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On the maximal Fejér operator for double Fourier series of functions in Hardy spaces

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Abstract. We consider the Fejér (or first arithmetic) means of double Fourier series of functions belonging to one of the Hardy spaces $H^{(1,0)}(\mathbb{T}^2)$, $H^{(0,1)}(\mathbb{T}^2)$, or $H^{(1,1)}(\mathbb{T}^2)$. We prove that the maximal Fejér operator is bounded from $H^{(1,0)}(\mathbb{T}^2)$ or $H^{(0,1)}(\mathbb{T}^2)$ into weak-$L^1(\mathbb{T}^2)$, and also bounded from $H^{(1,1)}(\mathbb{T}^2)$ into $L^1(\mathbb{T}^2)$. These results extend those by Jessen, Marcinkiewicz, and Zygmund, which involve the function spaces $L^1\log^+ L(\mathbb{T}^2)$, $L^1(\log^+ L)^2(\mathbb{T}^2)$, and $L^\mu(\mathbb{T}^2)$ with $0<\mu<1$, respectively. We establish analogous results for the maximal conjugate Fejér operators. On closing, we formulate two conjectures.

1. Introduction. Let $f(x,y)$ be a function, periodic in each variable and integrable in Lebesgue's sense on the two-dimensional torus $\mathbb{T}^2 := [-\pi,\pi)\times[-\pi,\pi)$; in symbols: $f\in L^1(\mathbb{T}^2)$. The double Fourier series of such a function f is defined by

$$\sum_{(j,k)\in\mathbb{Z}^2}\sum \widehat{f}(j,k)e^{i(jx+ky)}, \tag{1.1}$$

where

$$\widehat{f}(j,k) := \frac{1}{4\pi^2}\iint_{\mathbb{T}^2} f(u,v)e^{-i(ju+kv)}\,du\,dv.$$

We shall consider the Fejér (or first arithmetic) means $\sigma_{mn}(f)$ of the rectangular partial sums $s_{jk}(f)$ of the Fourier series (1.1) defined by

$$\begin{aligned}\sigma_{mn}(f;x,y) &:= \frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n} s_{jk}(f;x,y) \\ &= \sum_{j=-m}^{m}\sum_{k=-n}^{n}\left(1-\frac{|j|}{m+1}\right)\left(1-\frac{|k|}{n+1}\right)\widehat{f}(j,k)e^{i(jx+ky)}, \quad (m,n)\in\mathbb{N}^2.\end{aligned} \tag{1.2}$$

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We shall be interested in the behavior of the maximal Fejér operator $\sigma_*(f)$ defined by

$$\sigma_*(f;x,y) := \sup_{(m,n)\in\mathbb{N}^2} |\sigma_{mn}(f;x,y)|. \tag{1.3}$$

Jessen, Marcinkiewicz, and Zygmund [5] (see also [9, Vol. 2, p. 308]) proved that for all $0<\mu<1$ we have the following estimates:

$$\|\sigma_*(f)\|_{L^\mu} := \Big\{\int\int_{\mathbb{T}^2} |\sigma_*(f;x,y)|^\mu\,dx\,dy\Big\}^{1/\mu} \le C_\mu + C_\mu \int\int_{\mathbb{T}^2} |f(x,y)|(\log^+|f(x,y)|)\,dx\,dy, \tag{1.4}$$

$$\|\sigma_*(f)\|_{L^1} \le C_1 + C_1 \int\int_{\mathbb{T}^2} |f(x,y)|(\log^+|f(x,y)|)^2\,dx\,dy, \tag{1.5}$$

where by $\log^+|f|$ we mean $\log|f|$ whenever $|f|\ge e$, and 1 otherwise; and the constants C_μ depend only on μ.

For the sake of brevity, denote by $L^1(\log^+ L)^\beta(\mathbb{T}^2)$ the class of measurable functions f such that

$$\int\int_{\mathbb{T}^2} |f(x,y)|(\log^+|f(x,y)|)^\beta\,dx\,dy < \infty.$$

Actually, we shall use these classes only in the cases $\beta = 1$ or 2.

Marcinkiewicz and Zygmund [6] (see also [9, Vol. 2, p. 309]) later deduced that if $f\in L^1\log^+ L(\mathbb{T}^2)$, then the Fejér means $\sigma_{mn}(f;x,y)$ converge in Pringsheim's sense (that is, as $m\to\infty$ and $n\to\infty$ independently of each other) at almost all points $(x,y)\in\mathbb{T}^2$ (cf. Corollary 2 below).

2. Main results. We recall that the conjugate function $\widetilde f^{(1,0)}$ of a function $f\in L^1(\mathbb{T}^2)$ with respect to the first variable is defined by

$$\widetilde f^{(1,0)}(x,y) := (\text{P.V.})\frac{1}{2\pi}\int_{\mathbb{T}} f(x-u,y)\cot(u/2)\,du = -\frac{1}{2\pi}\lim_{\varepsilon\downarrow 0}\int_\varepsilon^\pi [f(x+u,y)-f(x-u,y)]\cot(u/2)\,du.$$

It is known that $\widetilde f^{(1,0)}(x,y)$ exists for almost all $(x,y)\in\mathbb{T}^2$, but $\widetilde f^{(1,0)}\notin L^1(\mathbb{T}^2)$ in general.

This gives rise to the definition of the two-dimensional hybrid Hardy space

$$H^{(1,0)}(\mathbb{T}^2) := \{f\in L^1(\mathbb{T}^2) : \widetilde f^{(1,0)}\in L^1(\mathbb{T}^2)\}$$

endowed with the norm

$$\|f\|_{H^{(1,0)}} := \|f\|_{L^1} + \|\widetilde f^{(1,0)}\|_{L^1}.$$

We note that the dyadic counterpart of this space $H^{(1,0)}$ has been introduced in [8].

It is also well known that if $f\in H^{(1,0)}(\mathbb{T}^2)$, then the double Fourier series of $\widetilde f^{(1,0)}$ coincides with the conjugate series to (1.1) with respect to the first variable, i.e. with the series

$$\sum\sum_{(j,k)\in\mathbb{Z}^2} (-i\,\text{sign}\,j)\widehat f(j,k)e^{i(jx+ky)}. \tag{2.1}$$

Our first main result states that the maximal Fejér operator $\sigma_*(f)$ is bounded from $H^{(1,0)}(\mathbb{T}^2)$ into weak-$L^1(\mathbb{T}^2)$, or briefly, $\sigma_*(f)$ is of type $(H^{(1,0)}$, weak-$L^1)$.

THEOREM 1. *If $f\in H^{(1,0)}(\mathbb{T}^2)$, then*

$$\sup_{\lambda>0}\lambda|\{(x,y)\in\mathbb{T}^2 : \sigma_*(f;x,y)>\lambda\}| \le C\|f\|_{H^{(1,0)}}, \tag{2.2}$$

where $|\cdot|$ means the Lebesgue measure on the plane, and the constant C does not depend on f.

We point out that inequality (2.2) is more general than (1.4), because

$$L^1\log^+ L(\mathbb{T}^2) \subset H^{(1,0)}(\mathbb{T}^2) \tag{2.3}$$

and this inclusion is strict. However, if $f\in L^1(\mathbb{T}^2)$ and $f\ge 0$, then $f\in L^1\log^+L(\mathbb{T}^2)$ also follows from $f\in H^{(1,0)}(\mathbb{T}^2)$. (See, e.g., [3, pp. 84–85].)

The symmetric counterpart of $H^{(1,0)}(\mathbb{T}^2)$ is the other hybrid Hardy space $H^{(0,1)}(\mathbb{T}^2)$ defined by

$$H^{(0,1)}(\mathbb{T}^2) := \{f\in L^1(\mathbb{T}^2) : \widetilde f^{(0,1)}\in L^1(\mathbb{T}^2)\}$$

endowed with the norm

$$\|f\|_{H^{(0,1)}} := \|f\|_{L^1} + \|\widetilde f^{(0,1)}\|_{L^1},$$

where

$$\widetilde f^{(0,1)}(x,y) := (\text{P.V.})\frac{1}{2\pi}\int_{\mathbb{T}} f(x,y-v)\cot(v/2)dv$$

is the conjugate function of $f\in L^1(\mathbb{T}^2)$ with respect to the second variable. In case $f\in H^{(0,1)}(\mathbb{T}^2)$, the double Fourier series of $\widetilde f^{(0,1)}$ coincides with the conjugate series to (1.1) with respect to the second variable, i.e. with the series

$$\sum\sum_{(j,k)\in\mathbb{Z}^2} (-i\,\text{sign}\,k)\widehat f(j,k)e^{i(jx+ky)}. \tag{2.4}$$

The symmetric counterpart of Theorem 1 reads as follows.

COROLLARY 1. *If* $f \in H^{(0,1)}(\mathbb{T}^2)$, *then*

$$\sup_{\lambda>0} \lambda |\{(x,y) \in \mathbb{T}^2 : \sigma_*(f;x,y) > \lambda\}| \le C\|f\|_{H^{(0,1)}},$$

with the same constant C *as in* (2.2).

The pointwise convergence follows from Theorem 1 and Corollary 1.

COROLLARY 2. *If* $f \in H^{(1,0)}(\mathbb{T}^2) \cup H^{(0,1)}(\mathbb{T}^2)$, *then*

$$\lim_{m,n\to\infty} \sigma_{mn}(f;x,y) = f(x,y) \tag{2.5}$$

at almost all points $(x,y) \in \mathbb{T}^2$.

Finally, we introduce the two-dimensional Hardy space $H^{(1,1)}(\mathbb{T}^2)$. To this end, we start with a function $f \in H^{(1,0)}(\mathbb{T}^2) \cap H^{(0,1)}(\mathbb{T}^2)$. Then it makes sense to define the conjugate function $\widetilde{f}^{(1,1)}$ of f with respect to both variables as follows:

$$\widetilde{f}^{(1,1)}(x,y) := (\widetilde{f}^{(1,0)})^{\sim(0,1)}(x,y),$$

which turns out to be equal to the following:

$$\begin{aligned}(\widetilde{f}^{(0,1)})^{\sim(1,0)}(x,y) &= (\mathrm{P.V.})\frac{1}{4\pi^2}\iint_{\mathbb{T}^2} f(x-u,y-v)\cot(u/2)\cot(v/2)\,du\,dv\\ &= \frac{1}{4\pi^2}\lim_{\substack{\delta\downarrow 0\\ \varepsilon\downarrow 0}}\int_\delta^\pi\int_\varepsilon^\pi [f(x+u,y+v)-f(x-u,y+v)\\ &\qquad - f(x+u,y-v)+f(x-u,y-v)]\cot(u/2)\cot(v/2)\,du\,dv\end{aligned}$$

for almost all $(x,y) \in \mathbb{T}^2$. Again, it is known that $\widetilde{f}^{(1,1)}(x,y)$ exists for almost all $(x,y) \in \mathbb{T}^2$, but $\widetilde{f}^{(1,1)} \notin L^1(\mathbb{T}^2)$ in general. Thus, we define the Hardy space $H^{(1,1)}(\mathbb{T}^2)$ as follows:

$$H^{(1,1)}(\mathbb{T}^2) := \{f \in L^1(\mathbb{T}^2) : \widetilde{f}^{(1,0)}, \widetilde{f}^{(0,1)}, \widetilde{f}^{(1,1)} \in L^1(\mathbb{T}^2)\}$$

endowed with the norm

$$\|f\|_{H^{(1,1)}} := \|f\|_{L^1} + \|\widetilde{f}^{(1,0)}\|_{L^1} + \|\widetilde{f}^{(0,1)}\|_{L^1} + \|\widetilde{f}^{(1,1)}\|_{L^1}.$$

It is also known that if $f \in H^{(1,1)}(\mathbb{T}^2)$, then the double Fourier series of $\widetilde{f}^{(1,1)}$ coincides with the conjugate series to (1.1) with respect to both variables, i.e. with the series

$$\sum\sum_{(j,k)\in\mathbb{Z}^2} (-i\,\mathrm{sign}\,j)(-i\,\mathrm{sign}\,k)\widehat{f}(j,k)e^{i(jx+ky)}. \tag{2.6}$$

We encountered the notion of the multi-parameter Hardy space $H^{(1,1)}(\mathbb{T}^2)$ the first time in [4], where we denoted it by $\mathcal{H}^1(\mathbb{T}\times\mathbb{T})$. Also in [4] we introduced the space $J(\mathbb{T}\times\mathbb{T})$, which is identical with $H^{(1,0)}(\mathbb{T}^2) \cap H^{(0,1)}(\mathbb{T}^2)$ in our present notation. We note that the roots of the spaces $H^{(1,0)}(\mathbb{T}^2)$, $H^{(0,1)}(\mathbb{T}^2)$, and $H^{(1,1)}(\mathbb{T}^2)$ in the notation of the present paper actually go back to the papers by G. H. Hardy and J. E. Littlewood, J. Marcinkiewicz and A. Zygmund, and R. Fefferman [2], etc.

Our second main result states that the maximal Fejér operator $\sigma_*(f)$ defined by (1.3) is bounded from $H^{(1,1)}(\mathbb{T}^2)$ into $L^1(\mathbb{T}^2)$, or briefly, $\sigma_*(f)$ is of type $(H^{(1,1)}, L^1)$.

THEOREM 2. *If* $f \in H^{(1,1)}(\mathbb{T}^2)$, *then*

$$\|\sigma_*(f)\|_{L^1} \le C\|f\|_{H^{(1,1)}}, \tag{2.7}$$

where the constant C *does not depend on* f.

We emphasize that inequality (2.7) is more general than (1.5), since the strict inclusion

$$L^1(\log^+ L)^2(\mathbb{T}^2) \subset H^{(1,1)}(\mathbb{T}^2)$$

can be proved by the same sort of argument as the analogous inclusion (2.3). (See, e.g., [3, pp. 84–85].)

3. Auxiliary results. We shall need the corresponding results for single Fourier series. So, we present a concise summary of them.

We remind the reader that the conjugate function $\widetilde{g}$ of a function $g \in L^1(\mathbb{T})$ is defined by

$$\widetilde{g}(x) := (\mathrm{P.V.})\frac{1}{2\pi}\int_{\mathbb{T}} g(x-u)\cot(u/2)\,du,$$

which exists for almost all $x \in \mathbb{T}$. The familiar Hardy space $H^1(\mathbb{T})$ is defined by

$$H^1(\mathbb{T}) := \{g \in L^1(\mathbb{T}) : \widetilde{g} \in L^1(\mathbb{T})\}$$

endowed with the norm

$$\|g\|_{H^1} := \|g\|_{L^1} + \|\widetilde{g}\|_{L^1}.$$

(See, e.g., [1, pp. 372–373, Theorem 6.14].)

We note that in this section, the integrals in the norms are taken over the one-dimensional torus $\mathbb{T} := [-\pi,\pi)$. For instance,

$$\|g\|_{L^1} := \int_{\mathbb{T}} |g(x)|\,dx.$$

If $g \in L^1(\mathbb{T})$, then its single (ordinary) Fourier series is given by

$$\sum_{j\in\mathbb{Z}} \widehat{g}(j)e^{ijx}, \quad \text{where} \quad \widehat{g}(j) := \frac{1}{2\pi}\int_{\mathbb{T}} g(u)e^{-iju}\,du. \tag{3.1}$$

We consider the Fejér means $\sigma_m(g)$ of the partial sums $s_j(g)$ of (3.1) defined by

$$\sigma_m(g;x) := \frac{1}{m+1}\sum_{j=0}^{m} s_j(g;x) = \sum_{j=-m}^{m}\Big(1-\frac{|j|}{m+1}\Big)\widehat{g}(j)e^{ijx}, \qquad m \in \mathbb{N}.$$

In Section 4, we shall rely on the following two results proved in [7].

LEMMA 1. *If $g \in L^1(\mathbb{T})$, then for all $\lambda > 0$ we have*

$$(3.2)\qquad |\{x \in \mathbb{T} : \sup_{m\in\mathbb{N}} |\sigma_m(g;x)| > \lambda\}| \le \frac{C_1}{\lambda}\|g\|_{L^1},$$

where this time $|\cdot|$ means the Lebesgue measure on the real line, and the constant C_1 does not depend on g or λ.

As a matter of fact, Lemma 1 is essentially already contained in [9, Vol. 1, pp. 154–156 and Vol. 2, p. 308], however not stated explicitly there.

LEMMA 2. *If $g \in H^1(\mathbb{T})$, then*

$$(3.3)\qquad \|\sup_{m\in\mathbb{N}} |\sigma_m(g)|\|_{L^1} \le C_2\|g\|_{H^1},$$

where the constant C_2 does not depend on g.

4. Proofs of the main results

Proof of Theorem 1. (i) We start with the representation

$$\sigma_{mn}(f;x,y) = \frac{1}{\pi^2}\int\int_{\mathbb{T}^2} f(u,v)K_m(x-u)K_n(y-v)\,du\,dv,$$

where

$$K_m(t) := \frac{2}{m+1}\Big(\frac{\sin(m+1)(t/2)}{2\sin(t/2)}\Big)^2, \qquad m,n \in \mathbb{N},$$

is the Fejér kernel (see, e.g., [9, Vol. 2, pp. 302–303]). Since this kernel is positive, we may estimate the maximal Fejér operator $\sigma_*(f)$ defined in (1.3) as follows:

$$(4.1)\qquad \sigma_*(f;x,y) \le \sup_{n\in\mathbb{N}}\frac{1}{\pi}\int_{\mathbb{T}}\Big\{\sup_{m\in\mathbb{N}}\Big|\frac{1}{\pi}\int_{\mathbb{T}} f(u,v)K_m(x-u)\,du\Big|\Big\}K_n(y-v)\,dv.$$

This motivates the introduction of the auxiliary function

$$(4.2)\qquad h(x,v) := \sup_{m\in\mathbb{N}}\Big|\frac{1}{\pi}\int_{\mathbb{T}} f(u,v)K_m(x-u)\,du\Big|.$$

The key point is that the right-hand side here can be viewed as the maximal operator of the Fejér means

$$(4.3)\qquad \sigma_m^{(1,0)}(f;\cdot,v) := \frac{1}{\pi}\int_{\mathbb{T}} f(u,v)K_m(\cdot-u)\,du$$

of the function $f(\cdot,v)$, which clearly belongs to $L^1(\mathbb{T})$ for almost all $v \in \mathbb{T}$.

(ii) We claim that $h(x,\cdot) \in L^1(\mathbb{T})$ for almost all $x \in \mathbb{T}$. To see this, we first observe that from the assumption $f \in H^{(1,0)}(\mathbb{T}^2)$ it follows that $f(\cdot,v) \in H^1(\mathbb{T})$ for almost all $v \in \mathbb{T}$. Consequently, we may apply Lemma 2 in the first variable of f. As a result of (3.3), we infer that the inequality

$$\int_{\mathbb{T}} |h(x,v)|\,dx \le C_2\Big\{\int_{\mathbb{T}} |f(x,v)|\,dx + \int_{\mathbb{T}} |\widetilde{f}^{(1,0)}(x,v)|\,dx\Big\}$$

holds true for almost all $v \in \mathbb{T}$. Integrating with respect to v yields

$$(4.4)\qquad \int\int_{\mathbb{T}^2} |h(x,v)|\,dx\,dv \le C_2\{\|f\|_{L^1} + \|\widetilde{f}^{(1,0)}\|_{L^1}\} = C_2\|f\|_{H^{(1,0)}} < \infty.$$

It remains to apply Fubini's theorem in order to conclude that $h(x,\cdot) \in L^1(\mathbb{T})$ for almost all $x \in \mathbb{T}$.

(iii) Returning to (4.1) and (4.2), we may write

$$(4.5)\qquad \sigma_*(f;x,y) \le \sup_{n\in\mathbb{N}}\frac{1}{\pi}\int_{\mathbb{T}} h(x,v)K_n(y-v)\,dv = \sup_{n\in\mathbb{N}}\sigma_n^{(0,1)}(h;x,y),$$

where

$$\sigma_n^{(0,1)}(h;x,\cdot) := \frac{1}{\pi}\int_{\mathbb{T}} h(x,v)K_n(\cdot-v)\,dv$$

can be viewed as the Fejér mean of the function $h(x,\cdot)$. Since $h(x,\cdot) \in L^1(\mathbb{T})$ for almost all $x \in \mathbb{T}$, we may apply Lemma 1 in the second variable of h. As a result of (3.2), we deduce that the inequality

$$(4.6)\qquad |\{y \in \mathbb{T} : \sup_{n\in\mathbb{N}}\sigma_n^{(0,1)}(h;x,y) > \lambda\}| \le \frac{C_1}{\lambda}\int_{\mathbb{T}} |h(x,v)|\,dv$$

holds true for almost all $x \in \mathbb{T}$ and for all $\lambda > 0$.

(iv) We recall that given a measurable subset E of $\mathbb{T}^2$, the planar measure of E can be computed by means of the linear measure of the cross-sections as follows:

$$|E| = \int_{\mathbb{T}} |\{y \in \mathbb{T} : (x,y) \in E\}|\,dx.$$

Combining (4.4)–(4.6) with Fubini's theorem yields

$$\begin{aligned}|\{(x,y)\in\mathbb{T}^2:\sigma_*(f;x,y)>\lambda\}| &\le |\{(x,y)\in\mathbb{T}^2:\sup_{n\in\mathbb{N}}\sigma_n^{(0,1)}(h;x,y)>\lambda\}|\\ &\le \frac{C_1}{\lambda}\int_{\mathbb{T}}dx\int_{\mathbb{T}}|h(x,v)|\,dv\le\frac{C_1C_2}{\lambda}\|f\|_{H^{(1,0)}}\end{aligned}$$

for all $\lambda>0$, which is (2.2) to be proved. ∎

Proof of Corollary 2. The two-dimensional trigonometric polynomials are dense in the Hardy spaces $H^{(1,0)}(\mathbb{T}^2)$ and $H^{(0,1)}(\mathbb{T}^2)$. The Cesàro mean $\sigma_{mn}(f;x,y)$ clearly converges at all points as $m,n\to\infty$ in the case where f is a trigonometric polynomial in its variables. Thus, we may apply the usual density argument due to Marcinkiewicz and Zygmund [6], which povides (2.5) to be proved. ∎

Proof of Theorem 2. It runs along the same lines as the proof of Theorem 1 above.

(v) We begin with inequality (4.1) and introduce the auxiliary function $h(x,v)$ defined in (4.2).

(vi) We claim that this time $h(x,\cdot)\in H^1(\mathbb{T})$ for almost all $x\in\mathbb{T}$. In part (ii) of the proof of Theorem 1, we have shown that $h(x,\cdot)\in L^1(\mathbb{T})$ for almost all $x\in\mathbb{T}$. Also, we have to prove that $\widetilde{h}^{(0,1)}(x,\cdot)\in L^1(\mathbb{T})$ for almost all $x\in\mathbb{T}$.

To see this, we first notice that from the assumption $f\in H^{(1,1)}(\mathbb{T}^2)$ it follows that $\widetilde{f}^{(0,1)}(\cdot,v)\in H^1(\mathbb{T})$ for almost all $v\in\mathbb{T}$. Consequently, we may apply Lemma 2 in the first variable of $\widetilde{f}^{(0,1)}$. As a result of (3.3), we infer that the inequality

$$\int_{\mathbb{T}}\{\sup_{m\in\mathbb{N}}|\sigma_m^{(1,0)}(\widetilde{f}^{(0,1)};x,v)|\}\,dx \le C_2\Big\{\int_{\mathbb{T}}|\widetilde{f}^{(0,1)}(x,v)|\,dx+\int_{\mathbb{T}}|\widetilde{f}^{(1,1)}(x,v)|\,dx\Big\}\tag{4.7}$$

holds true for almost all $v\in\mathbb{T}$. On the one hand,

$$\sigma_m^{(1,0)}(\widetilde{f}^{(0,1)};\cdot,v)=\frac{1}{\pi}\int_{\mathbb{T}}\widetilde{f}^{(0,1)}(u,v)K_m(\cdot-u)\,du$$

(cf. (4.3)). On the other hand, from (4.2) it follows that

$$\widetilde{h}^{(0,1)}(\cdot,v)=\sup_{m\in\mathbb{N}}\Big|\frac{1}{\pi}\int_{\mathbb{T}}\widetilde{f}^{(0,1)}(u,v)K_m(\cdot-u)\,du\Big|.$$

Consequently, (4.7) can be rewritten into the form

$$\int_{\mathbb{T}}\widetilde{h}^{(0,1)}(x,v)\,dx\le C_2\Big\{\int_{\mathbb{T}}|\widetilde{f}^{(0,1)}(x,v)|\,dx+\int_{\mathbb{T}}|\widetilde{f}^{(1,1)}(x,v)|\,dx\Big\},$$

which holds true for almost all $v\in\mathbb{T}$. Hence

$$\iint_{\mathbb{T}^2}|\widetilde{h}^{(0,1)}(x,v)|\,dx\,dv\le C_2\{\|\widetilde{f}^{(0,1)}\|_{L^1}+\|\widetilde{f}^{(1,1)}\|_{L^1}\}<\infty.\tag{4.8}$$

It remains to apply Fubini's theorem in order to conclude that $\widetilde{h}^{(0,1)}(x,\cdot)\in L^1(\mathbb{T})$ for almost all $x\in\mathbb{T}$. Thus, we have completed the proof of our claim that $h(x,\cdot)\in H^1(\mathbb{T})$ for almost all $x\in\mathbb{T}$.

(vii) Consequently, we may apply Lemma 2 again, this time in the second variable of h. As a result of (3.3), we deduce that the inequality

$$\int_{\mathbb{T}}\{\sup_{n\in\mathbb{N}}\sigma_n^{(0,1)}(h;x,y)\}\,dy\le C_2\{\int_{\mathbb{T}}|h(x,v)|\,dv+\int_{\mathbb{T}}|\widetilde{h}^{(0,1)}(x,v)|\,dv\}$$

holds true for almost all $x\in\mathbb{T}$. Taking into account (4.5), it follows that

$$\iint_{\mathbb{T}^2}\sigma_*(f;x,y)\,dx\,dy\le C_2\{\|h\|_{L^1}+\|\widetilde{h}^{(0,1)}\|_{L^1}\}.\tag{4.9}$$

Combining (4.4), (4.8), and (4.9) yields

$$\|\sigma_*(f)\|_{L^1}\le C_2^2\{\|f\|_{L^1}+\|\widetilde{f}^{(1,0)}\|_{L^1}+\|\widetilde{f}^{(0,1)}\|_{L^1}+\|\widetilde{f}^{(1,1)}\|_{L^1}\}=C_2^2\|f\|_{H^{(1,1)}},$$

which is (2.7) to be proved. ∎

5. On the maximal conjugate Fejér operators. Denote by $\widetilde{\sigma}_{mn}^{(1,0)}(f)$, $\widetilde{\sigma}_{mn}^{(0,1)}(f)$, and $\widetilde{\sigma}_{mn}^{(1,1)}(f)$ the Fejér means of the rectangular partial sums of the conjugate series (2.1), (2.4), and (2.6), respectively. We have mentioned in Section 2 that if $f\in H^{(1,0)}(\mathbb{T}^2)$, then the conjugate series (2.1) is the Fourier series of the conjugate function $\widetilde{f}^{(1,0)}$. Consequently,

$$\widetilde{\sigma}_{mn}^{(1,0)}(f)\equiv\sigma_{mn}(\widetilde{f}^{(1,0)}),\qquad(m,n)\in\mathbb{N}^2.$$

Hence, for the maximal conjugate Fejér operator $\widetilde{\sigma}_{mn}^{(1,0)}(f)$ defined by

$$\widetilde{\sigma}_*^{(1,0)}(f;x,y):=\sup_{(m,n)\in\mathbb{N}^2}|\widetilde{\sigma}_{mn}^{(1,0)}(f;x,y)|,$$

we have

$$\widetilde{\sigma}_*^{(1,0)}(f)\equiv\sigma_*(\widetilde{f}^{(1,0)}).\tag{5.1}$$

Analogous statements are valid also for the other two maximal conjugate Fejér operators $\widetilde{\sigma}_*^{(0,1)}(f)$ and $\widetilde{\sigma}_*^{(1,1)}(f)$.

Next, we introduce two functions r and s of one variable by letting

$$r(x) := \frac{1}{2\pi}\int_{\mathbb{T}} f(x,y)\,dy \quad\text{and}\quad s(y) := \frac{1}{2\pi}\int_{\mathbb{T}} f(x,y)\,dx. \tag{5.2}$$

It is plain that $r, s \in L^1(\mathbb{T})$; their Fourier series are given by

$$\sum_{j\in\mathbb{Z}} \widehat{f}(j,0)e^{ijx} \quad\text{and}\quad \sum_{k\in\mathbb{Z}} \widehat{f}(0,k)e^{iky},$$

respectively; and the corresponding conjugate functions $\widetilde{r}$ and $\widetilde{s}$ are given by

$$\widetilde{r}(x) = \frac{1}{2\pi}\int_{\mathbb{T}} \widetilde{f}^{(1,0)}(x,y)\,dy \quad\text{and}\quad \widetilde{s}(y) = \frac{1}{2\pi}\int_{\mathbb{T}} \widetilde{f}^{(0,1)}(x,y)\,dx$$

for almost all $x \in \mathbb{T}$ and for almost all $y \in \mathbb{T}$, respectively. Hence it follows immediately that if $f \in H^{(1,0)}(\mathbb{T}^2)$, then $r \in H^1(\mathbb{T})$; and if $f \in H^{(0,1)}(\mathbb{T}^2)$, then $s \in H^1(\mathbb{T})$.

By the completeness of the trigonometric system, it is not difficult to deduce that the relations

$$(\widetilde{f}^{(1,0)})^{\sim(1,0)}(x,y) = -f(x,y) + s(y), \tag{5.3}$$
$$(\widetilde{f}^{(0,1)})^{\sim(0,1)}(x,y) = -f(x,y) + r(x),$$
$$(\widetilde{f}^{(1,0)})^{\sim(0,1)}(x,y) = (\widetilde{f}^{(0,1)})^{\sim(1,0)}(x,y) = \widetilde{f}^{(1,1)}, \tag{5.4}$$
$$(\widetilde{f}^{(1,0)})^{\sim(1,1)}(x,y) = (\widetilde{f}^{(1,1)})^{\sim(1,0)}(x,y) = -f^{(0,1)}(x,y) + \widetilde{s}(y), \tag{5.5}$$
$$(\widetilde{f}^{(0,1)})^{\sim(1,1)}(x,y) = (\widetilde{f}^{(1,1)})^{\sim(0,1)}(x,y) = -\widetilde{f}^{(1,0)}(x,y) + \widetilde{r}(x),$$

and

$$(\widetilde{f}^{(1,1)})^{\sim(1,1)}(x,y) = f(x,y) - r(x) - s(y) + \widehat{f}(0,0)$$

hold true for almost all $(x,y) \in \mathbb{T}^2$, where r and s are defined in (5.2).

These relations have remarkable consequences:

(i) By (5.2) and (5.3), if $f \in H^{(1,0)}(\mathbb{T}^2)$, then $\widetilde{f}^{(1,0)} \in H^{(1,0)}(\mathbb{T}^2)$ as well, and

$$\|\widetilde{f}^{(1,0)}\|_{H^{(1,0)}} \le \|f\|_{H^{(1,0)}} + \|f\|_{L^1}. \tag{5.6}$$

(ii) Similarly, if $f \in H^{(0,1)}(\mathbb{T}^2)$, then $\widetilde{f}^{(0,1)} \in H^{(0,1)}(\mathbb{T}^2)$ as well.

(iii) If $f \in H^{(1,1)}(\mathbb{T}^2)$, then each of $\widetilde{f}^{(1,0)}$, $\widetilde{f}^{(0,1)}$, and $\widetilde{f}^{(1,1)}$ also belongs to $H^{(1,1)}(\mathbb{T}^2)$. The norms $\|\widetilde{f}^{(1,0)}\|_{H^{(1,1)}}$, $\|\widetilde{f}^{(0,1)}\|_{H^{(1,1)}}$, and $\|\widetilde{f}^{(1,1)}\|_{H^{(1,1)}}$ may increase in comparison with $\|f\|_{H^{(1,1)}}$, but each of them remains equivalent to $\|f\|_{H^{(1,1)}}$. For instance, by (5.3)–(5.5), we may estimate as follows:

$$\begin{aligned}\|\widetilde{f}^{(1,0)}\|_{H^{(1,1)}} &\le \|\widetilde{f}^{(1,0)}\|_{L^1} + \|f\|_{L^1} + \|s\|_{L^1} \\ &\quad + \|\widetilde{f}^{(1,1)}\|_{L^1} + \|\widetilde{f}^{(0,1)}\|_{L^1} + \|\widetilde{s}\|_{L^1} \\ &= \|f\|_{H^{(1,1)}} + \|s\|_{H^1} \le 2\|f\|_{H^{(1,1)}},\end{aligned} \tag{5.7}$$

since

$$\|s\|_{H^1} \le \|f\|_{L^1} + \|\widetilde{f}^{(0,1)}\|_{L^1}.$$

Now, Theorem 1 implies via (5.1) that the maximal conjugate Fejér operator $\widetilde{\sigma}_*^{(1,0)}(f)$ is of type $(H^{(1,0)}, \text{weak-}L^1)$, while Corollary 1 implies via the symmetric counterpart of (5.1) that $\widetilde{\sigma}_*^{(0,1)}(f)$ is of type $(H^{(0,1)}, \text{weak-}L^1)$.

Corollary 3. *If* $f \in H^{(1,0)}(\mathbb{T}^2)$, *then*

$$\sup_{\lambda>0} \lambda|\{(x,y)\in\mathbb{T}^2 : \widetilde{\sigma}_*^{(1,0)}(f;x,y) > \lambda\}| \le 2C\|f\|_{H^{(1,0)}};$$

while if $f \in H^{(0,1)}(\mathbb{T}^2)$, *then*

$$\sup_{\lambda>0} \lambda|\{(x,y)\in\mathbb{T}^2 : \widetilde{\sigma}_*^{(0,1)}(f;x,y) > \lambda\}| \le 2C\|f\|_{H^{(0,1)}},$$

with the same constant C *in both cases as in* (2.2) (cf. (5.6)).

The pointwise convergence follows from Corollary 3 exactly in the same way as in the cases of Theorem 1 and Corollary 1.

Corollary 4. *If* $f \in H^{(1,0)}(\mathbb{T}^2)$, *then*

$$\lim_{m,n\to\infty} \widetilde{\sigma}_{mn}^{(1,0)}(f;x,y) = \widetilde{f}^{(1,0)}(x,y)$$

at almost all points $(x,y)\in\mathbb{T}^2$; *while if* $f \in H^{(0,1)}(\mathbb{T}^2)$, *then*

$$\lim_{m,n\to\infty} \widetilde{\sigma}_{mn}^{(0,1)}(f;x,y) = \widetilde{f}^{(0,1)}(x,y) \tag{5.8}$$

at almost all points $(x,y)\in\mathbb{T}^2$.

Finally, Theorem 2 implies via (5.1) and its counterparts that each of the maximal conjugate Fejér operators $\widetilde{\sigma}_*^{(1,0)}(f)$, $\widetilde{\sigma}_*^{(0,1)}(f)$, and $\widetilde{\sigma}_*^{(1,1)}(f)$ is of type $(H^{(1,1)}, L^1)$.

Corollary 5. *Assume* $f \in H^{(1,1)}(\mathbb{T}^2)$, *and denote by* $\widetilde{\sigma}_*(f)$ *one of the maximal conjugate Fejér operators* $\widetilde{\sigma}_*^{(1,0)}(f)$, $\widetilde{\sigma}_*^{(0,1)}(f)$, *or* $\widetilde{\sigma}_*^{(1,1)}(f)$. *Then*

$$\|\widetilde{\sigma}^*(f)\|_{L^1} \le 3C\|f\|_{H^{(1,1)}},$$

with the same constant C *as in* (2.7) (cf. (5.7)).

Hence the pointwise convergence follows again.

Corollary 6. *If* $f \in H^{(1,1)}(\mathbb{T}^2)$, *then*

$$\lim_{m,n\to\infty} \widetilde{\sigma}_{mn}^{(1,1)}(f;x,y) = f(x,y) \tag{5.9}$$

at almost all points $(x,y)\in\mathbb{T}^2$.

On closing, we mention two conjectures concerning the maximal conjugate Fejér operators which we are unable to prove or disprove.

The first of them lies in the positive direction. We guess that $\widetilde{\sigma}_*^{(1,1)}(f)$ is bounded from $H^{(1,0)}(\mathbb{T}^2) \cap H^{(0,1)}(\mathbb{T}^2)$ into weak-$L^1(\mathbb{T}^2)$.

CONJECTURE 1. If $f \in H^{(1,0)}(\mathbb{T}^2) \cap H^{(0,1)}(\mathbb{T}^2)$, then we have

$$\sup_{\lambda>0} \lambda |\{(x,y) \in \mathbb{T}^2 : \widetilde{\sigma}_*^{(1,1)}(f;x,y) > \lambda\}| \le C(\|f\|_{H^{(1,0)}} + \|f\|_{H^{(0,1)}}),$$

where the constant C does not depend on f.

If this conjecture were true, then (5.9) would hold for almost all $(x,y) \in \mathbb{T}^2$ under the assumption that $f \in H^{(1,0)}(\mathbb{T}^2) \cap H^{(0,1)}(\mathbb{T}^2)$, which is clearly less restrictive than the requirement $f \in H^{(1,1)}(\mathbb{T}^2)$.

The second conjecture lies in the negative direction. We guess that Corollary 4 is the best possible in a certain sense.

CONJECTURE 2. There exists a function $f \in H^{(1,0)}(\mathbb{T}^2)$ such that each of the relations (5.8) and (5.9) is no longer true at almost all points $(x,y) \in \mathbb{T}^2$.

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Erratum to "Operators on spaces of analytic functions"

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by

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The author wishes to add the reference: S. Axler and P. Bourdon, *Finite codimensional invariant subspaces of Bergman spaces*, Trans. Amer. Math. Soc. 306 (1988), 805–817, which has an overlap with his Lemma 1 and Theorem 2. Actually the reference was given in the author's lecture notes, Department of Mathematics, University of Calgary, 1991 (page 9 of Lecture 3), but has been inadvertently omitted during the preparation of the article.

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