$, $0 < \delta \le 1$ and $n/(n+\delta) < p \le 1$. If $T$ is a $\delta$-Calderón–Zygmund operator in $L^q(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} Ta(x)\,dx = 0$ for each central $(p,q)$-atom $a(x)$, then $T$ can be extended to a bounded operator in $H\dot K_q^{\alpha,p}(\mathbb{R}^n)$, where $\alpha = n(1/p-1/q)$.*

Proof. Let $f$ be a central $(p,q)$-atom with support in $B(0,r)$ $(r>0)$. It suffices to show $Tf$ is a central $(p,q,0,\varepsilon)$-molecule for some $\varepsilon > 1/p-1$. Let $a = 1-1/p+\varepsilon$, $b = 1-1/q+\varepsilon$. Obviously, we only need to verify the size condition for molecules, that is,

$$\mathcal{R}_q(Tf) = \|Tf\|^{a/b}_{L^q(\mathbb{R}^n)}\||x|^{nb}(Tf)(x)\|^{1-a/b}_{L^q(\mathbb{R}^n)} \le c < \infty,$$

with $c$ independent of $f$. To do this, we first estimate $\||x|^{nb}(Tf)(x)\|_{L^q(\mathbb{R}^n)}$. In fact, we have

$$\begin{aligned} \int_{|x|\le 2r} |Tf(x)|^q|x|^{nbq}\,dx &\le cr^{nbq}\|Tf\|^q_{L^q(\mathbb{R}^n)} \le cr^{nbq}r^{n(1/q-1/p)q} \\ &= cr^{nq(1-1/p+\varepsilon)}. \end{aligned}$$

On the other hand, if $|x| > 2r$, from (4.1) and $\int_{\mathbb{R}^n} f(y)\,dy = 0$ it follows that

$$\begin{aligned} |Tf(x)| &= \Big|\int_{\mathbb{R}^n} K(x,y)f(y)\,dy\Big| = \Big|\int_{\mathbb{R}^n} (K(x,y)-K(x,0))f(y)\,dy\Big| \\ &\le c\int_{\mathbb{R}^n} \frac{|y|^\delta}{|x-y|^{n+\delta}}|f(y)|\,dy \le cr^{\delta+n(1-1/p)}|x|^{-(n+\delta)}. \end{aligned}$$

Hence, we get

$$\begin{aligned} \int_{|x|>2r} |Tf(x)|^q|x|^{nbq}\,dx &\le cr^{\delta q+n(1-1/p)q}\int_{|x|>2r} |x|^{-(n+\delta)q+nbq}\,dx \\ &\le cr^{nq(1-1/p+\varepsilon)}. \end{aligned}$$

That is,

$$\||x|^{nb}(Tf)(x)\|_{L^q(\mathbb{R}^n)} \le cr^{n(1-1/p+\varepsilon)}.$$

Thus,

$$\begin{aligned} \mathcal{R}_q(Tf) &= \|Tf\|^{a/b}_{L^q(\mathbb{R}^n)}\||x|^{nb}(Tf)(x)\|^{1-a/b}_{L^q(\mathbb{R}^n)} \le c\|f\|^{a/b}_{L^q(\mathbb{R}^n)}r^{na(1-a/b)} \\ &\le cr^{n(1/q-1/p)a/b}r^{na(1-a/b)} = c < \infty, \end{aligned}$$

where $c$ is independent of $f$.

This finishes the proof of Theorem 4.2.

More generally, using the atom-molecule theory of §2, we can get the following theorem, whose proof is similar to that of Theorem 4.2; we omit the details.

THEOREM 4.3. *Let $1 < q < \infty$ and $n/(n+m) < p \le n/(n+m-1)$, where $m \in \mathbb{N}$. Suppose $T$ satisfies the conditions of Definition* 4.1 *with* (4.1) *replaced by the following two conditions*:

$$|(\partial_y^\alpha K)(x,0)| = 0 \quad \text{for } |\alpha| \le m-1, \tag{4.3}$$

$$|(\partial_y^\alpha K)(x,y)| \le c|x-y|^{-n-m} \quad \text{for } |\alpha| = m. \tag{4.4}$$

*In addition, assume that $\int_{\mathbb{R}^n} Ta(x)x^\alpha\,dx = 0$ for each central $(p,q)$-atom $a(x)$ and for $|\alpha| \le m-1$. Then $T$ can be extended to a bounded operator in $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, where $\alpha = n(1/p - 1/q)$.*

We point out that there are results similar to Theorems 4.1–4.3 for the spaces $HK_q^{\alpha,p}(\mathbb{R}^n)$.

**Acknowledgements.** We thank the editors for a number of valuable comments which make the paper readable.

## References

[1] A. Baernstein II and E. T. Sawyer, *Embedding and multiplier theorems for $H^p(\mathbb{R}^n)$*, Mem. Amer. Math. Soc. 318 (1985).

[2] Y. Z. Chen and K. S. Lau, *On some new classes of Hardy spaces*, J. Funct. Anal. 84 (1989), 255–278.

[3] C. Fefferman and E. M. Stein, *$H^p$ spaces of several variables*, Acta Math. 129 (1972), 137–193.

[4] M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. 34 (1985), 777–799.

[5] —, —, *The $\varphi$-transform and applications to distribution spaces*, in: Lecture Notes in Math. 1302, Springer, 1988, 223–246.

[6] —, —, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. 93 (1990), 34–170.

[7] J. García-Cuerva, *Hardy spaces and Beurling algebras*, J. London Math. Soc. (2) 39 (1989), 499–513.

[8] S. Z. Lu and F. Soria, *On the Herz spaces with power weights*, in: Fourier Analysis and Partial Differential Equations, J. García-Cuerva, E. Hernández, F. Soria and J. L. Torrea (eds.), Stud. Adv. Math., CRC Press, Boca Raton, 1995, 227–236.

[9] S. Z. Lu and D. C. Yang, *The Littlewood–Paley function and $\varphi$-transform characterizations of a new Hardy space $HK_2$ associated with the Herz space*, Studia Math. 101 (1992), 285–298.

[10] —, —, *Some new Hardy spaces associated with the Herz spaces and their wavelet characterizations*, J. Beijing Normal Univ. (Natural Sci.) 29 (1993), 10–19.

[11] M. H. Taibleson and G. Weiss, *The molecular characterization of certain Hardy spaces*, Astérisque 77 (1980), 67–149.



[12] M. H. Taibleson and G. Weiss, *Certain function spaces associated with a.e. convergence of Fourier series*, in: Conference in Honor of A. Zygmund, Vol. I, Wadsworth, 1983, 95–113.

DEPARTMENT OF MATHEMATICS
BEIJING NORMAL UNIVERSITY
BEIJING 100875
THE PEOPLE'S REPUBLIC OF CHINA

*Received December 13, 1993*
*Revised version March 22, 1995* (3208)