des développements de Taylor en 0, $\frac{1}{p!}D^p g_k(0) = N_{pm+k}$. On déduit alors, d'après les hypothèses faites sur la suite $(N_p)_{p\geq 0}$, que les fonctions g_k, $0 \leq k \leq m-1$, n'appartiennent à $(p!N_p^l)_{[0,1]}$ pour aucun l réel, $l < m$.

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UNIVERSITÉ PARIS-SUD
MATHÉMATIQUES, BÂT. 425
91405 ORSAY CEDEX, FRANCE

UNIVERSITÉ DE LILLE
U.F.R. DE MATHÉMATIQUE
59655 VILLENEUVE D'ASCQ CEDEX, FRANCE

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An example of a non-topologizable algebra

by

R. FRANKIEWICZ and G. PLEBANEK (Wrocław)

Abstract. We present an example of an algebra that is generated by ω_1 elements, and cannot be made a topological algebra. This answers a problem posed by W. Żelazko.

A real or complex algebra $\mathbf{A}$ is said to be *topologizable* if there exists a topology τ on $\mathbf{A}$ such that $(\mathbf{A},\tau)$ is a Hausdorff topological vector space, and multiplication in $\mathbf{A}$ is jointly continuous (see [3]). While one can always find a vector space topology in which multiplication is separately continuous, there are algebras that are not topologizable.

Żelazko [4] showed that $\mathcal{L}(X)$, the algebra of all endomorphisms of a linear space X, is not topologizable as a locally convex algebra whenever X is of infinite dimension. Müller [2] gave an example of a commutative algebra that is not topologizable. He also noted that $\mathcal{L}(X)$ is not topologizable at all for infinite-dimensional X.

On the other hand, Żelazko [5] proved the following positive result on topologization of algebras ($\tau_{\max}^{\mathrm{LC}}$ denotes the maximal topology, i.e. the topology generated by all seminorms).

THEOREM 1. *Let $\mathbf{F}$ be a real or complex free algebra in variables $(t_i : i \in I)$. Then $(\mathbf{F},\tau_{\max}^{\mathrm{LC}})$ is a (complete) locally convex topological algebra if and only if the set of variables is at most countable. Consequently, every countably generated algebra can be topologized as a locally convex complete topological algebra.*

Żelazko [5] noted that, since the above-mentioned examples of non-topologizable algebras are 2^ω-generated, the result of Theorem 1 is best possible if the continuum hypothesis holds.

To show that the second statement of Theorem 1 is indeed optimal, at least concerning the number of generators, we present below an example of an ω_1-generated algebra that is not topologizable. For this we modify

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an idea due to Müller [2]. Roughly speaking, Müller's algebra is generated by the family of all functions $f : \omega \to \omega$ (here and below ω is the set of natural numbers, ω_1 stands for the first uncountable ordinal). To get an ω_1-generated example one might try to take just a family $\mathcal{F}$ of functions with $|\mathcal{F}| = \omega_1$. But to repeat an argument used by Müller one has to know that $\mathcal{F}$ is unbounded, that is, there is no function g which eventually dominates every $f \in \mathcal{F}$. This, however, cannot be assured without extra axioms (see e.g. [1]). Therefore we use functions with larger domains.

THEOREM 2. *There exists an algebra generated by ω_1 elements that is not topologizable.*

Proof. For every ordinal number $\alpha < \omega_1$ we choose an injective function $f_\alpha : \alpha \to \omega$. We let $\mathbf{A}$ be the linear space of formal linear combinations of the following:

- a fixed element c;
- elements x_α, where $\alpha < \omega_1$;
- elements a_α, where $\alpha < \omega_1$.

We define a multiplication in $\mathbf{A}$ by putting

$$x_\beta \cdot a_\alpha = a_\alpha \cdot x_\beta = \begin{cases} f_\alpha(\beta)c & \text{if } \beta < \alpha, \\ 0 & \text{if } \beta \ge \alpha, \end{cases}$$

and

$$c \cdot z = z \cdot c = x_\alpha \cdot x_\beta = a_\alpha \cdot a_\beta = 0,$$

for every $\alpha, \beta < \omega_1$ and for every $z \in \mathbf{A}$. These relations define a unique associative and commutative multiplication "$\cdot$" on $\mathbf{A}$.

Suppose now that $\mathbf{A}$ is topologizable. Then there is a system $\mathcal{V}$ of neighbourhoods of 0 such that $\bigcap \mathcal{V} = \{0\}$ and

- every $V \in \mathcal{V}$ is balanced (i.e. $tV \subseteq V$ for every scalar t with $|t| \le 1$);
- every $V \in \mathcal{V}$ is absorbent (i.e. $\bigcup_{n\in\omega} nV = \mathbf{A}$);
- for every $V \in \mathcal{V}$ there is $W \in \mathcal{V}$ with $W + W \subseteq V$;
- for every $V \in \mathcal{V}$ there is $W \in \mathcal{V}$ with $W \cdot W \subseteq V$.

Now fix $V \in \mathcal{V}$ such that $c \notin V$ and $W \in \mathcal{V}$ with $W \cdot W \subseteq V$. For every $\alpha < \omega_1$ there is $s(\alpha) \in \omega$ such that $x_\alpha \in s(\alpha)W$. This defines a function $s : \omega_1 \to \{1, 2, \ldots\}$, so there exist $k \ge 1$ and $\alpha < \omega_1$ such that the set $P = \{\beta < \alpha : s(\beta) = k\}$ is infinite. Next fix a (necessarily positive) number $m \in \omega$ such that $a_\alpha \in mW$.

Now for every $\beta \in P$ we have $x_\beta \in kW$ and

$$c = \frac{1}{f_\alpha(\beta)} a_\alpha x_\beta = \frac{mk}{f_\alpha(\beta)} \frac{a_\alpha}{m} \cdot \frac{x_\beta}{k} \in \frac{mk}{f_\alpha(\beta)} W \cdot W \subseteq \frac{mk}{f_\alpha(\beta)} V.$$

Since $c \notin V$, we get $mk \ge f_\alpha(\beta)$. But this means that f_α maps an infinite set P into $\{1, \ldots, mk\}$, so f_α cannot be injective, a contradiction. The proof is complete.

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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
P.O. BOX 137
00-950 WARSZAWA, POLAND
E-mail: RF@IMPAN.GOV.PL

INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW, POLAND
E-mail: GRZES@MATH.UNI.WROC.PL

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