On termine avec une remarque que, bien sûr, les poids puissances de Verbitski w_n^α, $\alpha \in \mathbb{R}$, vérifient les hypothèses du Théorème 5.11.3.

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Characterizing spectra of closed operators through existence of slowly growing solutions of their Cauchy problems

by

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Abstract. Let A be a closed linear operator in a Banach space E. In the study of the nth-order abstract Cauchy problem $u^{(n)}(t) = Au(t)$, $t \in \mathbb{R}$, one is led to considering the linear Volterra equation

$$\text{(AVE)} \qquad u(t) = p(t) + A\int_0^t a(t-s)u(s)\,ds, \qquad t \in \mathbb{R},$$

where $a(\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $p(\cdot)$ is a vector-valued polynomial of the form $p(t) = \sum_{j=0}^n \frac{1}{j!}x_j t^j$ for some elements $x_j \in E$. We describe the spectral properties of the operator A through the existence of slowly growing solutions of the (AVE). The main tool is the notion of Carleman spectrum of a vector-valued function. Moreover, an extension of a theorem of Pólya in complex analysis is obtained and applied to the individual "$Ax = 0$" and "$Tx = x$" problem.

1. Introduction. Let A be the generator of a bounded C_0-semigroup $(T(t))_{t\ge0}$ on a reflexive Banach space E. It is shown in [Vu, Cor. 2.6] that, if $(T^*(t))_{t\ge0}$ is not *asymptotically stable*, i.e., $\|T^*(t)f_0\| \not\to 0$ as $t \to \infty$ for some $f_0 \in E'$, and if $\sigma(A) \not\supseteq i\mathbb{R}$, then the full time Cauchy problem $u'(t) = Au(t)$, $t \in \mathbb{R}$, $u(0) = u_0$, has a mild, bounded solution for each $u_0 \in E$. This result implies (roughly speaking) that the spectral structure of the generator A can furnish a non-trivial, slowly growing (e.g., bounded) solution for the Cauchy problem $u'(t) = Au(t)$, $t \in \mathbb{R}$.

In this paper we study the converse aspect, i.e., describing the spectral properties of A through the existence of slowly growing solutions of the corresponding Cauchy problems. Generally, we take A to be a closed linear operator in a Banach space E. In the study of the nth-order abstract Cauchy problem $u^{(n)}(t) = Au(t)$, $t \in \mathbb{R}$, one is led to considering the linear Volterra

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equation (see [Pr] for more information)

$$\text{(AVE)}\qquad u(t) = p(t) + A\int_0^t a(t-s)u(s)\,ds, \qquad t\in\mathbb{R},$$

where $a(\cdot)\in L^1_{\text{loc}}(\mathbb{R})$ and $p(\cdot)$ is a vector-valued polynomial of the form $p(t)=\sum_{j=0}^n \frac{1}{j!}x_jt^j$ for some elements $x_j\in E$. In Theorem 2.5 we show, under certain hypotheses on $a(\cdot)$, that the approximate spectrum $\sigma_{\text{a}}(A)$ must intersect some curve described by $a(\cdot)$, whenever for some $p(\cdot)\neq 0$ the corresponding (AVE) has a solution $u(\cdot)$ satisfying certain growth conditions (e.g., $\|u(t)\|=O(e^{|t|^\alpha})$ as $t\to\infty$ for some $0\le\alpha<1$). In particular, we show that if the nth-order abstract Cauchy problem $u^{(n)}(t)=Au(t)$, $t\in\mathbb{R}$, has a non-trivial bounded solution, then $\sigma_{\text{a}}(A)\cap i^n\mathbb{R}\neq\emptyset$.

As consequences of these results we find that $\sigma_{\text{a}}(A)\cap i\mathbb{R}\neq\emptyset$ if A generates a C_0-group $\mathcal{T}=(T(t))_{t\in\mathbb{R}}$ satisfying either

$$\int_{-\infty}^{\infty}\frac{\log(1+\|T(t)x_0\|)}{1+t^2}\,dt<\infty \qquad \text{for some } 0\neq x_0\in E$$

or

$$\int_{-\infty}^{\infty}\frac{\log(1+\|T(t)'\mu_0\|)}{1+t^2}\,dt<\infty \qquad \text{for some } 0\neq \mu_0\in E'.$$

In the third section we consider the individual "$Ax=0$" problem. We show, for instance, that $A^kx_0=0$ if $\sigma(A)\subseteq\{0\}$ and (AVE) for $a(t)\equiv 1$ and $p(t)\equiv x_0$ has a solution $u(\cdot)$ satisfying $\|u(t)\|=o(t^k)$ as $t\to\infty$ and $\log^+\|u(t)\|=o(|t|^{1/2})$ as $t\to-\infty$. These results are based on a new theorem (Theorem 3.1) in complex analysis which extends old theorems of Pólya [H-P, Theorem 3.13.5 and 3.13.8] and Shah [Sh, Theorem 1], and is of independent interest especially in the operator theory context (see Zemánek [Ze]).

2. Existence of spectrum. We recall briefly some basic facts from complex analysis. As a basic tool we take the weighted Banach spaces $L_\omega(\mathbb{R})$ given by

$$(2.1)\qquad L_\omega(\mathbb{R}) := \Big\{f\in L^1(\mathbb{R}) : \|f\|_\omega := \int_{-\infty}^{\infty}|f(t)|\omega(t)\,dt<\infty\Big\}$$

for measurable, locally bounded functions $\omega:\mathbb{R}\to\mathbb{R}$ such that

$$(2.2)\qquad 1\le\omega(t) \quad\text{and}\quad \sup_{s\in\mathbb{R}}\omega(s+t)/\omega(s)<\infty \quad \text{for all } t\in\mathbb{R}.$$

For fixed ω the space $L_\omega(\mathbb{R})$ is the L^1-space $L^1(\mathbb{R},\omega(t)dt)$ and thus its dual, denoted by $L^\infty_{\omega^{-1}}(\mathbb{R})$, is the set of all measurable functions g such that

$$(2.3)\qquad \operatorname*{ess\,sup}_{t\in\mathbb{R}}|g(t)|/\omega(t)<\infty.$$

The *Fourier transform* of $f\in L_\omega(\mathbb{R})$ is now defined as (see [K, p. 120])

$$(2.4)\qquad \widehat{f}(s) := \int_{-\infty}^{\infty}f(t)e^{-ist}\,dt \qquad (s\in\mathbb{R}),$$

and if $\widehat{f}\in L^1(\mathbb{R})$ we have the inverse formula (see [K, p. 125])

$$(2.5)\qquad f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(s)e^{its}\,ds \qquad \text{for almost every } t\in\mathbb{R}.$$

Later we will be interested in the subspace

$$L_\omega(\mathbb{R})_0 := \{f\in L_\omega(\mathbb{R}) : \widehat{f} \text{ has compact support}\}.$$

Using (2.2) it is easily verified that $L_\omega(\mathbb{R})_0$ is invariant under translations and multiplication by functions $e^{i\lambda t}$ ($\lambda\in\mathbb{R}$). So, if $g\in L^\infty_{\omega^{-1}}(\mathbb{R})$ annihilates $L_\omega(\mathbb{R})_0$, then

$$\int_{-\infty}^{\infty}f(s+t)g(t)e^{i\lambda t}\,dt=0$$

for all $f\in L_\omega(\mathbb{R})_0$, $s\in\mathbb{R}$ and $\lambda\in\mathbb{R}$. Fix f and s. By the uniqueness theorem for the Fourier transform we find

$$f(s+t)g(t)=0 \qquad \text{for almost all } t\in\mathbb{R}.$$

Therefore the existence of $0\neq f_0\in L_\omega(\mathbb{R})_0$ implies $g=0$ and we have the following lemma.

LEMMA 2.1. *$L_\omega(\mathbb{R})_0$ is dense in $L_\omega(\mathbb{R})$ if and only if $L_\omega(\mathbb{R})_0\neq\{0\}$.*

Conditions guaranteeing that the subspace $L_\omega(\mathbb{R})_0$ is non-trivial have been found by Beurling and Malliavin.

LEMMA 2.2 ([B-M], Theorem I). *The space $L_\omega(\mathbb{R})_0$ is dense in $L_\omega(\mathbb{R})$ if the weight function ω satisfies condition* (2.2) *and*

$$(2.6)\qquad \int_{-\infty}^{\infty}\frac{\log\omega(t)}{1+t^2}\,dt<\infty.$$

For the sake of convenience we shall call a weight ω satisfying conditions (2.2) and (2.6) a weight of *Beurling–Malliavin type* and write $\omega\in\text{BM}(\mathbb{R})$. It should be pointed out that (2.6) combined with (2.2) implies

$$(2.7)\qquad \lim_{t\to\infty}\frac{1}{t}\log\omega(t)=0.$$

To see this, let $\varrho(t) := \log\omega(t)$, $t \in \mathbb{R}$. As shown in [B-M, p. 306] we can assume that $\varrho(0) = 0$ and

$$|\varrho(t) - \varrho(s)| \le M|t-s|, \qquad t, s \in \mathbb{R},$$

for some constant $M > 0$. Consider the function $f(t) := \varrho(t)/t$, $t \ge 1$. Then

$$|f(t) - f(s)| \le \frac{|\varrho(t) - \varrho(s)|}{t} + \left|\frac{s}{t} - 1\right| \cdot \frac{\varrho(s)}{s} \le 2M\left|\frac{s}{t} - 1\right|.$$

Thus $f(t)$ is a feebly oscillating function as $t \to \infty$ (cf. [A-P, Theorem 2.4]). Moreover, for $1 < s < t$, we have

$$\frac{1}{t}\int_1^t f(u)\,du \le \frac{1}{t}\int_1^s f(u)\,du + \int_s^t \frac{\varrho(u)}{u^2}\,du.$$

Fixing s and letting $t \to \infty$, we find

$$\limsup_{t\to\infty} \frac{1}{t}\int_1^t f(u)\,du \le \int_s^\infty \frac{\varrho(u)}{u^2}\,du.$$

Letting $s \to \infty$ and using (2.6) we find

$$\lim_{t\to\infty} \frac{1}{t}\int_1^t f(u)\,du = 0.$$

Therefore, by [A-P, Theorem 2.4] we have

$$0 = \lim_{t\to\infty} f(t) = \lim_{t\to\infty} \frac{\log\omega(t)}{t}.$$

Analogously, $\lim_{t\to-\infty}(\log\omega(t))/t = 0$. This gives (2.7).

We need some more notations. Let E be a Banach space and let $\mathbf{x}(\cdot)$ be a measurable E-valued function on $\mathbb{R}$ such that $\|x(\cdot)\|$ is dominated by some $\omega \in \mathrm{BM}(\mathbb{R})$. By (2.6) the *Carleman transform* of $\mathbf{x}$ is defined as

$$\widetilde{\mathbf{x}}(\lambda) := \begin{cases} \int_0^\infty e^{-\lambda t}x(t)\,dt, & \operatorname{Re}\lambda > 0, \\ -\int_{-\infty}^0 e^{-\lambda t}x(t)\,dt, & \operatorname{Re}\lambda < 0, \end{cases} \tag{2.8}$$

and is holomorphic in $\mathbb{C} \setminus i\mathbb{R}$. A point $\lambda_0 \in i\mathbb{R}$ is called a *regular point* of $\mathbf{x}$ if $\widetilde{\mathbf{x}}(\lambda)$ has a holomorphic extension in a neighborhood of λ_0. The *Carleman spectrum* of $\mathbf{x}$, denoted by $\mathrm{Sp}(\mathbf{x})$, is defined as the complement in $i\mathbb{R}$ of the set of regular points (see [Vu], [Pr, Sect. 0.5 and 0.6] and compare [K, pp. 179–181]).

LEMMA 2.3. *Under the above assumptions* $\mathrm{Sp}(\mathbf{x}) = \emptyset$ *implies* $x(t) = 0$ *almost everywhere.*

Proof. Let $f \in L_\omega(\mathbb{R})_0$. For $\delta > 0$, we have by (2.7) and Parseval's formula (see [K, p. 132])

$$\int_{-\infty}^\infty f(t)e^{-\delta|t|}x(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^\infty \widehat{f}(-s)\Big[\int_{-\infty}^\infty e^{-ist}e^{-\delta|t|}x(t)\,dt\Big]\,ds \tag{2.9}$$
$$= \frac{1}{2\pi}\int_{-\infty}^\infty \widehat{f}(-s)[\widetilde{\mathbf{x}}(\delta + is) - \widetilde{\mathbf{x}}(-\delta + is)]\,ds.$$

Since $\widehat{f}$ has compact support, the assumption $\mathrm{Sp}(\mathbf{x}) = \emptyset$ implies for $\delta \to 0^+$ that

$$\int_{-\infty}^\infty f(t)x(t)\,dt = 0 \qquad \text{for all } f \in L_\omega(\mathbb{R})_0. \tag{2.10}$$

Since the linear functional $f \mapsto \int_{-\infty}^\infty f(t)x(t)\,dt$ is continuous in $L_\omega(\mathbb{R})$ and the subspace $L_\omega(\mathbb{R})_0$ is dense in $L_\omega(\mathbb{R})$ (see Lemma 2.2), from (2.10) we see that $x(t) = 0$ almost everywhere. ■

Remark. The notion of Carleman spectrum has a long history; see [A1], [C], [D], [Pr, Section 0.5], [K] and [Vu]. In [Vu, Prop. 3.2] it is shown that the Carleman spectrum for $\mathbf{x} \in L^\infty(\mathbb{R}, E)$ coincides with the classical Beurling spectrum (see [K, Chap. VI, Theorem 8.2] for the scalar case) and thus the Carleman spectrum is not empty for such functions. In fact, according to Atzmon [A1, Lemma 2.3] the equality of the Carleman spectrum and the Beurling spectrum for $\mathbf{x}$ satisfying certain growth conditions can be dated back to Carleman [C] for the scalar case and to Domar [D] for the vector-valued case. The present Lemma 2.3 gives a new (and weaker) growth condition guaranteeing the non-voidness of the Carleman spectrum. It might be hoped that Lemma 2.3 remains true for weight functions ω only satisfying (2.2) and the weaker condition (2.7). But this is not true as the following example shows.

EXAMPLE 2.4. As seen in the proof of (2.7) we only need to consider ω for which the function $\varrho(t) := \log\omega(t)$ $(t \in \mathbb{R})$ is positive,

$$\int_{-\infty}^\infty \frac{\varrho(t)}{1+t^2}\,dt = \infty,$$

and $|\varrho(t) - \varrho(s)| \le M|t-s|$ for some $M > 0$, $t, s \in \mathbb{R}$. Moreover, we assume that ϱ is an even function. The following construction is given in [V, Lemma 3.1] to which we refer for more details. For $z := x + iy$, $y > 0$, define

$$u(z) := \frac{y}{\pi}\int_{-\infty}^\infty \left(\frac{1}{(t-x)^2 + y^2} - \frac{1}{t^2+1}\right)\varrho(t)\,dt.$$

Then u is harmonic in $y>0$ and has boundary values equal to ϱ on $\mathbb{R}$. For $y>0, h\in\mathbb{R}$, we have

$$u(iy+h)-u(iy)=\frac{y}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{(t-h)^2+y^2}-\frac{1}{t^2+y^2}\right)\varrho(t)\,dt$$
$$=\frac{y}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{t^2+y^2}-\frac{1}{(t+h)^2+y^2}\right)\varrho(t+h)\,dt$$
$$=\frac{y}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{(t+h)^2+y^2}-\frac{1}{t^2+y^2}\right)\varrho(t)\,dt.$$

We have used the fact that ϱ is even. Therefore,

$$|u(iy+h)-u(iy)|$$
$$=\left|\frac{y}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{t^2+y^2}-\frac{1}{(t+h)^2+y^2}\right)(\varrho(t+h)-\varrho(t))\,dt\right|$$
$$\le\frac{y}{2\pi}\int_{-\infty}^{\infty}\frac{M|h|}{t^2+y^2}\,dt+\frac{y}{2\pi}\int_{-\infty}^{\infty}\frac{M|h|}{(t+h)^2+y^2}\,dt=M|h|.$$

So

$$|u(iy+h)-u(iy)|\le M|h|,\qquad y>0,\ h\in\mathbb{R}.$$

Let v be the conjugate harmonic function of u satisfying $v(i)=0$ and define $G(z):=\exp(u(z)+iv(z))$, $y>0$. As a consequence of Fatou's Theorem the boundary values $G(x):=\lim_{y\to0^+}G(x+iy)$ exist a.e. and $|G(x)|=e^{\varrho(x)}=\omega(x)$. The condition $\int_{-\infty}^{\infty}\frac{\varrho(t)}{1+t^2}\,dt=\infty$ implies that $G(iy)$ $(y>0)$ decreases faster than exponentially and hence

$$\varphi(\lambda):=\int_0^{i\infty}e^{-\lambda z}G(z)\,dz$$

is an entire function. Through the contour integration used in [V], but using the inequality for u established above instead of using Lemma 3.2 and (1.6) of [V], we find that the Carleman transform $\widehat{G}(\lambda)$ of the boundary function $G(x)$ $(x\in\mathbb{R})$ is equal to $\varphi(\lambda)$ for λ with large $|\mathrm{Re}\,\lambda|$ and thus for all possible λ. This implies that $\widehat{G}$ has an analytic extension to $\mathbb{C}$. So $\mathrm{Sp}(G)=\emptyset$. ∎

Now we give some applications of Lemma 2.3. Consider the following linear Volterra equation (see [A-P] and [Pr] for more information):

$$\text{(AVE)}\qquad u(t)=p(t)+A\int_0^t a(t-s)u(s)\,ds,\qquad t\in\mathbb{R},$$

where A is a closed, not necessarily densely defined linear operator in a Banach space E with domain $D(A)$, $a(\cdot)\in L^1_{\mathrm{loc}}(\mathbb{R})$ and $p(\cdot)$ is an E-valued polynomial of the form

$$p(t)=\sum_{j=0}^{n}\frac{1}{j!}x_jt^j$$

for some $x_j\in E$. A function $u\in C(\mathbb{R},E)$ satisfying $a*u\in D(A)$ and (AVE) is called a *mild solution*. We make the following assumptions on $a(\cdot)$:

(h1) $a(\cdot)$ is subexponential, i.e., $\sup_{t\in\mathbb{R}}e^{-\delta|t|}|a(t)|<\infty$ for all $\delta>0$.

(h2) $\mathrm{Sp}(a)$ consists only of poles of $\widetilde{a}(z)$ and $\widetilde{a}(z)\ne0$ for all $z\in i\mathbb{R}\setminus\mathrm{Sp}(a)$.

(h3) $z_0=0$ is a pole of order $\le n+1$ of the function $\widetilde{a}(z)$.

Remark. Typical examples of $a(\cdot)$ are $a_n(t):=t^n/n!$ for $n=0,1,\ldots$

THEOREM 2.5. *Assume the hypotheses* (h1)–(h3). *If the equation* (AVE) *for some* $p(t)\not\equiv0$ *has a mild solution* $u(\cdot)$ *such that the norm function* $\|u(\cdot)\|$ *can be dominated by some* $\omega\in\mathrm{BM}(\mathbb{R})$, *then*

$$\sigma_{\mathrm{a}}(A)\cap\overline{\{\widetilde{a}(z)^{-1}:z\in i\mathbb{R}\setminus\mathrm{Sp}(a)\}}\ne\emptyset.$$

Proof. Assume conversely that

$$\text{(2.11)}\qquad \sigma_{\mathrm{a}}(A)\cap\overline{\{\widetilde{a}(z)^{-1}:z\in i\mathbb{R}\setminus\mathrm{Sp}(a)\}}=\emptyset.$$

Then by (h3) we find that

$$\text{(2.12)}\qquad 0\notin\sigma_{\mathrm{a}}(A).$$

For the polynomial $q(z):=\sum_{j=0}^{n}x_jz^{n-j}$ we have

$$\int_0^t e^{-zs}u(s)\,ds=\int_0^t e^{-zs}\Big[p(s)+A\int_0^s a(s-\tau)u(\tau)\,d\tau\Big]\,ds$$
$$=z^{-n-1}q(z)+A\int_0^t ds\int_0^s e^{-zs}a(s-\tau)u(\tau)\,d\tau-\int_t^{\infty}e^{-zs}p(s)\,ds$$
$$=z^{-n-1}q(z)+A\int_0^t d\tau\int_\tau^t e^{-zs}a(s-\tau)u(\tau)\,ds-\int_t^{\infty}e^{-zs}p(s)\,ds$$

$$= z^{-n-1}q(z) + A\int_0^t d\tau\Big[\int_0^{t-\tau} e^{-zs}a(s)\,ds\Big]e^{-z\tau}u(\tau)\,d\tau$$
$$-\int_t^\infty e^{-zs}p(s)\,ds.$$

Since $\|u(\cdot)\|$ is dominated by some $\omega \in \mathrm{BM}(\mathbb{R})$, by (2.7) we find that $\|u(\cdot)\|$ is subexponential. In the above equality we let $t \to \infty$ for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ and let $t \to -\infty$ for $z \in \mathbb{C}$ with $\operatorname{Re} z < 0$. Then, by the closedness of A and by (h1) (compare the proof of Theorem 1.3 of [Pr]) one has

$$\widetilde{u}(z) = z^{-n-1}q(z) + \widetilde{a}(z)A\widetilde{u}(z)$$

for all $z \in \mathbb{C}$ with $\operatorname{Re} z \neq 0$. Thus

$$(I - \widetilde{a}(z)A)\widetilde{u}(z) = z^{-n-1}q(z) \quad \text{for } \operatorname{Re} z \neq 0. \tag{2.13}$$

On the other hand, by (h2) and (h3) we find for $z_0 \in i\mathbb{R}$ that

$$\lim_{z\to z_0} \widetilde{a}(z)^{-1} = 0 \quad \text{and} \quad \lim_{z\to z_0} z^{-n-1}\widetilde{a}(z)^{-1} \text{ exists.}$$

Moreover, by (2.13) we have

$$(\widetilde{a}(z)^{-1} - A)\widetilde{u}(z) = z^{-n-1}\widetilde{a}(z)^{-1}q(z) \quad \text{for } \operatorname{Re} z \neq 0.$$

Hence by using (2.12) one can easily verify that $\widetilde{u}(z)$ is bounded in a neighborhood of z_0. This implies that $z_0 \notin \mathrm{Sp}(u)$ and thus $\mathrm{Sp}(u) = \emptyset$. But from Lemma 2.3 we derive a contradiction that $u(t) \equiv 0$ and hence $p(t) \equiv 0$. This finishes the proof. ∎

This result will now be applied to (AVE).

EXAMPLE 2.6. We take for $p(t)$ an E-valued polynomial of degree less than or equal to n and $a_n(t) = t^n/n!$ for some $n = 0, 1, 2, \ldots$ Then the corresponding (AVE) is

$$u(t) = p(t) + A\int_0^t \frac{(t-s)^n}{n!}u(s)\,ds, \qquad t \in \mathbb{R}. \tag{2.14}$$

We have $\widetilde{a}_n(z) = z^{-n-1}$ for all $z \neq 0$. Hence $a_n(\cdot)$ satisfies the hypotheses (h1)–(h3). If $\sigma_a(A) \cap i^{n+1}\mathbb{R} = \emptyset$, then Theorem 2.5 implies that equation (2.14) for $p(t) \not\equiv 0$ has no mild solution $u(\cdot)$ such that $\|u(t)\|$ $(t \in \mathbb{R})$ can be dominated by some $\omega \in \mathrm{BM}(\mathbb{R})$. In particular, (2.14) has no bounded solution for $p(t) \not\equiv 0$ if $\sigma_a(A) \cap i^{n+1}\mathbb{R} = \emptyset$. ∎

Another application is obtained by taking A to be the generator of a C_0-group $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ on a Banach space E. Denote by A' the weak-star generator of the weak-star continuous group $\mathcal{T}' = (T(t)')_{t\in\mathbb{R}}$ on the dual space E'. Then A' is closed and $\sigma(A') = \sigma(A)$ (see [N, p. 16]). For $x \in E$,

$u(t) := T(t)x$ $(t \in \mathbb{R})$ is the unique mild solution of the equation

$$u(t) = x + A\int_0^t u(s)\,ds, \qquad t \in \mathbb{R},$$

and for $\mu \in E'$, $v(t) := T(t)'\mu$ $(t \in \mathbb{R})$ is the unique mild solution of the equation

$$v(t) = \mu + A'\int_0^t v(s)\,ds, \qquad t \in \mathbb{R}.$$

For $x \in E$ and $\mu \in E'$, we consider the conditions

$$\int_{-\infty}^{\infty} \frac{\log(1 + \|T(t)x\|)}{1+t^2}\,dt < \infty \tag{2.15}$$

and

$$\int_{-\infty}^{\infty} \frac{\log(1 + \|T(t)'\mu\|)}{1+t^2}\,dt < \infty. \tag{2.16}$$

It is easily verified that (2.15) and (2.16) imply $\omega_1, \omega_2 \in \mathrm{BM}(\mathbb{R})$ for

$$\omega_1(t) := 1 + \|T(t)x\|, \qquad t \in \mathbb{R}, \text{ and}$$
$$\omega_2(t) := 1 + \|T(t)'\mu\|, \qquad t \in \mathbb{R}.$$

PROPOSITION 2.7. *Let A be the generator of a C_0-group $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ on a Banach space E. Then the following assertions hold.*

(i) *$\sigma_a(A) \cap i\mathbb{R} \neq \emptyset$ if either (2.15) or (2.16) holds for some non-zero element.*

(ii) *Assume that either (2.15) holds for all x in a weakly total subset $E_0 \subseteq E$ or that (2.16) holds for all μ in a weak-star total subset $F \subseteq E'$. Then A is bounded if and only if $\sigma(A)$ is bounded.*

Proof. (i) follows immediately from Theorem 2.5.

To show (ii), assume that $\sigma(A)$ is bounded. We only consider the case where (2.15) holds for all x in a weakly total subset E_0. The proof for the second case is similar. Since $\sigma(A)$ is bounded, by the spectral decomposition theorem [N, Theorem 3.3] we obtain a projection $P \in \mathcal{L}(E)$ commuting with $\mathcal{T}$ such that $AP \in \mathcal{L}(E)$ and $\sigma(A_1) = \emptyset$, where $A_1 := A_{|(I-P)E}$. For $x_0 \in E_0$, let $y_0 := (I-P)x_0$. Then $u(t) := T(t)y_0$ is the unique mild solution of the equation

$$u(t) = y_0 + A_1\int_0^t u(s)\,ds, \qquad t \in \mathbb{R}.$$

We have

$$\|u(t)\| = \|(I-P)T(t)x_0\| \le \|I-P\| \cdot \|T(t)x_0\|$$

for all $t \in \mathbb{R}$. This shows that $\|u(\cdot)\|$ is dominated by some weight of Beurling–Malliavin type. But $\sigma(A_1) = \emptyset$, hence by Theorem 2.5 (see also Ex. 2.6) we find that $y_0 = 0$, i.e., $(I - P)x = 0$ for all $x \in E_0$. Since E_0 is weakly total, this implies $I - P = 0$ and thus A is bounded. ∎

Remark. If the C_0-group $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ satisfies the growth condition

$$\int_{-\infty}^{\infty} \frac{\log(1 + \|T(t)\|)}{1+t^2}\,dt < \infty,$$

then it is shown in [Ly, p. 201, Lemma 3] (see also [N-H, Cor. 3.3]) that the generator A is bounded if and only if its spectrum is bounded. Our Proposition 2.7(ii) improves this result.

As an interesting application we have the following.

EXAMPLE 2.8. We consider a C_0-group $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ of disjointness-preserving operators on $C_0(X)$ given by

$$T(t)f := h_t \cdot f \circ \phi_t, \qquad f \in C_0(X).$$

(For more details we refer to [N, pp. 148–160]). For the generator A of $\mathcal{T}$ it is not known whether $\sigma(A) \neq \emptyset$. For $x \in X$ we consider the corresponding Dirac measure $\delta_x \in C_0(X)'$. We have

$$\|T(t)'\delta_x\| = \sup\{|h_t(x)f(\phi_t(x))| : \|f\| \le 1\} = |h_t(x)|.$$

By using Proposition 2.7 we find that

(i) $\sigma_{\mathrm{a}}(A) \cap i\mathbb{R} \neq \emptyset$ if for some $x_0 \in X$ the function $|h_t(x_0)|$ $(t \in \mathbb{R})$ is dominated by some $\omega \in \mathrm{BM}(\mathbb{R})$.

(ii) If $\sigma(A)$ is bounded and if for all x in a dense subset of X the function $|h_t(x)|$ $(t \in \mathbb{R})$ is dominated by some $\omega_x \in \mathrm{BM}(\mathbb{R})$, then A is bounded. Thus the flow $\{\phi_t\}$ is trivial and A is a bounded multiplication operator on $C_0(X)$. ∎

3. Existence of eigenvalues. We start with a new result on rational functions with values in a Banach space which yields Pólya's theorem [H-P, Theorem 3.13.5] as a consequence (see Theorem 3.2); it also improves the theorem of Shah [Sh, Theorem 1]. (I thank Prof. Zemánek for drawing my attention to the paper of Shah.)

THEOREM 3.1. *Let E be a Banach space and $g(z)$ an E-valued function which is holomorphic in the entire plane including infinity except for a subset S in the unit circle. Assume that there exists $k \in \mathbb{N}$ such that*

(i) $\|g(z)\| = o((1-|z|)^{-k})$ *as* $|z| \to 1^-$ *and*
(ii) $\log^+ \|g(z)\| = o((|z|-1)^{-1})$ *as* $|z| \to 1^+$.

We have the following assertions.

(1) *Every isolated point in S is a pole of $g(z)$ of order not exceeding k.*

(2) *If the set S is finite, say $S = \{z_1, \ldots, z_l\}$, then $g(z)$ is a rational function. More precisely, let*

$$g(z) := \begin{cases} \sum_{n=0}^{\infty} a_n z^n & \text{for small } |z|, \\ -b_0 - \sum_{n=-\infty}^{-1} a_n z^n & \text{for large } |z|, \end{cases}$$

be the expansion of $g(z)$ near the points $z = 0$ and $z = \infty$, respectively. Then there exist polynomials $p_j(z)$ of degree smaller than k and elements $y_j \in E$ for $j = 1, \ldots, l$ such that

$$a_n = \sum_{j=1}^{l} p_j(n) z_j^{-n} y_j \qquad \text{for all } |n| \ge 1.$$

Proof. First we prove (1). Let z_0 be an isolated point in the subset S of singularity. It is not hard to see that it suffices to show the result in the case $E = \mathbb{C}$ and $z_0 = 1$. We obtain the proof by modifying the proof of a theorem of Pólya given in [L, pp. 151–152].

Assume $E = \mathbb{C}$ and 1 is an isolated singularity of $g(z)$. Let $\mathcal{C}$ be the class of positively oriented Jordan curves around the point $z = 1$ but not enclosing the points $\{0\} \cup S \setminus \{1\}$. Using Cauchy's theorem we see that the integrals

$$\frac{1}{2\pi i} \int_C \frac{g(z)}{z^{w+1}}\,dz, \qquad C \in \mathcal{C},\ w \in \mathbb{C},$$

do not depend on the choice of the paths $C \in \mathcal{C}$. Let $G(w)$ be this common value. Then $G(w)$ is an entire function. We will show $G(w)$ is a polynomial of degree $<k$.

Let $\delta > 0$ be so small that the circle of radius δ around the point $z = 1$ belongs to $\mathcal{C}$. If we take C to be this path, then

$$|G(w)| = \left| \frac{1}{2\pi i} \int_C \frac{g(z)}{z^{w+1}}\,dz \right| = \frac{1}{2\pi} \left| \int_{|z|=\delta} g(1+z)(1+z)^{-w-1}\,dz \right|$$
$$\le \int_{|z|=\delta} |g(1+z)|(1+\delta)^{|w+1|}\,|dz| \le e^{\delta|w+1|} \int_{|z|=\delta} |g(1+z)|\,|dz|.$$

This shows that $G(w)$ is an entire function of exponential type zero.

Let $0 < a < 1 < b < 3$. Take $\theta_0 > 0$ to be so small that the path

$$C := \{ae^{i\theta} : |\theta| \le \theta_0\} \cup \{be^{i\theta} : |\theta| \le \theta_0\}$$
$$\cup \{te^{\theta_0} : a \le t \le b\} \cup \{te^{-\theta_0} : a \le t \le b\}$$

belongs to $\mathcal{C}$. Then for $w \in \mathbb{R}$ we have

$$|G(w)| = \left|\frac{1}{2\pi i}\int_C \frac{g(z)}{z^{w+1}}\,dz\right|$$
$$\le \frac{1}{2\pi}\int_{-\theta_0}^{\theta_0} (a^{-w}|g(ae^{i\theta})| + b^{-w}|g(be^{i\theta})|)\,d\theta$$
$$+ \frac{1}{2\pi}\int_a^b t^{-w-1}(|g(te^{-\theta_0})| + |g(te^{\theta_0})|)\,dt$$
$$\le b^{-w}\sup_{|z|=b}|g(z)| + a^{-w}\sup_{|z|=a}|g(z)| + \frac{1}{w}(a^{-w} - b^{-w})M,$$

where $M := \sup_{0<r<3}(|g(re^{\theta_0})| + |g(re^{-\theta_0})|) < \infty$. Consequently,

(3.1) $$|G(w)| \le b^{-w}\sup_{|z|=b}|g(z)| + a^{-w}\sup_{|z|=a}|g(z)| + \frac{1}{w}(a^{-w} - b^{-w})M \quad \text{for all } w \in \mathbb{R}.$$

Consider $w > 2$ and let $a := 1 - 1/w > 0$ and $b := 2$. It follows that

$$|G(w)| \le \mathrm{const}\cdot(1 + \sup_{|z|=1-1/w}|g(z)|).$$

Assumption (i) yields that

(3.2) $$|G(w)| = o(w^k) \quad \text{as } w \to \infty.$$

On the other hand, consider $w < -1$. Fix $1 < b < 2$. By letting $a \to 0^+$ in (3.1) we find

$$|G(w)| \le b^{-w}(M + \sup_{|z|=b}|g(z)|).$$

Fix $0 < \varepsilon < 1$, and consider $b = b_w := 1 + (\varepsilon/|w|)^{1/2}$. Then we obtain

$$|G(w)| \le e^{|\varepsilon w|^{1/2}}(M + \sup_{|z|=b_w}|g(z)|).$$

But assumption (ii) implies

$$\limsup_{w\to-\infty}\frac{\log^+\sup_{|z|=b_w}|g(z)|}{|w|^{1/2}} = 0.$$

So we have

$$\limsup_{w\to-\infty}\frac{\log^+|G(w)|}{|w|^{1/2}} \le \varepsilon^{1/2}$$

for all $0 < \varepsilon < 1$. Thus

(3.3) $$\log^+|G(w)| = o(|w|^{1/2}) \quad \text{as } w \to -\infty.$$

Estimates (3.2) and (3.3) make it possible to apply the result of Zarrabi [Z, Cor. 2.2] to conclude that $G(w)$ is a polynomial of degree $< k$.

To finish the proof of (1), let

$$g(z) = \sum_{n=-\infty}^{\infty} c_n(z-1)^n$$

be the Laurent series of $g(z)$ in some neighborhood of the point $z = 1$. We want to show that $c_n = 0$ for all $n \le -k$. To this end, we note for very small $\delta > 0$ that

$$G(w) = \frac{1}{2\pi i}\int_{|z|=\delta}\frac{g(1+z)}{(1+z)^{w+1}}\,dz$$
$$= \sum_{n=-\infty}^{\infty} c_n\frac{1}{2\pi i}\int_{|z|=\delta}\frac{z^n}{(1+z)^{w+1}}\,dz.$$

Since $G(w)$ is a polynomial of degree $< k$, its kth derivative is zero. Therefore,

$$\sum_{n=-\infty}^{\infty} c_n\frac{1}{2\pi i}\int_{|z|=\delta}\frac{z^n[\log(1+z)]^k}{(1+z)^{w+1}}\,dz \equiv 0.$$

It follows that $c_n = 0$ if $n \le -k-1$. Furthermore, using assumption (i) one has $c_{-k} = 0$. This finishes the proof of the assertion (1).

To prove (2), we take $\delta > 0$ so small that the positively oriented circles

$$C_j := \{z \in \mathbb{C} : |z - z_j| = \delta\}, \quad j = 1, \ldots, l,$$

are disjoint from each other and far from the origin. Let

$$p_j(w) := -\frac{1}{2\pi i}\int_{C_j}\frac{g(z)}{(zz_j^{-1})^{w+1}}\cdot\frac{dz}{z_j}, \quad j = 1, \ldots, l.$$

Then from (1) we find that each $p_j(w)$ is a polynomial of degree $< k$. Let $r > 1$ and take the path

$$C_r := \{|z| = r\} \cup \{|z| = r^{-1}\}.$$

Let $n \in \mathbb{Z}$. By Cauchy's theorem we have

$$\frac{1}{2\pi i}\int_{C_r}\frac{g(z)}{z^{n+1}}\,dz = \sum_{j=1}^{l}\frac{1}{2\pi i}\int_{C_j}\frac{g(z)}{z^{n+1}}\,dz = -\sum_{j=1}^{l} z_j^{-n}p_j(n).$$

On the other hand, by using the expansion of $g(z)$ near $z = 0$ and $z = \infty$ given in the theorem we have

$$a_n = -\frac{1}{2\pi i}\int_{C_r}\frac{g(z)}{z^{n+1}}\,dz = \sum_{j=1}^{l} z_j^{-n}p_j(n)$$

for all $n \in \mathbb{Z}$ with $|n| \ge 1$. Using this relation one sees that $g(z)$ is a rational function. ■

Remark. The example $g_0(z) := \exp\left(\frac{z+1}{z-1}\right)$ shows that (ii) of Theorem 3.1 cannot be weakened to an O-growth condition.

Theorem 3.1 has some important applications.

THEOREM 3.2. *Let $\mathbf{x} \equiv \{x_n : n \in \mathbb{Z}\}$ be elements in a Banach space E satisfying*

(i) $\|x_n\| = o(n^k)$ *as* $n \to \infty$ *for some* $k \in \mathbb{N}$ *and*
(ii) $\log^+ \|x_n\| = o(\sqrt{-n})$ *as* $n \to -\infty$.

If the Carleman transform

$$\widetilde{\mathbf{x}}(z) := \begin{cases} \sum_{n=1}^{\infty} x_n z^n, & |z| < 1, \\ -\sum_{n=-\infty}^{0} x_n z^n, & |z| > 1, \end{cases}$$

has a holomorphic extension to the unit circle except for a finite subset $S = \{z_1, \ldots, z_l\}$, then there exist polynomials $p_j(z)$ of degree smaller than k and elements $y_j \in E$ for $j = 1, \ldots, l$ such that

$$x_n = \sum_{j=1}^{l} p_j(n) z_j^{-n} y_j \quad \text{for all } n \in \mathbb{Z}.$$

Proof. The result will be obtained if Theorem 3.1(2) can be applied to the function $\widetilde{\mathbf{x}}(z)$. It is not hard to verify that condition (i) above implies condition (i) of Theorem 3.1 with $k+1$ replacing k. Moreover, (ii) implies condition (ii) of Theorem 3.1, and $\widetilde{\mathbf{x}}(z)$ is holomorphic at infinity (for a more careful computation we refer to the proof of Theorem 3.4). ■

Theorem 3.2 extends Theorem 3.13.5 of [H-P] due to Pólya and Hille. More interesting, we have the following discrete version of Corollary 2.2 of [Z] and Theorem 3.13.8 of [H-P] due to Pólya.

COROLLARY 3.3. *Let f be an E-valued entire function of exponential type zero such that the sequence $\{f(n)\}_{n\in\mathbb{Z}}$ satisfies*

(i) $\|f(n)\| = o(n^k)$ *as* $n \to \infty$ *for some* $k \in \mathbb{N}$ *and*
(ii) $\log^+ \|f(n)\| = o(\sqrt{-n})$ *as* $n \to -\infty$.

Then f is a polynomial of degree smaller than k.

Proof. Replacing f by the function

$$z \mapsto \frac{1}{z^k}\left(f(z) - \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} z^j\right)$$

we reduce the problem to the case $k = 1$. So we may assume $k = 1$ and $f(1) = 0$ and want to show that $f(z) \equiv 0$.

To this end, consider

$$F(w) := \frac{1}{2i} \int_{-i\infty}^{i\infty} \frac{z f(z)}{\sin \pi z} e^{wz}\, dz.$$

As shown in [L, p. 128], $F(\cdot)$ is holomorphic in the strip $\{w \in \mathbb{C} : |\mathrm{Im}\, w| < \pi\}$ and

$$F(w) = \begin{cases} -\sum_{n=1}^{\infty} (-1)^n n f(-n) e^{-nw}, & \mathrm{Re}\, w > 0, \\ \sum_{n=-\infty}^{1} (-1)^n n f(-n) e^{-nw}, & \mathrm{Re}\, w < 0. \end{cases}$$

Let $x_n := (-1)^n n f(-n)$, $n \in \mathbb{Z}$, and $\mathbf{x} := \{x_n : n \in \mathbb{Z}\}$. Then for the Carleman transform $\widetilde{\mathbf{x}}(z)$ of $\mathbf{x}$ we have $\widetilde{\mathbf{x}}(e^w) = -F(w)$ for $\mathrm{Re}\, w \neq 0$ and $|\mathrm{Im}\, w| < \pi$. This shows that $\widetilde{\mathbf{x}}(z)$ can be extended holomorphically to $\mathbb{C} \setminus \{-1\}$ by defining $\widetilde{\mathbf{x}}(e^{it}) := -F(it)$ for $t \in (-\pi, \pi)$. Therefore, Theorem 3.2 yields that $x_n = (-1)^n n y_0$ ($n \in \mathbb{Z}$) for some $y_0 \in E$. But $x_{-1} = f(1) = 0$ implies $y_0 = 0$, and hence $0 = x_n = (-1)^n n f(-n)$ for all $n \in \mathbb{Z}$. Turning to $F(\cdot)$ this implies $F(\cdot) \equiv 0$. By the uniqueness of Laplace transforms [H-P, p. 126] we find that $f(z) = 0$ for all $z \in i\mathbb{R}$. As a consequence of the Phragmén–Lindelöf principle we conclude that $f(z) \equiv 0$ as desired. ■

Remark. The reader should be aware that an essential ingredient of the proof of Theorem 3.1 is [Z, Corollary 2.2] which has been extended to our Corollary 3.3. So all of the results (Theorems 3.1–3.2, Cor. 3.3 and [Z, Cor. 2.2]) are equivalent. The following is another equivalent form for vector-valued functions on the real axis.

THEOREM 3.4. *Let $f(t)$ ($t \in \mathbb{R}$) be a continuous E-valued function satisfying*

(i) $\|f(t)\| = o(t^k)$ *as* $t \to \infty$ *for some* $k \in \mathbb{N}$ *and*
(ii) $\log^+ \|f(t)\| = o(\sqrt{n})$ *as* $t \to -\infty$.

If the Carleman transform

$$\widetilde{f}(\lambda) := \begin{cases} \int_0^{\infty} e^{-\lambda t} f(t)\, dt & \text{for } \mathrm{Re}\, \lambda > 0, \\ -\int_{-\infty}^{0} e^{-\lambda t} f(t)\, dt & \text{for } \mathrm{Re}\, \lambda < 0, \end{cases}$$

has a holomorphic extension to the imaginary axis except for a finite subset $S = \{\lambda_1, \ldots, \lambda_l\} \subset i\mathbb{R}$, then there exist polynomials $p_j(z)$ of degree smaller than k and elements $y_j \in E$ for $j = 1, \ldots, l$ such that

$$f(t) = \sum_{j=1}^{l} p_j(t) e^{\lambda_j t} y_j \quad \text{for all } t \in \mathbb{R}.$$

Proof. Let $\operatorname{Re}\lambda > 0$. By (i) we have

$$\|\widetilde{f}(\lambda)\| \le \text{const}\cdot \int_0^\infty t^k e^{-\operatorname{Re}\lambda t}\,dt = \text{const}\cdot(\operatorname{Re}\lambda)^{-k-1}.$$

Fix $\varepsilon > 0$. Then (ii) implies for $\operatorname{Re}\lambda < 0$ that

$$\begin{aligned}\|\widetilde{f}(\lambda)\| &\le \text{const}\cdot \int_0^\infty \exp(\operatorname{Re}\lambda t + \varepsilon t^{1/2})\,dt \\ &= \text{const}\cdot \int_0^\infty \exp\left(\frac{1}{2}\operatorname{Re}\lambda t + \varepsilon t^{1/2}\right)\exp\left(\frac{1}{2}\operatorname{Re}\lambda t\right)dt \\ &\le \text{const}\cdot \exp(\varepsilon^2(-\operatorname{Re}\lambda)^{-1})\int_0^\infty \exp\left(\frac{1}{2}\operatorname{Re}\lambda t\right)dt \\ &= \text{const}\cdot(-\operatorname{Re}\lambda)^{-1}\exp(\varepsilon^2(-\operatorname{Re}\lambda)^{-1}),\end{aligned}$$

where we use the inequality $-at + \varepsilon t^{1/2} \le \varepsilon^2/(4a)$ for $a, t > 0$.

To finish the proof we consider the function

$$g(z) := \widetilde{f}\left(\frac{1+z}{1-z}\right).$$

Then $g(z)$ is holomorphic in the entire plane excluding infinity except for finitely many points in the circle. Moreover, the above estimates imply that $g(z)$ satisfies conditions (i) and (ii) of Theorem 3.1 with $k+1$ replacing k. Theorem 3.1(2) can be applied. ∎

In conclusion we give two applications of the above results to operator theory. The following results refine previous theorems by Atzmon [A, Cor. 1], Esterle [E, Theorem 9.12], [Z, Theorem 4.3] and [Ze, Theorem 9].

COROLLARY 3.5. *Let $T \in \mathcal{L}(E)$ have spectrum in the unit circle. Let $x_0 \in E$. Assume that there exists some $k \in \mathbb{N}$ satisfying*

$$\lim_{n\to\infty}\left(\frac{\|T^n x_0\|}{n^k} + \frac{\log^+\|T^{-n}x_0\|}{\sqrt{n}}\right) = 0.$$

Then $(T-I)^k x_0 = 0$ if and only if the local resolvent $R(z,T)x_0$ ($|z| \ne 1$) has a holomorphic extension to $\mathbb{C}\setminus\{1\}$.

Proof. Let $x_n := T^{-n}x_0$ and consider $\mathbf{x} \equiv \{x_n : n \in \mathbb{Z}\}$. Assume that the local resolvent $R(z,T)x_0$ has a holomorphic extension to $\mathbb{C}\setminus\{1\}$. Since $\sigma(T)$ is contained in the unit circle, it follows that the Carleman transform $\widetilde{\mathbf{x}}(z)$ satisfies

$$\widetilde{\mathbf{x}}(z) = -zR(z,T)x_0 \quad \text{for } |z| \ne 1.$$

So $\widetilde{\mathbf{x}}(z)$ can be holomorphically extended to $\mathbb{C}\setminus\{1\}$ and Theorem 3.2 is applied to obtain a polynomial $p(z)$ of degree smaller than k and an element $y_0 \in E$ such that

$$T^{-n}x_0 = x_n = p(n)y_0 \quad \text{for all } n \in \mathbb{Z}.$$

To show that $(T-I)^k x_0 = 0$, one uses the equalities

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j j^l = 0, \quad l = 0, 1, \ldots, k-1.$$

They can be established by differentiating the equality

$$(1-t)^k = \sum_{j=0}^{k}\binom{k}{j}(-1)^j t^j.$$

Since the degree of $p(z)$ is smaller than k, from the above equalities we have

$$(T-I)^k x_0 = \sum_{j=0}^{k}\binom{k}{j}(-1)^j T^j x_0 = \sum_{j=0}^{k}\binom{k}{j}(-1)^j p(-j)y_0 = 0.$$

The converse implication is clear. ∎

COROLLARY 3.6. *Let A be a closed linear operator in a Banach space E with spectrum contained in the imaginary axis. Let $u(\cdot)$ be a mild solution of the equation*

$$u(t) = x_0 + A\int_0^t u(s)\,ds, \quad t \in \mathbb{R},$$

satisfying

$$\|u(t)\| = o(t^k) \quad \textit{as } t\to\infty \textit{ for some } k \in \mathbb{N} \textit{ and}$$
$$\log^+\|u(t)\| = o(|t|^{1/2}) \quad \textit{as } t \to -\infty.$$

If the local resolvent $R(z,A)x_0$ ($\operatorname{Re} z \ne 0$) can be extended holomorphically to $\mathbb{C}\setminus\{0\}$, then $x_0 \in D_\infty(A) := \bigcap_{n\in\mathbb{N}} D(A^n)$ and $A^k x_0 = 0$.

Proof. As seen in the proof of Theorem 2.5 (see also Ex. 2.6) we have

$$(z-A)\widetilde{u}(z) = x_0 \quad \text{for } \operatorname{Re} z \ne 0$$

and thus

$$\widetilde{u}(z) = R(z,A)x_0 \quad \text{for } \operatorname{Re} z \ne 0.$$

Since the local resolvent $R(z,A)x_0$ has a holomorphic extension to $\mathbb{C}\setminus\{0\}$, the above shows $\operatorname{Sp}(u) \subseteq \{0\}$. By Theorem 3.4 we can find a polynomial $p(t)$ of degree smaller than k such that

$$u(t) = p(t)x_0 \quad \text{for all } t \in \mathbb{R}.$$

From this one easily derives that $x_0 \in D_\infty(A)$ and $A^k x_0 = 0$. ∎

It is shown in [A-P, Theorem 3.11] that if A generates a C_0-group $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ such that $\sup_{t\in\mathbb{R}} \|T(t)x\| < \infty$ for all $x \in D_\infty(A)$, then $\sigma(A) = \{0\}$ implies $A = 0$. Using Corollary 3.6 we refine this result as follows.

COROLLARY 3.7. *Let A be a closed linear operator in a Banach space E such that $\sigma(A) \subseteq \{0\}$. If for each x_0 in a dense subset E_0 the equation*

$$u(t) = x_0 + A \int_0^t u(s)\,ds, \quad t \in \mathbb{R},$$

has a bounded mild solution $u(t)$, then $A = 0$.

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