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Nonatomic Lipschitz spaces

by

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Abstract. We abstractly characterize Lipschitz spaces in terms of having a lattice-complete unit ball and a separating family of pure normal states. We then formulate a notion of "measurable metric space" and characterize the corresponding Lipschitz spaces in terms of having a lattice complete unit ball and a separating family of normal states.

Let (X,d) be a metric space. Then the *Lipschitz space* $\mathrm{Lip}(X,d)$ is the Banach space consisting of all bounded scalar-valued Lipschitz functions on X, with norm

$$\|f\|_L = \max(\|f\|_\infty, L(f)).$$

Here $\|f\|_\infty$ denotes the sup norm of f and $L(f)$ denotes the Lipschitz number of f,

$$L(f) = \sup\{|f(x)-f(y)|/d(x,y) : x,y\in X,\ x\neq y\}.$$

Lipschitz spaces have been studied in [1], [3], [5], [8], [9], [10], [11], [12], [13].

The real part of the unit ball of $\mathrm{Lip}(X,d)$ is a completely distributive complete sublattice, and we showed in [11] that this fact characterizes Lipschitz spaces up to isomorphism. Our first aim here is to give another abstract characterization of Lipschitz spaces, this time in terms of order properties which may be more familiar. In the new characterization, complete distributivity of the unit ball is replaced by the existence of a separating family of pure normal states (Theorem 4).

This new result is somewhat unnatural, in that it juxtaposes pureness and normality, two properties not usually seen together. This is actually an advantage, because it suggests a direction for generalization.

To see this, consider the space l^∞ of bounded scalar-valued sequences. It too has a separating family of pure normal states, namely the coordinate evaluations. But l^∞ is merely a special example of the class of spaces $L^\infty(X,\mu)$, which generally have a separating family of normal states (given by integration against functions in $L^1(X,\mu)$). *Pure* normal states exist in

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abundance only when the measure μ is atomic (every point has positive measure).

Guided by this example, we formulate a definition of "measurable" metric spaces (X,μ,d) and characterize the corresponding "nonatomic" Lipschitz spaces $\mathrm{Lip}(X,\mu,d)$ in terms of having a lattice-complete unit ball and a separating family of normal states (Theorems 6 and 10).

In a subsequent paper we will use this result to show that the domain of any unbounded derivation of abelian von Neumann algebras is isomorphic to a nonatomic Lipschitz algebra. This corroborates Connes' suggestion that noncommutative metric spaces should be modelled by derivations [2].

1. An abstract characterization of $\mathrm{Lip}(X,d)$**.** Let $\mathcal{L}$ be an ordered Banach space, i.e. a real Banach space that is equipped with a partial order such that for $x,y,z\in\mathcal{L}$ and $a\in\mathbb{R}^+$, $x\geq y$ implies $ax\geq ay$ and $x+z\geq y+z$. (See [7] for general material on ordered Banach spaces.) We call $\mathcal{L}$ an *L-lattice* if every norm-bounded set of elements $(x_\alpha)\subset\mathcal{L}$ has a supremum $\bigvee x_\alpha$ and

$$\Big\|\bigvee x_\alpha\Big\|\leq \sup\|x_\alpha\|.$$

This implies the corresponding inequality for infima and is equivalent to saying that the unit ball of $\mathcal{L}$ is a complete sublattice.

For example, for any σ-finite measure space (X,μ), the space $L^\infty(X,\mu)$ is an L-lattice; indeed, any abelian von Neumann algebra is an L-lattice. (There do exist pathological measure spaces such that $L^\infty(X,\mu)$ is not an L-lattice, however. For example, let X be an uncountable set and let μ be counting measure on the class of countable and co-countable subsets of X. Then the collection of characteristic functions $\{\chi_x : x\in A\}$ has no supremum in $L^\infty(X,\mu)$ if $A\subset X$ is neither countable nor co-countable.)

Denote the supremum of the unit ball of $\mathcal{L}$ by e. A *state* on $\mathcal{L}$ is then a positive linear functional ϱ on $\mathcal{L}$ such that $\varrho(e)=1$. Note that this implies $\|\varrho\|=1$: we have $\|\varrho\|\geq 1$ since $\|e\|=1$ and $\|\varrho\|\leq 1$ since $\|x\|\leq 1$ implies $-e\leq x\leq e$ implies $-1\leq\varrho(x)\leq 1$. A state ϱ is *pure* if it is an extreme point of the set of all states on $\mathcal{L}$, i.e. it cannot be expressed in the form $a\varrho_1+(1-a)\varrho_2$ for ϱ_1,ϱ_2 distinct states and $0<a<1$. It is *normal* if for any norm-bounded directed subset $(x_\alpha)\subset\mathcal{L}$ we have

$$\varrho\Big(\bigvee x_\alpha\Big)=\sup\varrho(x_\alpha).$$

The following is an equivalent definition of pureness.

LEMMA 1 ([6], Lemma 3.4.6). *Let $\mathcal{L}$ be an L-lattice and ϱ a state on $\mathcal{L}$. Then ϱ is pure if and only if every positive linear functional $\tau\leq\varrho$ is a scalar multiple of ϱ.*

Now we want to prove a crucial result about pure states on L-lattices. The notation $|x|$ stands for $x\vee(-x)$ (the supremum of x and $-x$), x^+ for $x\vee 0$, and x^- for $(-x)\vee 0$. By [7], Theorem 11.7, we have $|x|=x^++x^-$, $x=x^+-x^-$, and $x^+\wedge x^-=0$ (where $\wedge$ denotes infimum). We require the following simple lemma.

LEMMA 2. *Let x,y,z,w be positive elements of an L-lattice. Then*

(a) $(x\wedge z)+(y\wedge w)\leq(x+y)\wedge(z+w)$;
(b) $(x+y)\wedge z\leq(x\wedge z)+(y\wedge z)$; *and*
(c) *if* $x\wedge y=0$ *then* $x\wedge ay=0$ *for all* $a\in\mathbb{R}^+$

Proof. (a) Since $x\wedge z\leq x$ and $y\wedge w\leq y$, we have

$$(x\wedge z)+(y\wedge w)\leq x+y.$$

Similarly

$$(x\wedge z)+(y\wedge w)\leq z+w,$$

so that

$$(x\wedge z)+(y\wedge w)\leq(x+y)\wedge(z+w).$$

(b) Observe first that

$$((x+y)\wedge z)-x=y\wedge(z-x)\leq y\wedge z.$$

Hence

$$((x+y)\wedge z)-(y\wedge z)\leq x,$$

and as the left side is also $\leq z$, it is therefore $\leq x\wedge z$. So

$$(x+y)\wedge z\leq(x\wedge z)+(y\wedge z).$$

(c) If $a\leq 1$ then

$$x\wedge ay\leq x\wedge y=0,$$

and if $a\geq 1$ then

$$x\wedge ay\leq ax\wedge ay=a(x\wedge y)=0.$$

Since x and ay are both positive, $x\wedge ay\geq 0$. Hence $x\wedge ay=0$. ■

PROPOSITION 3. *Let $\mathcal{L}$ be an L-lattice, $x\in\mathcal{L}$, and ϱ a pure state on $\mathcal{L}$. Then $\varrho(|x|)=|\varrho(x)|$. Pure states preserve the lattice operations $\vee$ and $\wedge$.*

Proof. For any positive element $y\in\mathcal{L}$ define

$$\tau(y)=\sup\{\varrho(y\wedge ax^+) : a\in\mathbb{R}^+\}.$$

The sup is positive and finite since $0\leq y\wedge ax^+\leq y$. For any $b>0$ we have

$$by\wedge ax^+=b(y\wedge(a/b)x^+),$$

so $\tau(by) = b\tau(y)$. Also, for any positive $y, z \in \mathcal{L}$ and $a, b \in \mathbb{R}^+$, Lemma 2(a) and (b) imply that

$$(y \wedge ax^+) + (z \wedge bx^+) \le (y+z) \wedge ((a+b)x^+)$$

and

$$(y+z) \wedge ax^+ \le (y \wedge ax^+) + (z \wedge ax^+),$$

from which it follows that $\tau(y+z) = \tau(y) + \tau(z)$.

We now define $\tau(y-z) = \tau(y) - \tau(z)$ for any positive $y, z \in \mathcal{L}$; this can be done consistently because if y, y', z, z' are positive and $y - z = y' - z'$ then $y + z' = y' + z$ hence $\tau(y) + \tau(z') = \tau(y') + \tau(z)$ hence $\tau(y) - \tau(z) = \tau(y') - \tau(z')$. Since every $y \in \mathcal{L}$ decomposes as $y = y^+ - y^-$, τ is now a positive linear functional defined on all of $\mathcal{L}$.

Now $\tau \le \varrho$ and $\tau(x^+) = \varrho(x^+)$. Thus by Lemma 1 either $\varrho(x^+) = 0$ or else $\tau = \varrho$, in which case $\varrho(x^-) = 0$ by Lemma 2(c). But if $\varrho(x^+) = 0$ then

$$\varrho(|x|) = \varrho(x^+ + x^-) = -\varrho(x^+ - x^-) = -\varrho(x),$$

and if $\varrho(x^-) = 0$ then

$$\varrho(|x|) = \varrho(x^+ + x^-) = \varrho(x^+ - x^-) = \varrho(x).$$

In either case we get $\varrho(|x|) = |\varrho(x)|$.

Preservation of the lattice operations now follows immediately from the formulas $x \vee y = (x + y + |x-y|)/2$ and $x \wedge y = (x + y - |x-y|)/2$. ∎

Now we present our new characterization of Lipschitz spaces. We say that a family of linear functionals is *separating* if the intersection of their kernels is 0. We assume real scalars.

THEOREM 4. *Every* $\mathrm{Lip}(X,d)$ *is an L-lattice and has a separating family of pure normal states. Conversely, any L-lattice which possesses a separating family of pure normal states is isomorphic to some* $\mathrm{Lip}(X,d)$.

Proof. Let (X,d) be a metric space. The fact that $\mathrm{Lip}(X,d)$ is an L-lattice is easy and standard [11]: for any collection of functions $(f_\alpha) \subset \mathrm{Lip}(X,d)$ with uniformly bounded Lipschitz norms, the pointwise supremum f satisfies

$$\|f\|_L \le \sup \|f_\alpha\|_L.$$

We claim that for each $x \in X$, the linear functional $\varrho_x : f \mapsto f(x)$ is a pure normal state on $\mathrm{Lip}(X,d)$. This will suffice to prove the first statement of the theorem, as this family is clearly separating.

It is easy to see that ϱ_x is a state, and it is normal since the supremum of Lipschitz functions is taken pointwise. To show that it is pure, we use Lemma 1. Thus let τ be a positive linear functional and suppose $\tau \le \varrho_x$. Then for any positive Lipschitz function f which vanishes at x we have

$$0 \le \tau(f) \le \varrho_x(f) = 0.$$

Taking linear combinations, we find that $\tau(f) = 0$ for any Lipschitz function which vanishes at x. Thus $\ker(\varrho_x) \subset \ker(\tau)$, so τ must be a scalar multiple of ϱ_x. By Lemma 1 it follows that ϱ_x is pure, so we have finished proving the first statement of the theorem.

For the second statement, let $\mathcal{L}$ be an L-lattice. Let X be the set of pure normal states on $\mathcal{L}$ and suppose X is separating. Now X is contained in the dual Banach space $\mathcal{L}^*$, from which it inherits a metric d, that is, $d(\varrho, \tau) = \|\varrho - \tau\|$. Furthermore we have a natural map $\Gamma : \mathcal{L} \to \mathrm{Lip}(X,d)$ defined by $\Gamma(x)(\varrho) = \varrho(x)$. We now aim to show that Γ is a surjective isomorphism.

It is easy to check that Γ is nonexpansive, and it is 1-1 since X is separating. Thus, by the open mapping theorem we need only show that it is onto. To do this choose $f \in \mathrm{Lip}(X,d)$ and without loss of generality suppose $\|f\|_L < 1$. Now for any distinct $\varrho, \tau \in X$ we have $|f(\varrho) - f(\tau)| < \|\varrho - \tau\|$, so there must exist $x \in \mathcal{L}$, $\|x\| < 1$, such that $|\varrho(x) - \tau(x)| = |f(\varrho) - f(\tau)|$. Possibly multiplying x by -1, we may assume that $\varrho(x) - \tau(x) = f(\varrho) - f(\tau)$; then the element

$$x_{\varrho\tau} = x + (f(\tau) - \tau(x))e$$

satisfies $\varrho(x_{\varrho\tau}) = f(\varrho)$, $\tau(x_{\varrho\tau}) = f(\tau)$, and $\|x_{\varrho\tau}\| < 3$. (Recall that e is the supremum of the unit ball of $\mathcal{L}$ and $\varrho(e) = \tau(e) = 1$.)

Finally, define

$$y = \bigvee_{\varrho} \bigwedge_{\tau} x_{\varrho\tau}.$$

Then $\|y\| \le 3$. It follows from Proposition 3 that any pure normal state preserves suprema and infima of norm-bounded subsets of $\mathcal{L}$. (That is, normality plus preservation of finite lattice operations implies preservation of infinite lattice operations.) Thus for any $\varrho_0 \in X$ we have

$$\varrho_0\Big(\bigwedge_{\tau} x_{\varrho_0\tau}\Big) = \bigwedge_{\tau} \varrho_0(x_{\varrho_0\tau}) = f(\varrho_0),$$

and for every $\varrho \in X$, $\varrho \ne \varrho_0$,

$$\varrho_0\Big(\bigwedge_{\tau} x_{\varrho\tau}\Big) \le \varrho_0(x_{\varrho\varrho_0}) = f(\varrho_0).$$

So

$$\varrho_0(y) = \bigvee_{\varrho} \varrho_0\Big(\bigwedge_{\tau} x_{\varrho\tau}\Big) = f(\varrho_0),$$

and we conclude that $\Gamma(y) = f$. This shows that Γ is onto and completes the proof. ∎

The second part of this proof is quite similar to the proof of Theorem 2 in [11].

Note that the proof shows that Γ is not only an isomorphism, but actually satisfies $\|x\| \geq \|\Gamma(x)\|_L \geq \|x\|/3$ for all $x \in \mathcal{L}$. The 1/3 lower bound is best possible; to see this let $\varepsilon > 0$ and let $\mathcal{L}$ be $\mathbb{R}^2$ with coordinatewise order and the norm whose unit sphere is a parallelogram with vertices at $(1,1)$, $(1,1-\varepsilon)$, $(-1,-1+\varepsilon)$, and $(-1,-1)$. Then $\mathcal{L}$ is an L-lattice and the pure normal states on $\mathcal{L}$ are the two coordinate projections; their distance from each other is ε. A simple calculation now shows that for $x = (1-\varepsilon, 1)$ we have $\|x\| = 3 - \varepsilon$ and $\|\Gamma(x)\|_L = 1$.

Finally, we remark that the characterization of Lipschitz spaces given in [11] is much deeper than Theorem 4. Indeed, the bulk of the proof of that result is essentially a matter of constructing pure normal states.

2. Measurable metric spaces. In this section we define the class of measurable metric spaces (X,μ,d) and discuss the corresponding spaces $\mathrm{Lip}(X,\mu,d)$.

Let (X,μ) be a measure space and let $\mathcal{M}$ be the class of μ-measurable sets of positive measure. We are really only interested in measure spaces for which $L^\infty(X,\mu)$ is the dual of $L^1(X,\mu)$, which is essentially equivalent to saying that $L^\infty(X,\mu)$ is a von Neumann algebra; this includes the σ-finite case and excludes pathologies of the sort mentioned in Section 1. However, the following definitions are valid in general. We write $A \sim A'$ if A and A' have null symmetric difference.

We define a *measurable pseudometric* on (X,μ) to be a function $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ with the following properties: $A \sim A'$ and $B \sim B'$ imply $d(A,B) = d(A',B')$; for all $A, B, C \in \mathcal{M}$,

(1) $$d(A,A) = 0,$$

(2) $$d(A,B) = d(B,A),$$

(3) $$d(A \cup B, C) = \min(d(A,C), d(B,C)),$$

(4) $$d(A,C) \leq \sup_{B' \subset B} (d(A,B') + d(B',C));$$

and for any collection $(A_\alpha) \subset \mathcal{M}$ and any $B \in \mathcal{M}$ there exists $A \in \mathcal{M}$ such that $\mu(A_\alpha - A) = 0$ for all α and

(5) $$d(A,B) = \inf d(A_\alpha, B).$$

(That $d(A,B) \leq \inf d(A_\alpha,B)$ follows from (3).) In the σ-finite case axioms (3) and (5) are equivalent to the single assertion that

$$d\Big(\bigcup A_n, B\Big) = \inf d(A_n, B)$$

for countable collections $(A_n) \subset \mathcal{M}$, although we omit the proof of this.

A *measurable metric* is a measurable pseudometric with the additional property that for any disjoint $A, B \in \mathcal{M}$ there exist $A' \subset A$, $B' \subset B$ such that $d(A', B') > 0$.

If μ is atomic, then any metric on X gives rise to a measurable metric by the familiar formula

$$d(A,B) = \inf\{d(x,y) : x \in A,\ y \in B\},$$

and it is not hard to see that in the atomic case every measurable metric arises in this way. Similarly, for some measurable functions $f : X^2 \to \mathbb{R}^+$ the formula

$$d(A,B) = \operatorname{ess\,inf}(f|_{A\times B})$$

defines a measurable metric on a nonatomic measure space. However, not even every measurable metric on $[0,1]$ with Lebesgue measure arises in this way from a metric on $[0,1]$. For example, consider the measurable metric defined by

$$d(A,B) = \begin{cases} 1 & \text{if } \mu(A \cap B) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

If $f : [0,1]^2 \to \mathbb{R}^+$ is any measurable function, then either it is ≥ 1 on a set of full measure, in which case $\operatorname{ess\,inf}(f|_{A\times A}) \geq 1$ for any A, or else there exist disjoint positive measure subsets $A, B \subset [0,1]$ such that $\operatorname{ess\,inf}(f|_{A\times B}) < 1$. In neither case does d arise from f in the above manner.

(In general, any finite diameter measurable pseudometric on any measure space (X,μ) does arise from a pseudometric on the spectrum of $L^\infty(X,\mu)$, i.e. the set of ultrafilters over the Boolean algebra of measurable subsets modulo null sets. The relevant pseudometric is defined by

$$d(\mathcal{U},\mathcal{V}) = \sup\{d(A,B) : A \in \mathcal{U},\ B \in \mathcal{V}\},$$

for any ultrafilters $\mathcal{U}$ and $\mathcal{V}$. However, we do not find this fact especially useful.)

We now wish to define $\mathrm{Lip}(X,\mu,d)$. Let (X,μ,d) be a measurable metric space and let $f \in L^\infty(X,\mu)$. The *essential range* of f is the set of $a \in \mathbb{C}$ such that $f^{-1}(U)$ has positive measure, for every neighborhood U of a. Then for $A, B \in \mathcal{M}$ let $d_f(A,B)$ be the distance (in $\mathbb{C}$) between the essential ranges of $f|_A$ and $f|_B$. Alternately,

$$d_f(A,B) = \operatorname{ess\,inf}(F \circ (f \times f)|_{A\times B}),$$

where $F(x,y) = |x-y|$. Then we define the *Lipschitz number* of f to be

$$L(f) = \sup\{d_f(A,B)/d(A,B) : A, B \in \mathcal{M},\ d(A,B) > 0\},$$

and we say f is *Lipschitz* if $L(f)$ is finite. Also, let $\mathrm{Lip}(X,\mu,d)$ be the set of essentially bounded scalar-valued Lipschitz functions, with norm $\|f\|_L = \max(\|f\|_\infty, L(f))$. In the atomic case all of these definitions agree with the usual ones. We call $\mathrm{Lip}(X,\mu,d)$ a *nonatomic Lipschitz space.*

The following lemma and theorem establish one direction of the abstract characterization of nonatomic Lipschitz spaces. Here we assume real scalars.

LEMMA 5. *Let (X, μ, d) be a measurable metric space and suppose $A, B \subset X$ have positive measure. Then for every $\varepsilon > 0$ there exists a positive measure subset $A' \subset A$ such that $A'' \subset A'$ implies $d(A'', B) \leq d(A, B) + \varepsilon$.*

Proof. Let (C_α) be the collection of all positive measure subsets of A with the property that $d(C_\alpha, B) > d(A, B) + \varepsilon$. Then by axiom (5) there exists $C \subset X$ such that $\mu(C_\alpha - C) = 0$ for all α and

$$d(C, B) = \inf(d(C_\alpha, B)) \geq d(A, B) + \varepsilon.$$

Therefore also $d(A \cap C, B) \geq d(A, B) + \varepsilon$ by axiom (3), hence $A' = A - C$ has positive measure. It follows that $d(A'', B) \leq d(A, B) + \varepsilon$ for any positive measure subset A'' of A', since A' is disjoint from C. ■

THEOREM 6. *Let (X, μ, d) be a measurable metric space and suppose $L^\infty(X, \mu)$ is the dual of $L^1(X, \mu)$. Then $\mathrm{Lip}(X, \mu, d)$ is an L-lattice and has a separating family of normal states.*

Proof. Let S be a subset of the unit ball of $\mathrm{Lip}(X, \mu, d)$. It has a supremum f in $L^\infty(X, \mu)$ by [6], Theorem 5.2.1 and Exercise 5.7.14. We must show that $L(f) \leq 1$.

Let $A, B \in \mathcal{M}$, let $\varepsilon > 0$, and choose $A' \subset A$ as in Lemma 5, with $d(A', B) \leq d(A, B) + \varepsilon$. By shrinking A' if necessary, we may suppose that f varies by at most ε on A', i.e. for some $a \in \mathbb{R}$ we have $a \leq f(x) \leq a + \varepsilon$ for all $x \in A'$. By shrinking further we may also suppose that there exists $g \in S$ such that $g(x) \geq a - \varepsilon$ for all $x \in A'$.

Reversing the roles of A' and B, we can now find $B' \subset B$, $b \in \mathbb{R}$, and $h \in S$ such that $b \leq f(y) \leq b + \varepsilon$ and $h(y) \geq b - \varepsilon$ for all $y \in B'$, and $d(A', B') \leq d(A, B) + 2\varepsilon$.

Without loss of generality suppose that $a \geq b$ and note that $g(y) \leq f(y) \leq b + \varepsilon$ for almost every $y \in B'$. We conclude that

$$\frac{d_f(A, B) - 3\varepsilon}{d(A, B) + 2\varepsilon} \leq \frac{d_f(A', B') - 3\varepsilon}{d(A', B')} \leq \frac{a - b - 2\varepsilon}{d(A', B')} \leq \frac{d_g(A', B')}{d(A', B')} \leq L(g) \leq 1.$$

Taking $\varepsilon \to 0$ and letting A and B range over all positive measure subsets of X shows that $L(f) \leq 1$, as desired. Thus, the unit ball of $\mathrm{Lip}(X, \mu, d)$ is a complete sublattice.

We have not yet shown that $\mathrm{Lip}(X, \mu, d)$ is a Banach space. This follows from lattice-completeness by Proposition 1 of [11], and thus the proof that $\mathrm{Lip}(X, \mu, d)$ is an L-lattice is complete.

Finally, we must show that $\mathrm{Lip}(X, \mu, d)$ has a separating family of normal states. This immediately follows from the fact that $L^\infty(X, \mu)$ has such a family, namely the states $\varrho(f) = \int fg\, d\mu$ where $g \in L^1(X, \mu)$, $g \geq 0$, and $\int g\, d\mu = 1$. ■

3. An abstract characterization of $\mathrm{Lip}(X, \mu, d)$. We now wish to prove a converse to Theorem 6. To do this we need to use a standard operator algebraic technique, the GNS construction. Thus, our proof requires the existence of algebraic structure and the result is therefore only approximately a generalization of Theorem 4. Very likely the exact generalization of Theorem 4 is true, namely that any L-lattice with a separating family of normal states is isomorphic to some $\mathrm{Lip}(X, \mu, d)$. On the other hand, while our result assumes the existence of algebraic structure, isomorphism of that structure is also part of its conclusion. Therefore it is probably the most useful version for applications.

By "Banach algebra" we mean a complex Banach space which is also an algebra, such that $\|xy\| \leq C\|x\|\|y\|$ for some constant C and all x, y. The stronger version of the Banach algebra law, with $C = 1$, is clearly unacceptable here as it does not hold in $\mathrm{Lip}(X, \mu, d)$. (In general the best constant for $\mathrm{Lip}(X, \mu, d)$ is $C = 2$.)

Let $\mathcal{L}$ be a commutative Banach $*$-algebra. We define $x \leq y$ to mean that $\omega(y - x) \geq 0$ for every $*$-homomorphism $\omega : \mathcal{L} \to \mathbb{C}$. By a "preorder" we mean a relation which is reflexive and transitive, but not necessarily antisymmetric, and "preordered Banach space" is defined in the obvious way (cf. the definition of ordered Banach space in Section 1). Let $\mathrm{Re}(\mathcal{L})$ denote the self-adjoint part of $\mathcal{L}$.

LEMMA 7. *Let $\mathcal{L}$ be a commutative Banach $*$-algebra. Then the relation $\leq$ is a preorder and with it $\mathrm{Re}(\mathcal{L})$ a preordered Banach space. For every $x \in \mathcal{L}$ we have $xx^* \geq 0$, and if $x, y \geq 0$ then $xy \geq 0$.*

Proof. It is clear that $x \leq x$ for all $x \in \mathrm{Re}(\mathcal{L})$. If $x \leq y$ and $y \leq z$ then for every complex $*$-homomorphism ω,

$$\omega(z - x) = \omega(z - y) + \omega(y - x) \geq 0,$$

so that $x \leq z$. So $\leq$ is a preorder.

If $x, y, z \in \mathrm{Re}(\mathcal{L})$, $a \in \mathbb{R}^+$, and $x \leq y$, then $\omega(y - x) \geq 0$ implies that $\omega(ay - ax) \geq 0$ and $\omega((y + z) - (x + z)) \geq 0$, so $ax \leq ay$ and $x + z \leq y + z$, and we conclude that $\mathrm{Re}(\mathcal{L})$ is a preordered Banach space.

Finally, the last two assertions follow from the calculations $\omega(xx^*) = \omega(x)\overline{\omega(x)} = |\omega(x)|^2$ and $\omega(xy) = \omega(x)\omega(y)$. ■

We call a unital commutative Banach $*$-algebra $\mathcal{L}$ an *L-algebra* if $\leq$ makes $\mathrm{Re}(\mathcal{L})$ an L-lattice and its unit is e (the supremum of the unit ball).

Note that the first condition includes the assertion that $\leq$ is a partial order, not just a preorder. This is equivalent to saying that for every nonzero $x \in \mathcal{L}$ there is a complex $*$-homomorphism ω such that $\omega(x) \neq 0$, and is therefore a sort of semisimplicity assumption.

It follows immediately from Theorem 6 that (with complex scalars) $\mathrm{Lip}(X, \mu, d)$ is an L-algebra, provided $L^\infty(X,\mu)$ is the dual of $L^1(X,\mu)$.

We will call a linear functional on $\mathcal{L}$ a *normal state* if its restriction to $\mathrm{Re}(\mathcal{L})$ is a normal state in the original sense. In the following two lemmas let $\mathcal{L}$ be a fixed L-algebra and ϱ a fixed normal state on $\mathcal{L}$. Let $\langle x, y\rangle$ denote the pre-inner product on $\mathcal{L}$ defined by $\langle x, y\rangle = \varrho(xy^*)$, and let H denote the Hilbert space formed by factoring out by null elements with respect to this inner product, and then taking the completion.

LEMMA 8. *For any $x \in \mathcal{L}$, the equation $\pi(x)y = xy$ defines a bounded linear operator $\pi(x)$ on H. Its norm is at most $\sqrt{C}\|x\|$.*

The map $\pi : \mathcal{L} \to B(H)$ is a positive unital $*$-homomorphism of norm at most $\sqrt{C}$. If $(x_\alpha) \subset \mathrm{Re}(\mathcal{L})$ is a norm-bounded directed subset and $x = \bigvee x_\alpha$, then $(\pi(x_\alpha)) \subset \mathrm{Re}(B(H))$ is a norm-bounded directed subset and $\pi(x) = \bigvee \pi(x_\alpha)$.

Proof. That $\pi(x)$ is well defined and that it is bounded follow from the same calculation: $xx^* \leq \|xx^*\|e$, hence for all $y \in \mathcal{L}$ Lemma 7 implies $(\|xx^*\|e - xx^*)yy^* \geq 0$. Thus

$$\|xx^*\|\langle y, y\rangle - \langle xy, xy\rangle = \varrho((\|xx^*\|e - xx^*)yy^*) \geq 0,$$

and so $\langle \pi(x)y, \pi(x)y\rangle \leq \|xx^*\|\langle y, y\rangle$ for all y, which implies that $\|\pi(x)\| \leq \|xx^*\|^{1/2} \leq \sqrt{C}\|x\|$. Linearity of $\pi(x)$ is clear.

It is easy to check that π is a unital $*$-homomorphism, i.e.

$$\pi(e)x = x, \quad \pi(xy)z = \pi(x)\pi(y)z, \quad \langle \pi(x)y, z\rangle = \langle y, \pi(x^*)z\rangle$$

for all $x, y, z \in \mathcal{L}$, and the norm estimate follows from the estimate on $\pi(x)$ just given. The map π is positive because if $x \geq 0$ then $\langle \pi(x)y, y\rangle = \varrho(xyy^*) \geq 0$ for all $y \in \mathcal{L}$ by Lemma 7, hence $\pi(x) \geq 0$.

Now let $(x_\alpha) \subset \mathrm{Re}(\mathcal{L})$ be a norm-bounded directed subset and let $x = \bigvee x_\alpha$. Then $(\pi(x_\alpha))$ is contained in $\mathrm{Re}(B(H))$ since π is a $*$-homomorphism, it is bounded in norm since π is bounded, and it is directed since π is positive. Since $\bigwedge(x - x_\alpha) = 0$, for all $y \in \mathcal{L}$ we have

$$0 \leq \bigwedge(x - x_\alpha)yy^* \leq \|yy^*\| \bigwedge(x - x_\alpha) = 0.$$

Thus $\bigwedge(x - x_\alpha)yy^* = 0$ and so by normality of ϱ,

$$\inf\langle \pi(x - x_\alpha)y, y\rangle = \inf \varrho((x - x_\alpha)yy^*) = 0$$

for all y, which implies that $\bigwedge \pi(x - x_\alpha) = 0$, i.e. $\bigvee \pi(x_\alpha) = \pi(x)$. ∎

LEMMA 9. *Let $x, y, z \in \mathrm{Re}(\mathcal{L})$. Then $x \wedge y = 0$ implies $xy = 0$, and $\varrho(z^-) > 0$ implies $\pi(z) \ngeq 0$.*

Proof. Suppose $x \wedge y = 0$; then x and y are both positive and we have $xy \geq 0$ by Lemma 7. Conversely, for $a = \max(\|x\|, \|y\|)$ we have $x \leq ae$ and $y \leq ae$, hence

$$xy \leq ax \wedge ay = a(x \wedge y) = 0.$$

Thus $xy = 0$.

Now suppose $\varrho(z^-) > 0$. Then $\langle z^-, e\rangle > 0$, so z^- is a nonzero element of H. Thus $\langle \pi(z^-)z^-, e\rangle = \langle z^-, z^-\rangle > 0$, so that $\pi(z^-)z^-$ is also a nonzero element of H. But since $\pi(z^-) \geq 0$ (by positivity of π), it then follows that $\varrho((z^-)^3) = \langle \pi(z^-)z^-, z^-\rangle > 0$. Finally, since $z^+ \wedge z^- = 0$, the first part of this lemma implies that $z^+z^- = 0$, hence $zz^- = -(z^-)^2$, so

$$\langle \pi(z)z^-, z^-\rangle = -\varrho((z^-)^3) < 0.$$

This shows that $\pi(z) \ngeq 0$. ∎

With these preliminaries at hand we can now give the remaining direction of our characterization of nonatomic Lipschitz algebras.

THEOREM 10. *Let $\mathcal{L}$ be an L-algebra which has a separating family of normal states. Then $\mathcal{L}$ is isomorphic to some $\mathrm{Lip}(X, \mu, d)$.*

Proof. As above, for each normal state ϱ form the corresponding Hilbert space H_ϱ and $*$-homomorphism $\pi_\varrho : \mathcal{L} \to B(H_\varrho)$. Let H be the direct sum of all the H_ϱ's and $\pi : \mathcal{L} \to B(H)$ the direct sum of all the π_ϱ's; then π is a positive $*$-homomorphism of norm at most $\sqrt{C}$. Since the family of normal states is separating, for every nonzero $x \in \mathcal{L}$ we have $\langle \pi_\varrho(x)e, e\rangle = \varrho(x) \neq 0$ for some ϱ, so π is 1-1. Also, if x is not positive then x^- is not zero, so $\varrho(x^-) > 0$ for some normal state ϱ, which implies by Lemma 9 that $\pi(x)$ is not positive. So $x \geq 0$ if and only if $\pi(x) \geq 0$.

Since the range of π is a commutative $*$-subalgebra of $B(H)$, we may consider π as taking $\mathcal{L}$ into the weak operator closure of $\pi(\mathcal{L})$, which is an abelian von Neumann algebra, without loss of generality $L^\infty(X, \mu)$ by [4], Theorem 7.3.1.

Now we wish to show that π preserves suprema. Thus choose $x, y \in \mathrm{Re}(\mathcal{L})$ and let $z = x \vee y$. Now $\pi(z) \geq \pi(x) \vee \pi(y)$ since π is a positive map. Suppose $\pi(z) > \pi(x) \vee \pi(y)$; then as the latter belongs to the C*-algebra generated by (= norm closure of) $\pi(\mathcal{L})$, there must exist $w \in \mathrm{Re}(\mathcal{L})$ such that $\pi(w) \geq \pi(x) \vee \pi(y)$ but $\pi(w) \ngeq \pi(z)$. Since π is an order-isomorphism, this implies that $w \geq x \vee y$ but $w \ngeq z$, a contradiction. So $\pi(x \vee y) = \pi(x) \vee \pi(y)$. By the last part of Lemma 8, it then follows that π preserves the supremum of any norm-bounded set.

By the preceding facts, we are free to consider $\mathcal{L}$ as being contained in $L^\infty(X,\mu)$, with the inherited lattice structure. We continue to write $\|\cdot\|$ for the norm on $\mathcal{L}$ and $\|\cdot\|_\infty$ for the norm on $L^\infty(X,\mu)$. Now define a measurable pseudometric on (X,μ) by setting

$$d(A,B)=\sup\{d_f(A,B): f\in \mathrm{Re}(\mathcal{L}),\ \|f\|\le 1\},$$

where $d_f(A,B)$ is the distance in $\mathbb{R}$ between the essential ranges of $f|_A$ and $f|_B$. Since $\mathcal{L}$ is weak* dense in $L^\infty(X,\mu)$, d is a measurable metric, not just a pseudometric.

It is easy to see that the identity map takes $\mathrm{Re}(\mathcal{L})$ into $\mathrm{Lip}(X,\mu,d)$ and that $L(f)\le\|f\|$, hence $\|f\|_L\le\sqrt{C}\|f\|$ for all $f\in\mathrm{Re}(\mathcal{L})$. In fact, it takes all of $\mathcal{L}$ into $\mathrm{Lip}(X,\mu,d)$ with norm at most $\sqrt{2C}$:

$$\begin{aligned}\|f\|_L&\le\sqrt{2}\cdot\max(\|\mathrm{Re}(f)\|_L,\|\mathrm{Im}(f)\|_L)\\&\le\sqrt{2C}\cdot\max(\|\mathrm{Re}(f)\|,\|\mathrm{Im}(f)\|)\le\sqrt{2C}\|f\|\end{aligned}$$

for all $f\in\mathcal{L}$. Thus, by the open mapping theorem, to complete the proof it will suffice to show that the identity map is onto, i.e. every function in $\mathrm{Lip}(X,\mu,d)$ belongs to $\mathcal{L}$.

For each positive measure subset $A\subset X$, let f_A denote the function

$$f_A=\bigvee\{f\in\mathrm{Re}(\mathcal{L}): f|_A=0 \text{ and } \|f\|\le 1+\sqrt{C}\}.$$

Then $f_A\in\mathrm{Re}(\mathcal{L})$, $f_A|_A=0$ almost everywhere, and $\|f_A\|\le 1+\sqrt{C}$. We claim that, in addition, for any positive measure subset $B\subset X$ we have $f_A|_B\ge d(A,B)$ almost everywhere. To see this choose $\varepsilon>0$ and find $g\in\mathrm{Re}(\mathcal{L})$ such that $\|g\|\le 1$ and $d_g(A,B)\ge d(A,B)-\varepsilon$. Let R be the essential range of $g|_A$ and note that $R\subset[-\sqrt{C},\sqrt{C}]$. For each $a\in R$ the function

$$g_a=|g-ae|=(g-ae)\vee(ae-g)$$

has norm at most $1+\sqrt{C}$ in $\mathcal{L}$ and satisfies $g_a|_B\ge d(A,B)-\varepsilon$ almost everywhere. Thus $f=\bigwedge\{g_a: a\in R\}$ has norm at most $1+\sqrt{C}$ and satisfies $f|_A=0$ and $f|_B\ge d(A,B)-\varepsilon$ almost everywhere. Then f is in the supremum which defines f_A, hence $f_A|_B\ge d(A,B)-\varepsilon$. As this holds for all positive ε the claim is proved.

Now let $f\in\mathrm{Lip}(X,\mu,d)$ and suppose f is real-valued and $\|f\|_L\le 1$. For each positive measure subset $A\subset X$ let $k_A=\mathrm{ess\,inf}(f|_A)$; then $|k_A|\le 1$ and so $\|k_Ae-f_A\|\le 2+\sqrt{C}$. Also,

$$f=\bigvee_{A\subset X}(k_Ae-f_A)$$

and this shows that f belongs to $\mathcal{L}$ and has norm at most $2+\sqrt{C}$. This suffices to show that the identity map from $\mathcal{L}$ into $\mathrm{Lip}(X,\mu,d)$ is onto and completes the proof. ∎

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