STUDIA MATHEMATICA 115 (3) (1995)

# Second order unbounded parabolic equations in separated form

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**Abstract.** We prove existence and uniqueness of viscosity solutions of Cauchy problems for fully nonlinear unbounded second order Hamilton–Jacobi–Bellman–Isaacs equations defined on the product of two infinite-dimensional Hilbert spaces $H' \times H''$, where $H''$ is separable. The equations have a special "separated" form in the sense that the terms involving second derivatives are everywhere defined, continuous and depend only on derivatives with respect to $x'' \in H''$, while the unbounded terms are of first order and depend only on derivatives with respect to $x' \in H'$.

**1. Introduction.** Over the last decade substantial progress in the theory of Hamilton–Jacobi–Bellman-Isaacs equations in infinite-dimensional spaces has been made due to the introduction of the notion of viscosity solution. In particular, very general results have been obtained for equations with "unbounded" terms (see [7]–[10], [12], [22]–[26] for first order equations and [13], [17], [20], [21] for second order ones). In this paper we deal with special Cauchy problems defined on products of Hilbert spaces.

Suppose then that a real Hilbert space $H$ is written as the product of two Hilbert spaces, $H = H' \times H''$, where $H''$ is separable. Let $A$ be a maximal monotone (equivalently, $m$-accretive) operator in $H'$. Then $-A$ generates a strongly continuous semigroup $S(t)$ of contractions on $\overline{D(A)}$, where $D(A) \subseteq H'$ denotes the domain of $A$. We refer the reader to [1] or [3] for the information on nonlinear semigroups. For the sake of simplicity we will always assume that $0 \in A0$. Given $x \in H$ we will write $x = (x', x'')$, where $x' \in H'$ and $x'' \in H''$. Let $T > 0$.

Our Cauchy problem has the form

$$\text{(CP)}\quad \begin{cases} u_t + \langle Ax', D_{x'}u\rangle + F(t, x, D_x u, D^2_{x''}u) = 0 \quad \textbf{(E)} \\ \qquad\qquad\qquad \text{for } (t,x) \in (0,T) \times \Omega, \\ u(0,x) = \psi(x) \quad \text{for } x \in \Omega, \end{cases}$$

1991 *Mathematics Subject Classification*: Primary 49L25.

Research of the first author supported in part by Army Research Office Contract DAAL03-90-G-0102 and National Science Foundation Grant DMS90-02331.

where $\Omega$ is a relatively open subset of $D = \overline{D(A)} \times H'' \subseteq H$. All derivatives are understood in the Fréchet sense and therefore $F$ is defined on $[0,T] \times \Omega \times H \times \mathcal{S}(H)$, where $\mathcal{S}(H)$ stands for the space of all self-adjoint bounded operators on $H''$.

The definition of solution suitable for equations containing terms $\langle Ax', D_{x'}u\rangle$ with no additional assumptions on $A$ besides maximal monotonicity has been introduced by D. Tataru [22] and later refined by M. G. Crandall and P. L. Lions [9]. The definition uses test functions which are merely Lipschitz and this makes it impossible to extend it to second order equations. Equation **(E)** is of second order but the second order terms appear in the variable $x''$ which is "separated" from $x'$, appearing in the term $\langle Ax', D_{x'}u\rangle$ causing trouble. This suggests that Tataru's definition can be combined with the usual definition of viscosity solution to give an effective tool for investigating **(CP)**. Our definition of solutions of **(E)** is based on this observation.

In this paper we show that under appropriate assumptions problem **(CP)** has a unique viscosity solution. To prove this result we first develop appropriate tools to deal with second order terms. A key role is played by a version of a basic lemma of Lions (see [16], [6] and [13]). A perturbed optimization technique appropriate for **(E)** was obtained in [15]. Once these tools are available, the proof relies on techniques of Tataru, Crandall and Lions designed for first order unbounded problems ([22], [9]) combined with (by now standard) finite-dimensional methods, as extended to infinite dimensions by Lions in [16], and Ishii's method of constructing solutions ([12]). In order to make this paper self-contained we tried to present full proofs, even though some arguments follow closely those given in [9] and [12].

Finally, we remark that problems **(CP)** in separated form arise in finance theory. We will come back to this issue and study specific examples in a future publication.

**2. Definitions.** As explained in [22] and [9], for $\varphi \in C^1(H')$ it is natural to interpret the unbounded term $\langle Ax', D_{x'}\varphi\rangle$ in terms of the derivatives of $\varphi$ along the trajectories of $S(t)$. This motivates the following definition.

Definition 1. For a function $\Phi$ defined on $(0,T) \times \Omega$, where $\Omega \subseteq D$, and $(\widehat{t}, \widehat{x}) \in (0,T) \times \Omega$, define

$$\mathcal{D}_A^- \Phi(\widehat{t}, \widehat{x}) = \liminf_{\substack{(t,x)\to(\widehat{t},\widehat{x})\\ h\downarrow 0}} \frac{\Phi(t,x',x'') - \Phi(t,S(h)x',x'')}{h}$$

and



$$\mathcal{D}_A^+ \Phi(\widehat{t}, \widehat{x}) = \limsup_{\substack{(t,x)\to(\widehat{t},\widehat{x})\\ h\downarrow 0}} \frac{\Phi(t,x',x'') - \Phi(t,S(h)x',x'')}{h}.$$

Note that $B = \{0\} \times A \times \{0\}$ is an $m$-accretive operator in $\mathbb{R} \times H$ generating the semigroup $I \times S(t) \times I$ and then $\mathcal{D}_A^\pm = D_B^\pm$, where $D^\pm$ are defined as in [9]. Therefore, for example, for any function $\Phi$, $\mathcal{D}_A^-\Phi$ is lower semicontinuous.

Let $P$ denote the projection in $H'$ onto $\overline{D(A)}$. We write $B_r^K(x)$ for the closed ball in $K$ of radius $r$ centered at $x$. $a \vee b$ and $a \wedge b$ stand for $\max(a,b)$ and $\min(a,b)$. We will use standard notation to denote various function spaces. For instance, BUC stands for the space of all bounded uniformly continuous functions, and Lip denotes the class of Lipschitz continuous ones. Given a Lipschitz continuous function $\psi$, $L(\psi)$ will denote its Lipschitz constant. For less standard spaces we put

$$\mathrm{BUC}_x([0,T] \times \Omega) = \{u \in \mathrm{BUC}([0,T] \times \Omega) : u(t,\cdot) \text{ are uniformly continuous in } x, \text{ uniformly in } t \in [0,T]\},$$

$$C^{1,1,2}((0,T) \times H' \times H'') = \{u : u \text{ is once continuously differentiable in } t, x' \text{ and twice continuously differentiable in } x''\}.$$

Definition 2. We say that $\Phi(t,x) = \varphi(t,x) + \psi(x') \in C^{1,1,2}((0,T) \times H) + \mathrm{Lip}(H')$ is a *subtest* (respectively *supertest*) *function* if

$$(1) \quad \varphi(t,Px',x'') \le \varphi(t,x',x'') \quad\text{and}\quad \psi(Px') \le \psi(x') \quad \text{for } (t,x) \in (0,T) \times H,$$

respectively

$$(2) \quad \varphi(t,Px',x'') \ge \varphi(t,x',x'') \quad\text{and}\quad \psi(Px') \ge \psi(x') \quad \text{for } (t,x) \in (0,T) \times H.$$

In order to interpret the term "$D_{x'}\psi$" for merely Lipschitz $\psi$, for $(t,x,p,q,X) \in (0,T) \times \Omega \times H' \times H'' \times \mathcal{S}(H)$ and $\lambda > 0$ put

$$F_\lambda(t,x,p,q,X) = \inf\{F(t,x,p+r,q,X) : r \in H',\ \|r\| \le \lambda\}$$

and

$$F^\lambda(t,x,p,q,X) = \sup\{F(t,x,p+r,q,X) : r \in H',\ \|r\| \le \lambda\}.$$

The definition of viscosity solution, taken from [9], which we now adjust to the current setting, is the following.

Definition 3. Let $\Omega$ be a relatively open subset of $D$. Then $u \in \mathrm{USC}((0,T)\times\Omega)$ ($u \in \mathrm{LSC}((0,T)\times\Omega)$, respectively) is a *viscosity subsolution* (*supersolution*) of **(E)** in $(0,T) \times \Omega$ if for every subtest (supertest) function $\Phi = \varphi + \psi \in C^{1,1,2}((0,T) \times H' \times H'') + \mathrm{Lip}(H')$ and a local maximum (minimum) $(\widehat{t}, \widehat{x})$ of $u - \Phi$ over $(0,T) \times \Omega$ we have

(3) $$\varphi_t(\widehat{t},\widehat{x})+\mathcal{D}_A^-\Phi(\widehat{t},\widehat{x})+F_{L(\psi)}(\widehat{t},\widehat{x},D_x\varphi(\widehat{t},\widehat{x}),D^2_{x''}\varphi(\widehat{t},\widehat{x}))\le 0,$$

respectively

(4) $$\varphi_t(\widehat{t},\widehat{x})+\mathcal{D}_A^+\Phi(\widehat{t},\widehat{x})+F^{L(\psi)}(\widehat{t},\widehat{x},D_x\varphi(\widehat{t},\widehat{x}),D^2_{x''}\varphi(\widehat{t},\widehat{x}))\ge 0.$$

A function $u\in C((0,T)\times\Omega)$ is a *solution* of **(E)** if it is both a subsolution and a supersolution.

**3. Assumptions about** $F$. Now we state various assumptions on the function $F$ appearing in **(E)**. Suppose that $\Omega\subseteq D$ and $F:\ [0,T]\times\Omega\times H\times\mathcal{S}(H)\to\mathbb{R}$.

**F1.** $F$ is degenerate elliptic, that is, $F$ is nonincreasing in the last variable with respect to the standard ordering on $\mathcal{S}(H)$.

**F2.** $F$ is uniformly continuous on bounded subsets of $[0,T]\times\Omega\times H\times\mathcal{S}(H)$. Let $\omega_F$ denote the local modulus of continuity of $F$.

**F3.** There exists an increasing sequence $\{H_N\}_{N=1}^\infty$ of finite-dimensional subspaces of $H''$ such that $\bigcup_{N\ge1}H_N$ is dense in $H''$ and for every $(t,x,p)\in(0,T)\times\Omega\times H$ and $R>0$,

$$\sup\{|F(t,x,p,X+\lambda Q_N)-F(t,x,p,X)|:\ \|X\|\le R,\ |\lambda|\le R,\ X=P_NXP_N\}\to 0$$

as $N\to\infty$, where $P_N$ and $Q_N$ denote the orthogonal projections onto $H_N$ and $H_N^\perp$, respectively.

**F4.** There exists a modulus $\sigma_F$ such that for $t\in(0,T)$, $x,y\in\Omega$ and $\alpha>0$,

$$F(t,y,\alpha(x-y),-Y)-F(t,x,\alpha(x-y),X)\le\sigma_F(\|x-y\|+\alpha\|x-y\|^2)$$

whenever

$$-3\alpha\begin{pmatrix}I&0\\0&I\end{pmatrix}\le\begin{pmatrix}X&0\\0&Y\end{pmatrix}\le3\alpha\begin{pmatrix}I&-I\\-I&I\end{pmatrix}.$$

**F5.** There exists $\mu\in C^2(H)$ radial, nondecreasing, nonnegative, such that $D_x\mu$, $D^2_{x''}\mu$ are bounded and $\lim_{\|x\|\to\infty}\mu(x)=+\infty$, and a local modulus $\sigma_\infty$ such that for $(t,x,p,X)\in(0,T)\times\Omega\times H\times\mathcal{S}(H)$ and $r\in\mathbb{R}$,

$$|F(t,x,p,X)-F(t,x,p+rD_x\mu(x),X+rD^2_{x''}\mu(x))|\le\sigma_\infty(|r|,\|p\|+\|X\|).$$

Condition **F3** was introduced by Lions in [16]. **F4** is needed to handle the $x$-dependence, and **F5**, or some version of it, is required in proofs of comparison on unbounded domains to localize things in order to produce local extrema. Note that by Remark 3.4 of [5], **F4** implies **F1**, and that given **F2**, **F3** is equivalent to the following condition:

$$\begin{cases}\text{for every }(t,x,p)\in(0,T)\times\Omega\times H,\\ F(t,x,p,\cdot)-F(t,x,p,P_N\cdot P_N)\to0\ \text{ as }N\to\infty\\ \text{uniformly on bounded subsets of }\mathcal{S}(H'').\end{cases}$$

Let us fix a sequence $H_1\subseteq H_2\subseteq\ldots$ as in **F3**. In what follows, $P_N$ and $Q_N$ will always refer to the decomposition $H''=H_N\times H_N^\perp$.



## 4. Main result

THEOREM 4. *Suppose that $F$ satisfies conditions* **F1–F5**, *where* $\Omega=D$.

Comparison: *Let $u,-v\in\mathrm{USC}([0,T)\times D)$ and let $u,v$ be respectively a sub- and a supersolution of* (**E**) *on $(0,T)\times D$. Let $\eta\in\mathrm{UC}(D)$ and $\omega_1$ be a local modulus such that*

(5) $$u(t,x)-\eta(x)\le\omega_1(t,\|x\|),\qquad v(t,x)-\eta(x)\ge-\omega_1(t,\|x\|)$$

*for $(t,x)\in[0,T)\times D$. If $u,-v$ are bounded from above then $u\le v$ on $[0,T]\times D$. Moreover, there are a modulus $\omega_2$ and a local modulus $\omega_3$ such that*

(6) $$u(t,x)-v(s,y)\le\omega_2(\|x-y\|)+\omega_3(|t-s|,\|x\|+\|y\|)$$

*for $t,s\in[0,T)$, $x,y\in D$.*

Existence: *Assume that $\psi\in\mathrm{BUC}(D)$ satisfies*

(7) $$\psi(S(t)x',x'')\ \textit{is uniformly continuous in } t,\ \textit{uniformly for bounded } x\in D,$$

*and for every $R>0$,*

(8) $$\sup\{|F(t,x,p,X)|:(t,x)\in[0,T]\times D,\ \|p\|,\|X\|\le R\}=K_R<\infty.$$

*Then there exists a unique solution $u\in\mathrm{BUC}_x([0,T]\times D)$ of* (**CP**). *Moreover, for every $R>0$, $u$ is uniformly continuous in $t$ uniformly for $x\in D$, $\|x\|\le R$.*

**5. Tools.** Here we collect some tools and technical lemmas which will be required in the proof of the main Theorem 4. It is assumed that the reader is familiar with basic notions of the theory of viscosity solutions. For the finite-dimensional theory see [5], for "bounded" second order equations in infinite dimensions see [16].

LEMMA 5. *Suppose that a Hilbert space $K$ is written as a product $K=Z\times W$, where $Z$ is finite-dimensional. Let $P$ and $Q$ denote the orthogonal projections onto $Z$ and $W$, respectively. Let $u,-v\in\mathrm{USC}(\mathbb{R}\times K)$ and $\alpha,\beta>0$. Suppose that the map*

$$(t,x,s,y)\mapsto u(t,x)-v(s,y)-\tfrac12\alpha\|x-y\|^2-\tfrac12\beta(t-s)^2$$

*has a local maximum over* $\mathbb{R}\times K\times\mathbb{R}\times K$ *at* $(\widehat{t},\widehat{x},\widehat{s},\widehat{y})$. *Then there exist* $X,Y\in\mathcal{S}(K)$ *satisfying*

$$X=PXP,\qquad Y=PYP$$

*and*

$$-3\alpha\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}\le\begin{pmatrix} X & 0\\ 0 & Y\end{pmatrix}\le 3\alpha\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}$$

*such that*

$$(\beta(\widehat{t}-\widehat{s}),\alpha(\widehat{x}-\widehat{y}),X+2\alpha Q)\in\overline{\mathcal{P}}^{2,+}u(\widehat{t},\widehat{x}),$$
$$(\beta(\widehat{t}-\widehat{s}),\alpha(\widehat{x}-\widehat{y}),-Y-2\alpha Q)\in\overline{\mathcal{P}}^{2,-}v(\widehat{s},\widehat{y}).$$

For the definition of the second order parabolic jets $\overline{\mathcal{P}}^{2,+}$ and $\overline{\mathcal{P}}^{2,-}$ see [5]. This lemma is proved in [13]. The proof relies on a result fundamental for the theory of bounded second order equations in infinite-dimensional spaces, which can be found in [16] and [6]. Lemma 5 can also be deduced from a general theorem in [6].

The next tool we will need to investigate equations in separated form is an appropriate perturbation technique. For $(x',y')\in H'\times\overline{D(A)}$ define the *Tataru distance* $d$ (see [22], [25] and [9]) by

$$d(x',y')=\inf_{t\ge 0}\{\,t+\|x'-S(t)y'\|\}. \tag{9}$$

$d$ is almost a metric (it lacks symmetry). It is Lipschitz but not differentiable, and therefore is not suitable for second order equations. The following perturbation technique is a combination of Tataru's ([22]) and Ekeland–Lebourg–Stegall's ([11], [19]) and can be used to generate perturbed minima of a function defined on a product space. Its virtue is the property that the perturbation is smooth in the $x''$ variable. The following result is proved in [15]:

LEMMA 6. *Suppose that* $T\subset D$ *is closed and the projection of* $T$ *onto* $H''$ *is bounded, that is,* $\sup\{\|x''\|:x\in T\}<\infty$. *Let* $\Phi:T\to\mathbb{R}$ *be lower semicontinuous and bounded from below. Then for every* $\varepsilon>0$ *there exist* $\widehat{x}\in T$ *and* $p\in H''$ *satisfying* $\|p\|<\varepsilon$ *such that* $\Phi(\widehat{x})<\inf_T\Phi+\varepsilon$ *and the map*

$$x\mapsto\Phi(x)+\varepsilon d(x',\widehat{x}')+\langle p,x''\rangle$$

*has a strict global minimum over* $T$ *at* $\widehat{x}$.

In what follows we will apply this lemma repeatedly with $H''$ replaced by $\mathbb{R}\times H''$.

The second order jets $\overline{\mathcal{P}}^{2,+}$ and $\overline{\mathcal{P}}^{2,-}$ are designed for "bounded" problems and their elements cannot be just plugged in into "unbounded" equations. More precise information about these jets given by the next lemma will, however, enable us to do that to some extent in certain cases.



LEMMA 7. *Suppose that* $u,-v\in\mathrm{USC}((0,T)\times\Omega)$, *where* $\Omega\subseteq D$ *is relatively open. Suppose that* $\alpha,\beta>0$ *and that the map*

$$(t,x,s,y)\mapsto u(t,x)-v(s,y)-\tfrac12\alpha\|x''-y''\|^2-\tfrac12\beta(t-s)^2 \tag{10}$$

*has a strict local maximum over* $(0,T)\times\Omega\times(0,T)\times\Omega$ *at* $(\widehat{t},\widehat{x},\widehat{s},\widehat{y})$. *Then for every* $N\ge1$ *there exist* $X,Y\in\mathcal{S}(H)$ *satisfying*

$$X=P_NXP_N,\qquad Y=P_NYP_N \tag{11}$$

*and*

$$-3\alpha\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}\le\begin{pmatrix} X & 0\\ 0 & Y\end{pmatrix}\le 3\alpha\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}, \tag{12}$$

*and*

$$(\widehat{t}_n,\widehat{x}_n),\ (\widehat{s}_n,\widehat{y}_n)\in(0,T)\times\Omega,\quad p_n,q_n\in H'',\quad a_n,b_n\in\mathbb{R},$$
$$\gamma_n>0,\quad \varphi_n,\psi_n\in C^{1,2}(\mathbb{R}\times H'')$$

*such that, as* $n\to\infty$,

$$(\widehat{t}_n,\widehat{x}_n)\to(\widehat{t},\widehat{x}),\qquad(\widehat{s}_n,\widehat{y}_n)\to(\widehat{s},\widehat{y}), \tag{13}$$
$$u(\widehat{t}_n,\widehat{x}_n)\to u(\widehat{t},\widehat{x}),\qquad v(\widehat{s}_n,\widehat{y}_n)\to v(\widehat{s},\widehat{y}), \tag{14}$$
$$\|p_n\|\vee\|q_n\|\vee|a_n|\vee|b_n|\vee\gamma_n\to0, \tag{15}$$
$$(\varphi_n)_t(\widehat{t}_n,\widehat{x}''_n),(\psi_n)_t(\widehat{s}_n,\widehat{y}''_n)\to\beta(\widehat{t}-\widehat{s}), \tag{16}$$
$$D_{x''}\varphi_n(\widehat{t}_n,\widehat{x}''_n),D_{x''}\psi_n(\widehat{s}_n,\widehat{y}''_n)\to\alpha(\widehat{x}''-\widehat{y}''), \tag{17}$$
$$D^2_{x''}\varphi_n(\widehat{t}_n,\widehat{x}''_n)\to X+2\alpha Q_N,\qquad D^2_{x''}\psi_n(\widehat{s}_n,\widehat{y}''_n)\to -Y-2\alpha Q_N, \tag{18}$$

*and for every* $n=1,2,\ldots$,

(19) *the map* $(t,x)\mapsto u(t,x)-\gamma_n d(x',\widehat{x}'_n)-\langle p_n,x''\rangle-a_nt-\varphi_n(t,x'')$ *has a strict local maximum over* $(0,T)\times\Omega$ *at* $(\widehat{t}_n,\widehat{x}_n)$,

(20) *the map* $(s,y)\mapsto v(s,y)+\gamma_n d(y',\widehat{y}'_n)+\langle q_n,y''\rangle+b_ns-\psi_n(s,y'')$ *has a strict local minimum over* $(0,T)\times\Omega$ *at* $(\widehat{s}_n,\widehat{y}_n)$.

Proof. This proof uses ideas of [6], which was inspired by and simplifies [16]. Choose $r>0$ sufficiently small so that the map (10) has a strict global maximum at $(\widehat{t},\widehat{x},\widehat{s},\widehat{y})$ over

$$K=B_r^{\mathbb{R}}(\widehat{t})\times(B_r^H(\widehat{x})\cap D)\times B_r^{\mathbb{R}}(\widehat{s})\times(B_r^H(\widehat{y})\cap D).$$

For $(t,x'')\in B_r^{\mathbb{R}}(\widehat{t})\times B_r^{H''}(\widehat{x}'')$ and $(s,y'')\in B_r^{\mathbb{R}}(\widehat{s})\times B_r^{H''}(\widehat{y}'')$ define

$$\widehat{u}(t,x'')=\sup\{u(t,x',x''):x'\in\overline{D(A)},\ (x',x'')\in B_r^H(\widehat{x})\},$$
$$\widehat{v}(s,y'')=\inf\{v(s,y',y''):y'\in\overline{D(A)},\ (y',y'')\in B_r^H(\widehat{y})\}.$$

Then the map

$$(t,x'',s,y'')\mapsto\widehat{u}^*(t,x'')-\widehat{v}_*(s,y'')-\tfrac12\alpha\|x''-y''\|^2-\tfrac12\beta(t-s)^2$$

has a strict local maximum over $B_r^{\mathbb{R}}(\widehat{t}) \times B_r^{H''}(\widehat{x}'') \times B_r^{\mathbb{R}}(\widehat{s}) \times B_r^{H''}(\widehat{y}'')$ at $(\widehat{t}, \widehat{x}'', \widehat{s}, \widehat{y}'')$, where $\widehat{u}^*$ and $\widehat{v}_*$ denote the upper and lower semicontinuous envelopes of $\widehat{u}$ and $\widehat{v}$, respectively. Note that

$$\widehat{u}^*(\widehat{t}, \widehat{x}'') = u(\widehat{t}, \widehat{x}), \quad \widehat{v}_*(\widehat{s}, \widehat{y}'') = v(\widehat{s}, \widehat{y}). \tag{21}$$

By Lemma 5 there exist

$$\begin{gathered}(t_n, x_n'', s_n, y_n'') \in B_r^{\mathbb{R}}(\widehat{t}) \times B_r^{H''}(\widehat{x}'') \times B_r^{\mathbb{R}}(\widehat{s}) \times B_r^{H''}(\widehat{y}''),\\ \varphi_n, \psi_n \in C^{1,2}(\mathbb{R} \times H''), \quad X, Y \in \mathcal{S}(H),\end{gathered} \tag{22}$$

such that (11) and (12) hold, and as $n \to \infty$,

$$(t_n, x_n'', s_n, y_n'') \to (\widehat{t}, \widehat{x}'', \widehat{s}, \widehat{y}''), \tag{23}$$

$$\widehat{u}^*(t_n, x_n'') \to \widehat{u}^*(\widehat{t}, \widehat{x}''), \quad \widehat{v}_*(s_n, y_n'') \to \widehat{v}_*(\widehat{s}, \widehat{y}''), \tag{24}$$

$$D_{x''}\varphi_n(t_n, x_n''), D_{x''}\psi_n(s_n, y_n'') \to \alpha(\widehat{x}'' - \widehat{y}''), \tag{25}$$

$$(\varphi_n)_t(t_n, x_n''), (\psi_n)_t(s_n, y_n'') \to \beta(\widehat{t} - \widehat{s}), \tag{26}$$

$$D^2_{x''}\varphi_n(t_n, x_n'') \to X + 2\alpha Q_N, \quad D^2_{x''}\psi_n(s_n, y_n'') \to -Y - 2\alpha Q_N, \tag{27}$$

(28) the map $(t, x'') \mapsto \widehat{u}^*(t, x'') - \varphi_n(t, x'')$ has a strict maximum over $B_r^{\mathbb{R}}(\widehat{t}) \times B_r^{H''}(\widehat{x}'')$ at $(t_n, x_n'')$,

(29) the map $(s, y'') \mapsto \widehat{v}_*(s, y'') - \psi_n(s, y'')$ has a strict minimum over $B_r^{\mathbb{R}}(\widehat{s}) \times B_r^{H''}(\widehat{y}'')$ at $(s_n, y_n'')$.

We will see that $\varphi_n$, $\psi_n$, $X$ and $Y$ from (22) have the desired properties. Given $n$, for $\gamma > 0$ use Lemma 6 to find $a, b \in \mathbb{R}$, $p, q \in H''$, and $(\widetilde{t}_n, \widetilde{x}_n, \widetilde{s}_n, \widetilde{y}_n) \in K$ such that

$$|a| \vee |b| \vee \|p\| \vee \|q\| < \gamma, \tag{30}$$

and the map

$$\begin{aligned}(t, x, s, y) \mapsto\ & u(t, x) - v(s, y) - \varphi_n(t, x'') + \psi_n(s, y'')\\ & - \gamma d(x', \widetilde{x}_n') - \gamma d(y', \widetilde{y}_n') - at - bs - \langle p, x''\rangle - \langle q, y''\rangle\end{aligned} \tag{31}$$

has a strict maximum over $K$ at $(\widetilde{t}_n, \widetilde{x}_n, \widetilde{s}_n, \widetilde{y}_n)$. For all $(t, x, s, y) \in K$ we have

$$\begin{aligned}\widehat{u}^*(\widetilde{t}_n, \widetilde{x}_n'') - \widehat{v}_*(\widetilde{s}_n, \widetilde{y}_n'') - \varphi_n(\widetilde{t}_n, \widetilde{x}_n'') + \psi_n(\widetilde{s}_n, \widetilde{y}_n'')\\ \ge u(\widetilde{t}_n, \widetilde{x}_n) - v(\widetilde{s}_n, \widetilde{y}_n) - \varphi_n(\widetilde{t}_n, \widetilde{x}_n'') + \psi_n(\widetilde{s}_n, \widetilde{y}_n'')\\ \ge u(t, x) - v(s, y) - \varphi_n(t, x'') + \psi_n(s, y'')\\ - \langle p, x'' - \widetilde{x}_n''\rangle - \langle q, y'' - \widetilde{y}_n''\rangle - a(t - \widetilde{t}_n) - b(s - \widetilde{s}_n)\\ - \gamma d(x', \widetilde{x}_n') - \gamma d(y', \widetilde{y}_n')\\ \ge u(t, x) - v(s, y) - \varphi_n(t, x'') + \psi_n(s, y'') - C\gamma,\end{aligned} \tag{32}$$

where $C = C(r) > 0$. Take supremum over $(x', y')$ in (32) and then apply semicontinuous envelopes to obtain

$$\begin{aligned}\widehat{u}^*(\widetilde{t}_n, \widetilde{x}_n'') - \widehat{v}_*(\widetilde{s}_n, \widetilde{y}_n'') - \varphi_n(\widetilde{t}_n, \widetilde{x}_n'') + \psi_n(\widetilde{s}_n, \widetilde{y}_n'')\\ \ge u(\widetilde{t}_n, \widetilde{x}_n) - v(\widetilde{s}_n, \widetilde{y}_n) - \varphi_n(\widetilde{t}_n, \widetilde{x}_n'') + \psi_n(\widetilde{s}_n, \widetilde{y}_n'')\\ \ge \widehat{u}^*(t_n, x_n'') - \widehat{v}_*(s_n, y_n'') - \varphi_n(t_n, x_n'') + \psi_n(s_n, y_n'') - C\gamma.\end{aligned} \tag{33}$$

Now the strictness of the extrema in (28) and (29) together with (33) implies that

$$(\widetilde{t}_n, \widetilde{x}_n'', \widetilde{s}_n, \widetilde{y}_n'') \to (t_n, x_n'', s_n, y_n'') \tag{34}$$

and

$$\widehat{u}^*(\widetilde{t}_n, \widetilde{x}_n'') \to \widehat{u}^*(t_n, x_n''), \quad \widehat{v}_*(\widetilde{s}_n, \widetilde{y}_n'') \to \widehat{v}_*(s_n, y_n'') \tag{35}$$

as $\gamma \downarrow 0$. This, together with the middle inequality in (36), implies that

$$u(\widetilde{t}_n, \widetilde{x}_n) \to \widehat{u}^*(t_n, x_n''), \quad v(\widetilde{s}_n, \widetilde{y}_n) \to \widehat{v}_*(s_n, y_n'') \quad \text{as } \gamma \downarrow 0. \tag{36}$$

Now for each $n$ choose sufficiently small $\gamma_n < 1/n$ so that $(\widetilde{t}_n, \widetilde{x}_n'', \widetilde{s}_n, \widetilde{y}_n'')$ is at least $1/n$-close to $(t_n, x_n'', s_n, y_n'')$, and corresponding $a_n, b_n, p_n, q_n$ satisfying (30). Put $(\widehat{t}_n, \widehat{x}_n, \widehat{s}_n, \widehat{y}_n) = (\widetilde{t}_n, \widetilde{x}_n, \widetilde{s}_n, \widetilde{y}_n)$. Then (16)–(18) follow from (34) and (25)–(27). By construction and (23),

$$(\widehat{t}_n, \widehat{x}_n'', \widehat{s}_n, \widehat{y}_n'') \to (\widehat{t}, \widehat{x}'', \widehat{s}, \widehat{y}'') \quad \text{as } n \to \infty. \tag{37}$$

Now (21), (24) and (36) yield (14). Finally, the strictness of $(\widehat{t}, \widehat{x}, \widehat{s}, \widehat{y})$ with respect to (10) together with (37) and (14) implies (13). Finally, (19) and (20) follow from the extremization property of (31) and the proof of Lemma 7 is complete. ∎

**6. Doubling lemma.** Lemma 8 below is crucial for the proof of uniqueness of solutions of **(CP)**. It can be viewed as a restricted second order version of the Doubling Theorem 3.1 from [9], with a very specific subtest function for the doubled equation. See [14] for a general Doubling Theorem.

Suppose that $\Omega \subseteq D$ and $F, G : (0, T) \times \Omega \times H \times \mathcal{S}(H) \to \mathbb{R}$. For $\alpha > 0$, define a map $H(\alpha) : (0, T) \times \Omega \times (0, T) \times \Omega \times H \times H \to \mathbb{R}$ via

$$\begin{aligned}H(\alpha)(t, x, s, y, p, q) = \inf\{F(t, x, p, X) - G(s, y, q, -Y) :\\ X, Y \in \mathcal{S}(H) \text{ satisfy (11) and (12) for some } N \ge 1\}.\end{aligned} \tag{38}$$

Note that if $F$ and $G$ satisfy condition **F2** then for every $\alpha > 0$, $H(\alpha)$ is uniformly continuous on bounded subsets of $(0, T) \times \Omega \times (0, T) \times \Omega \times H \times H$.

LEMMA 8. *Suppose that $\Omega$ is relatively open in $D$, $\alpha, \beta > 0$ and that $F$ and $G$ satisfy conditions* **F1–F3**. *Let $u, -v \in \mathrm{USC}((0, T) \times \Omega)$ solve*

(39) $u_t + \langle Ax', D_{x'}u\rangle + F(t,x,D_xu,D^2_{x''}u) \le 0$ *for* $(t,x)\in(0,T)\times\Omega$

*and*

(40) $v_t + \langle Ax', D_{x'}v\rangle + G(t,x,D_xv,D^2_{x''}v) \ge 0$ *for* $(t,x)\in(0,T)\times\Omega$.

*Suppose that* $a,b\in\mathbb{R}$, $p,q\in H''$ *and* $\psi\in \mathrm{Lip}(H'\times H')$ *satisfies*

$$\psi(Px',Py')\le\psi(x',y') \quad \textit{for } x',y'\in H.$$

*If the map*

$$(41)\qquad (t,x,s,y)\mapsto u(t,x)-v(s,y)-\tfrac12\alpha\|x-y\|^2-\tfrac12\beta(t-s)^2 \\ -at-bs-\langle p,x''\rangle-\langle q,y''\rangle-\psi(x',y')$$

*has a strict local maximum over* $(0,T)\times\Omega\times(0,T)\times\Omega$ *at* $(\widehat t,\widehat x,\widehat s,\widehat y)$, *then*

$$(42)\qquad a+b+D^-_{A\times A}\psi(\widehat x',\widehat y') \\ +H(\alpha)_{L(\psi)}(\widehat t,\widehat x,\widehat s,\widehat y,\alpha(\widehat x-\widehat y)+p,\alpha(\widehat x-\widehat y)-q)\le 0,$$

*where* $H(\alpha)$ *is defined by* (38) *and for* $(t,x,s,y,p,q)\in(0,T)\times\Omega\times(0,T)\times\Omega\times H\times H$ *and* $\lambda>0$,

$$H(\alpha)_\lambda(t,x,s,y,p,q) \\ =\inf\{H(\alpha)(t,x,s,y,p+r,q+z): r,z\in H',\ \|r\|,\|z\|\le\lambda\}.$$

Proof. We will follow the notation from the proof of Theorem 3.1 in [9] as closely as possible. First of all choose $r>0$ small enough so that

$$B^{\mathbb{R}^2}_{2r}(\widehat t,\widehat s)\subseteq(0,T)\times(0,T) \quad\text{and}\quad B^{H^2}_{2r}(\widehat x,\widehat y)\cap\Omega^2=B^{H^2}_{2r}(\widehat x,\widehat y)\cap D^2$$

and the map (41) has at $(\widehat t,\widehat x,\widehat s,\widehat y)$ a strict global maximum over $B^{\mathbb{R}^2}_{2r}(\widehat t,\widehat s)\times(B^{H^2}_{2r}(\widehat x,\widehat y)\cap D^2)$. Put

$$I=B^{\mathbb{R}^2}_{2r}(\widehat t,\widehat s),\quad N=B^{H^2}_{2r}(\widehat x,\widehat y)\cap\Omega^2 \quad\text{and}\quad N'=B^{H'^2}_{2r}(\widehat x',\widehat y')\cap\overline{D(A)}^2.$$

$N'$ can be desribed as the projection of $N$ onto $H'\times H'$.

For $\varepsilon,\delta>0$, $(t,s)\in I$, $(x,y)\in N$ and $(\xi,\eta)\in N'$ define

$$u_1(t,x)=u(t,x)-\langle p,x''\rangle-at,\quad v_1(s,y)=v(s,y)+\langle q,y''\rangle+bs,$$
$$\Psi(t,x,s,y)=u_1(t,x)-v_1(s,y)-\tfrac12\alpha\|x-y\|^2-\tfrac12\beta(t-s)^2-\psi(x',y'),$$
$$\Psi_\varepsilon(t,x,s,y,\xi,\eta)=u_1(t,x)-v_1(s,y)-\tfrac12\alpha\|x''-y''\|^2-\tfrac12\beta(t-s)^2 \\ -\tfrac12\varepsilon^{-1}(\|x'-\xi\|^2+\|y'-\eta\|^2)-\tfrac12\alpha\|\xi-\eta\|^2-\psi(\xi,\eta),$$
$$\Psi_{\varepsilon,\delta}(t,x,s,y,\xi,\eta)=\Psi_\varepsilon(t,x,s,y,\xi,\eta)-\delta\|A^\circ\xi\|-\delta\|A^\circ\eta\|,$$

where

$$\|A^\circ\xi\|=\begin{cases}\min\{\|y\|:(\xi,y)\in A\} & \text{if } \xi\in D(A),\\ \infty & \text{otherwise.}\end{cases}$$



Recall that the map $\xi\mapsto\|A^\circ\xi\|$ is lower semicontinuous. Put

$$M=\sup_{I\times N}\Psi=\Psi(\widehat t,\widehat x,\widehat s,\widehat y),\quad M_\varepsilon=\sup_{I\times N\times N'}\Psi_\varepsilon,\quad M_{\varepsilon,\delta}=\sup_{I\times N\times N'}\Psi_{\varepsilon,\delta}.$$

Clearly $M\le M_\varepsilon$ and $M_{\varepsilon,\delta}\le M_\varepsilon$. Exactly as in [9] one verifies that

$$(43)\qquad M_{\varepsilon,\delta}\uparrow M_\varepsilon \quad\text{as } \delta\downarrow 0 \quad\text{and}\quad M_\varepsilon\downarrow M \quad\text{as } \varepsilon\downarrow 0.$$

For every $\varepsilon>0$ use (43) and perturbed optimization (Lemma 6) to find $\delta=\delta(\varepsilon)>0$, $a_\varepsilon,b_\varepsilon\in\mathbb{R}$, $p_\varepsilon,q_\varepsilon\in H''$, $(t_\varepsilon,s_\varepsilon)\in I$ and $(x_\varepsilon,y_\varepsilon,\xi_\varepsilon,\eta_\varepsilon)\in N\times N'$ such that $|a_\varepsilon|\vee|b_\varepsilon|\vee\|p_\varepsilon\|\vee\|q_\varepsilon\|\le\varepsilon$, $\delta(\varepsilon)\downarrow 0$ as $\varepsilon\downarrow 0$,

$$(44)\qquad M_\varepsilon-\varepsilon^3\le\Psi_{\varepsilon,\delta(\varepsilon)}(t_\varepsilon,x_\varepsilon,s_\varepsilon,y_\varepsilon,\xi_\varepsilon,\eta_\varepsilon)$$

and the map

$$(45)\qquad (t,x,s,y,\xi,\eta)\mapsto\Psi_{\varepsilon,\delta(\varepsilon)}(t,x,s,y,\xi,\eta)-\langle p_\varepsilon,x''\rangle-\langle q_\varepsilon,y''\rangle-a_\varepsilon t-b_\varepsilon s \\ -\varepsilon d(x',x'_\varepsilon)-\varepsilon d(y',y'_\varepsilon)-\varepsilon d(\xi,\xi_\varepsilon)-\varepsilon d(\eta,\eta_\varepsilon)$$

has a strict maximum over $I\times N\times N'$ at $(t_\varepsilon,x_\varepsilon,s_\varepsilon,y_\varepsilon,\xi_\varepsilon,\eta_\varepsilon)$. Note that $\xi_\varepsilon,\eta_\varepsilon\in D(A)$.

Next one shows exactly as in [9] that

$$(46)\qquad (t_\varepsilon,x_\varepsilon,s_\varepsilon,y_\varepsilon,\xi_\varepsilon,\eta_\varepsilon)\to(\widehat t,\widehat x,\widehat s,\widehat y,\widehat x',\widehat y') \quad\text{as } \varepsilon\downarrow 0.$$

Therefore from now on we can assume that $\varepsilon>0$ is chosen so small that

$$(t_\varepsilon,s_\varepsilon)\in B^{\mathbb{R}^2}_r(\widehat t,\widehat s),\qquad (x_\varepsilon,y_\varepsilon)\in B^{H^2}_r(\widehat x,\widehat y)\cap D^2$$

and

$$(\xi_\varepsilon,\eta_\varepsilon)\in B^{H'^2}_r(\widehat x',\widehat y')\cap D(A)^2.$$

Now fix $\xi=\xi_\varepsilon$ and $\eta=\eta_\varepsilon$ in (45) and observe that the map

$$(t,x,s,y)\mapsto\big(u_1(t,x)-\varepsilon d(x',x'_\varepsilon)-\langle p_\varepsilon,x''\rangle-a_\varepsilon t-\tfrac12\varepsilon^{-1}\|x'-\xi_\varepsilon\|^2\big) \\ -\big(v_1(s,y)+\varepsilon d(y',y'_\varepsilon)+\langle q_\varepsilon,y''\rangle+b_\varepsilon s+\tfrac12\varepsilon^{-1}\|y'-\eta_\varepsilon\|^2\big) \\ -\tfrac12\alpha\|x''-y''\|^2-\tfrac12\beta(t-s)^2$$

has a strict local maximum over $(0,T)\times\Omega\times(0,T)\times\Omega$ at $(t_\varepsilon,x_\varepsilon,s_\varepsilon,y_\varepsilon)$. From Lemma 7 for every $N\ge1$ there exist $X,Y\in\mathcal{S}(H)$ satisfying (11) and (12) and

$$(t_n,x_n),(s_n,y_n)\in(0,T)\times\Omega,\quad p_n,q_n\in H'',\quad a_n,b_n\in\mathbb{R}, \\ \gamma_n>0,\quad \varphi_n,\psi_n\in C^{1,2}(\mathbb{R}\times H'')$$

such that, as $n\to\infty$,

$$(47)\qquad (t_n,x_n,s_n,y_n)\to(t_\varepsilon,x_\varepsilon,s_\varepsilon,y_\varepsilon),$$
$$(48)\qquad \|p_n\|\vee\|q_n\|\vee|a_n|\vee|b_n|\vee\gamma_n\to 0,$$
$$(49)\qquad (\varphi_n)_t(t_n,x''_n),(\psi_n)_t(s_n,y''_n)\to\beta(t_\varepsilon-s_\varepsilon),$$
$$(50)\qquad D_{x''}\varphi_n(t_n,x''_n),D_{x''}\psi_n(s_n,y''_n)\to\alpha(x''_\varepsilon-y''_\varepsilon),$$

(51) $$D^2_{x''}\varphi_n(t_n,x''_n)\to X+2\alpha Q_N,\quad D^2_{x''}\psi_n(s_n,y''_n)\to -Y-2\alpha Q_N,$$

and such that for every $n=1,2,\ldots$ the map

$$\begin{aligned}(t,x)\mapsto u(t,x)&-\langle p,x''\rangle-at\\&-\varepsilon d(x',x'_\varepsilon)-\langle p_\varepsilon,x''\rangle-a_\varepsilon t-\tfrac12\varepsilon^{-1}\|x'-\xi_\varepsilon\|^2\\&-\gamma_n d(x',x'_n)-\langle p_n,x''\rangle-a_nt-\varphi_n(t,x'')\end{aligned}$$

has a (strict) local maximum over $(0,T)\times\Omega$ at $(t_n,x_n)$ and the map

$$\begin{aligned}(s,y)\mapsto v(s,y)&+\langle q,y''\rangle+bs\\&+\varepsilon d(y',y'_\varepsilon)+\langle q_\varepsilon,y''\rangle+b_\varepsilon s+\tfrac12\varepsilon^{-1}\|y'-\eta_\varepsilon\|^2\\&+\gamma_n d(y',y'_n)+\langle q_n,y''\rangle+b_ns-\psi_n(s,y'')\end{aligned}$$

has a local minimum at $(s_n,y_n)$.

From the extremization of $u$ and (39) ($v$ and (40), respectively) and Lemma 2.2 of [9] we obtain

$$\begin{aligned}&(\varphi_n)_t(t_n,x''_n)+a+a_\varepsilon+a_n+\varepsilon^{-1}\langle A^\circ\xi_\varepsilon,x'_n-\xi_\varepsilon\rangle\\&+F_{\varepsilon+\gamma_n}(t_n,x_n,\varepsilon^{-1}(x'_n-\xi_\varepsilon)+D_{x''}\varphi_n(t_n,x''_n)+p+p_\varepsilon+p_n,D^2_{x''}\varphi_n(t_n,x''_n))\\&\hspace{20em}\le\varepsilon+\gamma_n\end{aligned}$$

and

$$\begin{aligned}&(\psi_n)_t(s_n,y''_n)-b-b_\varepsilon-b_n+\varepsilon^{-1}\langle A^\circ\eta_\varepsilon,\eta_\varepsilon-y'_n\rangle\\&+G^{\varepsilon+\gamma_n}(s_n,y_n,\varepsilon^{-1}(\eta_\varepsilon-y'_n)+D_{x''}\psi_n(t_n,x''_n)-q-q_\varepsilon-q_n,D^2_{x''}\psi_n(t_n,x''_n))\\&\hspace{20em}\ge-\varepsilon-\gamma_n.\end{aligned}$$

Let $n\to\infty$ and use (47)–(51) to conclude that

(52) $$\begin{aligned}&\beta(t_\varepsilon-s_\varepsilon)+a+a_\varepsilon+\varepsilon^{-1}\langle A^\circ\xi_\varepsilon,x'_\varepsilon-\xi_\varepsilon\rangle\\&\quad+F_\varepsilon(t_\varepsilon,x_\varepsilon,\varepsilon^{-1}(x'_\varepsilon-\xi_\varepsilon)+\alpha(x''_\varepsilon-y''_\varepsilon)+p+p_\varepsilon,X+2\alpha Q_N)\le\varepsilon\end{aligned}$$

and

(53) $$\begin{aligned}&\beta(t_\varepsilon-s_\varepsilon)-b-b_\varepsilon+\varepsilon^{-1}\langle A^\circ\eta_\varepsilon,\eta_\varepsilon-y'_\varepsilon\rangle\\&\quad+G^\varepsilon(s_\varepsilon,y_\varepsilon,\varepsilon^{-1}(\eta_\varepsilon-y'_\varepsilon)+\alpha(x''_\varepsilon-y''_\varepsilon)-q-q_\varepsilon,-Y-2\alpha Q_N)\ge-\varepsilon.\end{aligned}$$

From (44) and $\Psi_{\varepsilon,\delta}\le\Psi_\varepsilon$ we obtain

$$\Psi_\varepsilon(t_\varepsilon,x,s_\varepsilon,y,\xi,\eta)\le\Psi_\varepsilon(t_\varepsilon,x_\varepsilon,s_\varepsilon,y_\varepsilon,\xi_\varepsilon,\eta_\varepsilon)+\varepsilon^3\quad\text{for }(x,y,\xi,\eta)\in N\times N'.$$

Putting $x=x_\varepsilon$, $y=y_\varepsilon$ in this inequality, using the fact that $\psi$ is a subtest function, and arguing as in [9] (Lemma A.8 in [9]) we get

(54) $$\limsup_{\varepsilon\downarrow0}\|\varepsilon^{-1}(x'_\varepsilon-\xi_\varepsilon,y'_\varepsilon-\eta_\varepsilon)-\alpha(x'_\varepsilon-y'_\varepsilon)\|\le L(\psi).$$

From (54) it follows that $\varepsilon^{-1}(x'_\varepsilon-\xi_\varepsilon,y'_\varepsilon-\eta_\varepsilon)$ remains bounded as $\varepsilon\downarrow0$. Using this, **F2**, and **F3** in (52) yields



(55) $$\begin{aligned}&\beta(t_\varepsilon-s_\varepsilon)+a+a_\varepsilon+\varepsilon^{-1}\langle A^\circ\xi_\varepsilon,x'_\varepsilon-\xi_\varepsilon\rangle\\&\quad+F(t_\varepsilon,x_\varepsilon,\varepsilon^{-1}(x'_\varepsilon-\xi_\varepsilon)+\alpha(x''_\varepsilon-y''_\varepsilon)+p,X)\le\varepsilon+\sigma_1(\varepsilon)+\sigma_2(\varepsilon,N),\end{aligned}$$

where for any fixed $\varepsilon$, $\sigma_2(\varepsilon,N)\to0$ as $N\to\infty$ and $\sigma_1(\varepsilon)\to0$ as $\varepsilon\downarrow0$ and is independent of $N$. Similarly,

(56) $$\begin{aligned}&\beta(t_\varepsilon-s_\varepsilon)-b-b_\varepsilon+\varepsilon^{-1}\langle A^\circ\eta_\varepsilon,\eta_\varepsilon-y'_\varepsilon\rangle\\&\quad+G(s_\varepsilon,y_\varepsilon,\varepsilon^{-1}(\eta_\varepsilon-y'_\varepsilon)+\alpha(x''_\varepsilon-y''_\varepsilon)-q,-Y)\ge-\varepsilon-\sigma_1(\varepsilon)-\sigma_2(\varepsilon,N).\end{aligned}$$

Putting $t=t_\varepsilon$, $x=x_\varepsilon$, $s=s_\varepsilon$ and $y=y_\varepsilon$ in (45) yields

$$\begin{aligned}\psi(\xi,\eta)&+\tfrac12\alpha\|\xi-\eta\|^2+\tfrac12\varepsilon^{-1}(\|x'_\varepsilon-\xi\|^2+\|y'_\varepsilon-\eta\|^2)\\&\quad+\delta(\varepsilon)(\|A^\circ\xi\|+\|A^\circ\eta\|)+\varepsilon d(\xi,\xi_\varepsilon)+\varepsilon d(\eta,\eta_\varepsilon)\\&\ge\psi(\xi_\varepsilon,\eta_\varepsilon)+\tfrac12\alpha\|\xi_\varepsilon-\eta_\varepsilon\|^2+\tfrac12\varepsilon^{-1}(\|x'_\varepsilon-\xi_\varepsilon\|^2+\|y'_\varepsilon-\eta_\varepsilon\|^2)\\&\quad+\delta(\varepsilon)(\|A^\circ\xi_\varepsilon\|+\|A^\circ\eta_\varepsilon\|)\end{aligned}$$

for $(\xi,\eta)\in N'\times N'$. Arguing as in Step 5 of the proof of Theorem 3.1 in [9] this implies that

(57) $$D^-_{A\times A}\psi(\xi_\varepsilon,\eta_\varepsilon)\le\varepsilon^{-1}(\langle A^\circ\xi_\varepsilon,x'_\varepsilon-\xi_\varepsilon\rangle+\langle A^\circ\eta_\varepsilon,y'_\varepsilon-\eta_\varepsilon\rangle)+2\varepsilon.$$

Subtracting (56) from (57) and using (57) leads to

$$\begin{aligned}&a+a_\varepsilon+b+b_\varepsilon+D^-_{A\times A}\psi(\xi_\varepsilon,\eta_\varepsilon)\\&+H(\alpha)(t_\varepsilon,x_\varepsilon,s_\varepsilon,y_\varepsilon,\varepsilon^{-1}(x'_\varepsilon-\xi_\varepsilon)+\alpha(x''_\varepsilon-y''_\varepsilon)+p,\varepsilon^{-1}(\eta_\varepsilon-y'_\varepsilon)+\alpha(x''_\varepsilon-y''_\varepsilon)-q)\\&\hspace{20em}\le4\varepsilon+2\sigma_1(\varepsilon)+2\sigma_2(\varepsilon,N).\end{aligned}$$

But combining this with (54) yields

(58) $$\begin{aligned}&a+a_\varepsilon+b+b_\varepsilon+D^-_{A\times A}\psi(\xi_\varepsilon,\eta_\varepsilon)\\&\quad+H(\alpha)_{L(\psi)+\sigma_3(\varepsilon)}(t_\varepsilon,x_\varepsilon,s_\varepsilon,y_\varepsilon,\alpha(x_\varepsilon-y_\varepsilon)+p,\alpha(x_\varepsilon-y_\varepsilon)-q)\\&\hspace{20em}\le4\varepsilon+2\sigma_1(\varepsilon)+2\sigma_2(\varepsilon,N),\end{aligned}$$

where $\lim_{\varepsilon\downarrow0}\sigma_3(\varepsilon)=0$. Since $D^-_{A\times A}\psi$ is lower semicontinuous, from (46) we have

(59) $$D^-_{A\times A}\psi(\widehat{x}',\widehat{y}')\le\liminf_{\varepsilon\downarrow0}D^-_{A\times A}\psi(\xi_\varepsilon,\eta_\varepsilon).$$

Now let $N\to\infty$ in (58) and then $\varepsilon\downarrow0$. (42) follows from (46) and (59). ■

**7. Proof of comparison.** It will be convenient to state the following simple lemma separately.

LEMMA 9. *Suppose that $\mu\in C^2(H)$ is as in* **F5** *and $\delta>0$. Suppose that $u$ and $v$ are a subsolution and a supersolution of* (**E**), *respectively. Define*

(60) $$\widetilde{u}(t,x)=u(t,x)-\delta\mu(x),\quad\widetilde{v}(t,x)=v(t,x)+\delta\mu(x).$$

*Then*

$$\widetilde{u}_t + \langle Ax', D_{x'}\widetilde{u}\rangle + \overline{F}(t,x,D_x\widetilde{u},D^2_{x''}\widetilde{u}) \le 0 \quad \text{for } (t,x)\in(0,T)\times\Omega$$

*and*

$$\widetilde{v}_t + \langle Ax', D_{x'}\widetilde{v}\rangle + \underline{F}(t,x,D_x\widetilde{v},D^2_{x''}\widetilde{v}) \ge 0 \quad \text{for } (t,x)\in(0,T)\times\Omega,$$

*where for* $(t,x,p,X)\in(0,T)\times D\times H\times \mathcal{S}(H)$,

$$\overline{F}(t,x,p,X) := F(t,x,p+\delta D_x\mu(x), X+\delta D^2_{x''}\mu(x)) \tag{61}$$

*and*

$$\underline{F}(t,x,p,X) := F(t,x,p-\delta D_x\mu(x), X-\delta D^2_{x''}\mu(x)). \tag{62}$$

Proof. The lemma follows immediately from the results of [9], where it is shown that $\mu$ is a subtest function and $\mathcal{D}^-_A\mu\ge 0$ etc. ■

Proof of Theorem 4. *Comparison.* We argue by contradiction. Suppose that for some $(\tau,z)\in(0,T)\times D$,

$$u(\tau,z)-v(\tau,z)=2\gamma>0. \tag{63}$$

Choose $c_0>0$ such that

$$2c_0T\le \tfrac12\gamma. \tag{64}$$

For $\nu>0$ put

$$u_\nu(t,x)=u(t,x)-\frac{\nu}{T-t},\qquad v_\nu(t,x)=v(t,x)+\frac{\nu}{T-t}.$$

Notice that $u_\nu$ and $v_\nu$ are a sub- and a supersolution of **(E)**, respectively. For $\alpha,\beta,\delta>0$ define $\Phi:[0,T]\times D\times[0,T]\times D\to\mathbb{R}\cup\{-\infty\}$ by

$$\Phi(t,x,s,y)=u_\nu(t,x)-v_\nu(s,y)-\tfrac12\alpha\|x-y\|^2-\tfrac12\beta(t-s)^2 - \delta\mu(x)-\delta\mu(y)-c_0(t+s),$$

where $\mu$ is provided by **F5**. In view of (63) and (65), for $\delta,\nu>0$ sufficiently small

$$\sup\Phi\ge\Phi(\tau,z,\tau,z)>\gamma. \tag{65}$$

$\Phi$ is bounded from above by assumption. Since $\mu(x)\to\infty$ as $\|x\|\to\infty$, for every $\delta>0$ there exists $R=R(\delta)>0$ such that $\Phi(t,x,s,y)\le-\gamma$ whenever $t,s\in[0,T]$ and $x,y\in D$ satisfy $\|x\|\vee\|y\|\ge R$. Put $B=[0,T]\times(D\cap B^H_R(0))$. For every $\varepsilon>0$, by perturbed optimization (Lemma 6), there exist $(\widehat t,\widehat x),(\widehat s,\widehat y)\in B$, $p,q\in H''$ and $a,b\in\mathbb{R}$ satisfying $\|p\|\vee\|q\|\vee|a|\vee|b|<\varepsilon$ such that

$$\Phi(\widehat t,\widehat x,\widehat s,\widehat y)\ge\sup_{B\times B}\Phi-\tfrac12\varepsilon \tag{66}$$



and the map

$$B\times B\ni(t,x,s,y)\mapsto\Phi(t,x,s,y)-\tfrac12\varepsilon d(x',\widehat x')-\tfrac12\varepsilon d(y',\widehat y') - \langle p,x''\rangle-\langle q,y''\rangle-at-bs \tag{67}$$

has a strict maximum over $B\times B$ at $(\widehat t,\widehat x,\widehat s,\widehat y)$.

If $\varepsilon<\gamma$ then from (65) and (66), $\Phi(\widehat t,\widehat x,\widehat s,\widehat y)\ge\gamma-\frac12\varepsilon>\frac12\gamma$. Therefore

$$\tfrac12\gamma+\tfrac12\alpha\|\widehat x-\widehat y\|^2+\tfrac12\beta(\widehat t-\widehat s)^2+\delta\mu(\widehat x)+\delta\mu(\widehat y)\le u(\widehat t,\widehat x)-v(\widehat s,\widehat y)\le C, \tag{68}$$

where $C=\sup_{[0,T)\times D}u-\inf_{[0,T)\times D}v<\infty$. Obviously $0\le\widehat t,\widehat s<T$. If $\varepsilon>0$ is small enough (depending on $R=R(\delta)$) then $\|\widehat x\|\vee\|\widehat y\|<R$. We will show that for any fixed $\delta$, if $\alpha$ and $\beta$ are sufficiently large, then $\widehat t,\widehat s>0$. Suppose for instance that $\widehat t=0$. From the middle inequality in (68) and (5),

$$\tfrac12\gamma\le\eta(\widehat x)-v(\widehat s,\widehat y)\le\eta(\widehat x)-\eta(\widehat y)+\omega_1(\widehat s,\|\widehat y\|) \le\omega_\eta(\|\widehat x-\widehat y\|)+\omega_1(\widehat s,\|\widehat y\|), \tag{69}$$

where $\omega_\eta$ denotes the modulus of continuity of $\eta$. From construction it follows that for fixed $\delta>0$, $\widehat y$ remains bounded, uniformly in $\alpha,\beta,\varepsilon$, and from (68),

$$\|\widehat x-\widehat y\|,|\widehat t-\widehat s|\to0\quad\text{as }\alpha,\beta\to\infty.$$

Hence (69) leads to a contradiction for $\alpha,\beta$ large, whence we can assume that $\widehat t>0$. Similarly one shows that $\widehat s>0$. We have shown that for $\alpha,\beta$ sufficiently large and $\delta,\nu,\varepsilon$ sufficiently small, $\widehat t,\widehat s\in(0,T)$ and therefore the map (67) has a local interior maximum at $(\widehat t,\widehat x,\widehat s,\widehat y)$, relative to $B\times B$. From Lemma 9, Doubling Lemma 8 (applied with $u=\widetilde u_\nu$, $v=\widetilde v_\nu$, $F=\overline F$ and $G=\underline F$ defined as in Lemma 9, etc.) and computations in Lemma 2.2 of [9] it follows that

$$a+b+2c_0-\varepsilon+H(\alpha)_\varepsilon(\widehat t,\widehat x,\widehat s,\widehat y,\alpha(\widehat x-\widehat y)+p,\alpha(\widehat x-\widehat y)-q)\le0,$$

where $H(\alpha)$ is given by (38) with $F=\overline F$ and $G=\underline F$ as in Lemma 9. Using **F2**, **F4** and **F5** to estimate $H(\alpha)_\varepsilon$ leads to

$$\begin{aligned}a+b+2c_0-\varepsilon &\le 2\sigma_\infty(\delta,\alpha\|\widehat x-\widehat y\|+2\varepsilon+6\alpha)+\sigma_F(\|\widehat x-\widehat y\|+\alpha\|\widehat x-\widehat y\|^2)\\ &\quad+2\omega_F(2\varepsilon+|\widehat t-\widehat s|,2\|\widehat x\|+2\|\widehat y\|+2\alpha\|\widehat x-\widehat y\|+2\varepsilon+12\alpha+2T).\end{aligned}\tag{70}$$

By a standard argument (see e.g. [13]) one shows that

$$\lim_{\alpha\to\infty}\limsup_{\delta\downarrow0}\limsup_{\beta\to\infty}\limsup_{\varepsilon\downarrow0}\{\alpha\|\widehat x-\widehat y\|^2+\beta(\widehat t-\widehat s)^2\}=0.$$

We reach a contradiction with $c_0>0$ by taking an iterated lim sup in (70) as $\varepsilon\downarrow0$, $\beta\to\infty$, $\delta\downarrow0$, $\alpha\to\infty$, in this order.

The proof of (6) is standard if one notices that we have actually shown that if $c_0 = 0$ then

$$\lim_{\alpha\to\infty} \limsup_{\delta\downarrow 0} \limsup_{\beta\to\infty} \limsup_{\varepsilon\downarrow 0} \Phi(\widehat{t},\widehat{x},\widehat{s},\widehat{y}) \le 0.$$

We refer the reader to [12], [13], [9] or [20]. ■

## 8. Proof of existence

Proof of Theorem 4. *Existence.* We produce a solution of **(CP)** by Perron's method as it was presented in [9] and which certainly applies to the current situation. We will therefore be using without proof (see [9] for the proof which, when combined with the usual Perron process, adapts to our setting in an obvious way) the fact that if $\Omega$ is a relatively open subset of $D$ then

(71) the upper (lower) semicontinuous envelope of the supremum (infimum) of a locally bounded family of sub(super)solutions of **(E)** on $(0,T)\times\Omega$ is a sub(super)solution of **(E)** on $(0,T)\times\Omega$,

and that

(72) if $u, -v \in \mathrm{USC}((0,T)\times\Omega)$, $u, v$ are bounded, $u$ is a sub- and $v$ is a supersolution of **(E)** on $(0,T)\times\Omega$ then $\sup\{w\in\mathrm{USC}((0,T)\times\Omega) : w$ is a subsolution of **(E)** and $u\le w\le v\}$ is a solution of **(E)** on $(0,T)\times\Omega$.

In what follows we imitate techniques used in [12]. To begin with we need two straightforward lemmas.

LEMMA 10. *Let $\Phi = \varphi + \psi \in C^{1,1,2}((0,T)\times D) + \mathrm{Lip}(H')$ satisfy* (2) (*respectively* (1)) *and*

$$\varphi_t(t,x) + \mathcal{D}_A^+\Phi(t,x) + F^{L(\psi)}(t,x,D\varphi(t,x),D^2_{x''}\varphi(t,x)) \le 0,$$

*respectively*

$$\varphi_t(t,x) + \mathcal{D}_A^-\Phi(t,x) + F_{L(\psi)}(t,x,D\varphi(t,x),D^2_{x''}\varphi(t,x)) \ge 0,$$

*for $(t,x)\in(0,T)\times\Omega$. Then $\Phi$ is a viscosity sub(super)solution of* **(E)**.

Proof. See the proof of Lemma 1.4 in [10] which adapts with minor modifications to our case. ■

LEMMA 11. *Let $\psi\in\mathrm{UC}(D)$. Then the following are equivalent*:

(i) *$\psi(S(t)x',x'')$ is uniformly continuous on bounded sets in $t$, uniformly for bounded $x\in D$.*

(ii) *There is a local modulus $\varrho$ such that*

$$|\psi(x)-\psi(y)| \le \varrho(d(x',y')+\|x''-y''\|, R) \quad \textit{for } x,y\in D,\ \|y\|\le R.$$



Proof. (i)⇒(ii). The assumptions imply that there exist a local modulus $\sigma_1$ and a modulus $\sigma_2$ such that

$$|\psi(S(t)y',y'')-\psi(y)| \le \sigma_1(t,R), \qquad |\psi(x)-\psi(y)| \le \sigma_2(\|x-y\|)$$

for $x,y\in D$, $\|y\|\le R$. Given such $x,y$, choose $t$ such that $d(x',y') = t+\|x'-S(t)y'\|$. Then

$$\begin{aligned}|\psi(x)-\psi(y)| &\le |\psi(x)-\psi(x',y'')| + |\psi(x',y'')-\psi(S(t)y',y'')| \\ &\quad + |\psi(S(t)y',y'')-\psi(y)| \\ &\le \sigma_2(\|x''-y''\|) + \sigma_2(\|(x',y'')-(S(t)y',y'')\|) + \sigma_1(t,R) \\ &\le \sigma_2(\|x''-y''\|) + \sigma_2(d(x',y')) + \sigma_1(d(x',y'),R).\end{aligned}$$

(ii)⇒(i). For $x\in D$, $\|x\|\le R$, we have

$$|\psi(S(t)x',x'')-\psi(x',x'')| \le \varrho(d(S(t)x',x'),R) \le \varrho(t,R),$$

which completes the proof since $S(t)$ is a semigroup of contractions. ■

Having these preliminaries in hand we can now start to construct a solution. Let $\|\psi\|_\infty = M$ and let $K=\max(M,K_0,1)$. Define $u_0, v_0 : [0,T]\times D\to\mathbb{R}$ by

$$u_0(t,x) = -K(1+t), \qquad v_0(t,x) = K(1+t).$$

Then, by Lemma 10, $u_0, v_0$ are respectively a sub- and a supersolution of **(E)** on $(0,T)\times D$ and

$$-K(1+T) \le u_0(t,x) \le \psi(x) \le v_0(t,x) \le K(1+T). \tag{73}$$

Fix $R > K(2+T)$. Lemma 11 implies that for every $0<\varepsilon<1$ there is $\alpha=\alpha(\varepsilon,R)>1$ such that

$$|\psi(x)-\psi(y)| \le \varepsilon + \alpha(\varepsilon,R)(d(x',y')+\|x''-y''\|^2) \tag{74}$$

if $x,y\in D$ and $\|y\|\le R$. Define

$$B^d_{R,r} = \{x\in D : d(x', B_R^{H'}(0)\cap\overline{D(A)}) < r,\ \|x''\| < R+r\}.$$

For $0<\varepsilon<1$, $\beta>0$ and $\widehat{x}\in B_R^H(0)\cap D$ we define

$$\begin{aligned}u_1(t,x;\varepsilon,\beta,\widehat{x}) &= \psi(\widehat{x}) - \varepsilon - \alpha(\varepsilon,R)(d(x',\widehat{x}')+\|x''-\widehat{x}''\|^2) - \beta t,\\ v_1(t,x;\varepsilon,\beta,\widehat{x}) &= \psi(\widehat{x}) + \varepsilon + \alpha(\varepsilon,R)(d(x',\widehat{x}')+\|x''-\widehat{x}''\|^2) + \beta t.\end{aligned} \tag{75}$$

The functions $u_1, v_1$ satisfy respectively (2) and (1). If $(t,x)\in(0,T)\times B^d_{R,2R}$, then

$$\begin{aligned}(u_1)_t + \mathcal{D}_A^+u_1 + F^{\alpha(\varepsilon,R)}(t,x,-2\alpha(\varepsilon,R)(x''-\widehat{x}''),-2\alpha(\varepsilon,R)I_{H''})&\\ \le -\beta+\alpha(\varepsilon,R)+K_{9R\alpha(\varepsilon,R)} \le 0&\end{aligned} \tag{76}$$

if $\beta=\beta(\varepsilon,R)$ is sufficiently large. We can obtain a similar estimate for $v_1$ and hence, by Lemma 10, $u_1$ is a subsolution and $v_1$ is a supersolution of **(E)** on $(0,T)\times B^d_{R,2R}$ for some $\beta=\beta(\varepsilon,R)$. If $(t,x)\in[0,T]\times(D\setminus B^d_{R,R})$ then, by (73),

$$u_1(t,x;\varepsilon,\beta(\varepsilon,R),\widehat{x})\le\psi(\widehat{x})-\varepsilon-\alpha(\varepsilon,R)(d(x',\widehat{x}')+\|x''-\widehat{x}''\|^2)$$
$$\le K-R\le -K(1+T)\le u_0(t,x).$$

Similarly,

$$v_0(t,x)\le v_1(t,x;\varepsilon,\beta(\varepsilon,R),\widehat{x})\quad\text{ for } (t,x)\in[0,T]\times(D\setminus B^d_{R,R}).$$

Define

$$u_R(t,x)=u_0(t,x)\vee(\sup_{\substack{0<\varepsilon<1\\ \widehat{x}\in B^H_R(0)\cap D}}u_1(t,x;\varepsilon,\beta(\varepsilon,R),\widehat{x}))^*,$$
$$v_R(t,x)=v_0(t,x)\wedge(\inf_{\substack{0<\varepsilon<1\\ \widehat{x}\in B^H_R(0)\cap D}}v_1(t,x;\varepsilon,\beta(\varepsilon,R),\widehat{x}))_*$$

for $(t,x)\in[0,T]\times D$, where the upper and lower asterisks denote respectively the upper and lower semicontinuous envelopes on $[0,T]\times D$. We have just shown that, if $(t,x)\in(0,T)\times(D\setminus B^d_{R,R})$, then $u_R(t,x)=u_0(t,x)$ and $v_R(t,x)=v_0(t,x)$. Since $D\setminus B^d_{R,R}$ is a neighborhood (relative to $D$) of $D\setminus B^d_{R,2R}$, using the above and (71), we see that $u_R$ and $v_R$ are respectively a subsolution and a supersolution of **(E)**. Finally, we define

$$\widetilde{u}(t,x)=(\sup_{R>K(2+T)}u_R(t,x))^*,\qquad \widetilde{v}(t,x)=(\inf_{R>K(2+T)}v_R(t,x))_*$$

for $(t,x)\in[0,T]\times D$. Because of (73), 74), and the way $\widetilde{u}$ and $\widetilde{v}$ have been defined, it is clear that

$$-K(1+T)\le u_R(t,x)\le\widetilde{u}(t,x)\le\psi(x)\le\widetilde{v}(t,x)\le v_R(t,x)\le K(1+T).\tag{77}$$

Also, if $(t,x)\in[0,T]\times(B^H_R(0)\cap D)$, we get

$$\begin{aligned}\widetilde{u}(t,x)&\ge u_1(t,x;\varepsilon,\beta(\varepsilon,R),x)\ge\psi(x)-\varepsilon-\beta(\varepsilon,R)t,\\ \widetilde{v}(t,x)&\le v_1(t,x;\varepsilon,\beta(\varepsilon,R),x)\le\psi(x)+\varepsilon+\beta(\varepsilon,R)t.\end{aligned}\tag{78}$$

Now, it follows from (77) and (72) that there exists a solution $u$ of **(E)** on $(0,T)\times D$ such that $\widetilde{u}\le u\le\widetilde{v}$. Therefore, from (78),

$$|u(t,x)-\psi(x)|\le\varepsilon+\beta(\varepsilon,R)t\quad \text{if}(t,x)\in[0,T]\times(B^H_R(0)\cap D),$$

which implies that the assumptions of comparison are satisfied. Thus, in virtue of (5), $u$ has the required properties.

**Acknowledgments.** The authors would like to thank Prof. P. L. Lions for bringing the problem to their attention and Prof. M. G. Crandall for his helpful comments and suggestions.



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*Received December 5, 1994* (3382)