COROLLARY 3. *If α has unbounded partial quotients then a generic C^1 cocycle of degree zero is weakly mixing and the associated skew-product diffeomorphism admits c.a. with speed $o(1/n)$. In particular it is rank-1 and has a nondiscrete singular simple spectrum with the only eigenvalues of the form* $\exp(2\pi i n\alpha)$.

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Compressible operators and the continuity of homomorphisms from algebras of operators

by

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Abstract. The notion of a compressible operator on a Banach space, E, derives from automatic continuity arguments. It is related to the notion of a cartesian Banach space. The compressible operators on E form an ideal in $\mathcal{B}(E)$ and the automatic continuity proofs depend on showing that this ideal is large. In particular, it is shown that each weakly compact operator on the James' space, J, is compressible, whence it follows that all homomorphisms from $\mathcal{B}(J)$ are continuous.

Introduction. It is shown by B. E. Johnson in [BJo] that all homomorphisms and derivations from $\mathcal{B}(E)$, the algebra of bounded, linear operators on the Banach space E, are continuous if E satisfies one of two decomposability conditions. The first of these conditions is that E be *cartesian*, that is, satisfy $E \cong F \oplus E \oplus E$ for some Banach space F, and the second that E have *continued bisection*, that is, $E \cong E_2 \oplus E_2$, where $E_2 \cong E_3 \oplus E_3$ and so on.

The continuity of derivations can be proved for some spaces which do not satisfy either of these decomposability conditions and this is done in [L&W] for the James space, J, and for $C(\Omega)$, where Ω is the space of ordinals less than or equal to the first uncountable ordinal. (However, it is not resolved in [L&W] whether all homomorphisms from $\mathcal{B}(J)$ and $\mathcal{B}(C(\Omega))$ are continuous.) There are spaces E, whose construction is based on that of J, such that all derivations from $\mathcal{B}(E)$ are continuous but for which there exist discontinuous homomorphisms from $\mathcal{B}(E)$, see [D,L&W]. There are also spaces for which there are discontinuous derivations from $\mathcal{B}(E)$, see [Re]. The existence of discontinuous derivations implies the existence of discontinuous homomorphisms.

The question of the continuity of homomorphisms from $\mathcal{B}(C(\Omega))$ has been treated by C. Ogden, who has shown that they are all continuous, see [Og].

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She has further shown that the same is true for all the Banach spaces $C(\omega_\eta)$, where ω_η is the ηth infinite cardinal.

It is shown here that all homomorphisms from $\mathcal{B}(J)$ are continuous. As in [L&W], the method of proof is based on that of Johnson. It turns out that, although J is not cartesian, many operators on J can be factored as though J were cartesian. This gives rise to the notion of a compressible operator. The notion of compressible operator is thus a generalisation of that of cartesian space and, in particular, if E is cartesian then the identity operator on E is compressible. Compressible operators do not appear to be at all related to spaces with continued bisection. A key point in the proof that all homomorphisms from $\mathcal{B}(J)$ are continuous is the fact that all weakly compact operators on J are compressible.

Compressible operators are defined in the first section of the paper and some connections with the structure of Banach spaces are noted. Facts about compressible operators are used in the second section to prove the automatic continuity of homomorphisms from $\mathcal{B}(J)$ and certain other algebras. Throughout, elements of a Banach space will be denoted by $x, y, \ldots$, elements of its dual space by $x^*, y^*, \ldots$ and elements of the second dual by $x^{**}, y^{**}, \ldots$ We shall also make use of the idea of the continuity ideal of a homomorphism, as defined in [Da].

Compressible operators

DEFINITION. An operator, T, on a Banach space, E, will be said to be *compressible* if there are an integer, n, and a sequence $\{Q_k\}_{k=1}^\infty$ of projections on E^n such that

(i) $Q_m Q_k = 0$ if $m \neq k$; and
(ii) T factors through Q_k for each k.

PROPOSITION 1. *The operator T in $\mathcal{B}(E)$ is compressible if and only if there are an integer, n, and closed subspaces $F_1, F_2, \ldots$ and $D_1, D_2, \ldots$ of E^n such that*

(i) *$E^n = F_1 \oplus D_1$ and $F_k = F_{k+1} \oplus D_{k+1}$ for $k = 1, 2, \ldots$, and*
(ii) *T factors through D_k for each k.*

Proof. Suppose that T is compressible and let n and $\{Q_k\}_{k=1}^\infty$ be as in the definition. Setting $D_k = Q_k(E^n)$, $F_1 = (I - Q_1)(E^n)$ and $F_{k+1} = (I - Q_{k+1})(F_k)$, we find that (i) and (ii) are satisfied.

Now suppose that (i) and (ii) are satisfied. Let Q_k be the projection of E^n onto D_k with kernel $F_k \oplus D_{k-1} \oplus \ldots \oplus D_1$. Then the sequence $\{Q_k\}_{k=1}^\infty$ satisfies the conditions for compressibility of T. ■

NOTATION. Denote the set of all compressible operators on E by $\mathcal{M}(E)$.

It is easily checked that $\mathcal{M}(E)$ is an ideal in $\mathcal{B}(E)$. Note that, if E is finite-dimensional, then $\mathcal{M}(E) = (0)$. Before proceeding to the automatic continuity results, we make some observations about the ideal of compressible operators and its connection with the structure of Banach spaces.

PROPOSITION 2. *The identity operator, I, on E is compressible if and only if there is an integer n such that*

$$E^n \cong D \oplus E^{n+1}$$

for some Banach space D.

Proof. Suppose that $E^n \cong D \oplus E^{n+1}$. Then $E^n \cong (D \oplus E^n) \oplus E^{(1)}$, where $E^{(1)} \cong E$. Set Q_1 to be the projection onto $E^{(1)}$ with kernel $D \oplus E^n$. Now, in turn, $E^n \cong (D \oplus D \oplus E^n) \oplus E^{(2)} \oplus E^{(1)}$, where $E^{(2)} \cong E$, and we set Q_2 to be the projection onto $E^{(2)}$ with kernel $(D \oplus D \oplus E^n) \oplus E^{(1)}$. Continuing in this way we find that I is compressible.

Conversely, suppose that I is compressible. Then, by Proposition 1, there is an integer n such that $E^n \cong F_{n+1} \oplus D_{n+1} \oplus D_n \oplus \ldots \oplus D_1$, where I factors through D_k for each k.

This means that $I = S_k T_k$, where $T_k : E \to D_k$ and $S_k : D_k \to E$. Put

$$T = \bigoplus_{k=1}^{n+1} T_k : E^{n+1} \to \bigoplus_{k=1}^{n+1} D_k \quad \text{and} \quad S = \bigoplus_{k=1}^{n+1} S_k : \bigoplus_{k=1}^{n+1} D_k \to E^{n+1}.$$

Then, regarding $\bigoplus_{k=1}^{n+1} D_k$ as a subspace of E^n we have $T : E^{n+1} \to E^n$ and $S : E^n \to E^{n+1}$ such that ST is the identity on E^{n+1}. It follows that TS is a projection on E^n with range isomorphic to E^{n+1}. ■

It follows from this proposition that, if E is cartesian, then the identity operator on E is compressible. The notion of a compressible operator thus extends that of a cartesian space. The classical Banach spaces are cartesian, as is the recently constructed space which is not isomorphic to its square but is isomorphic to its cube, thus solving the Schröder–Bernstein problem, see [Go] and [GM2]. Presumably, there are non-cartesian spaces for which the identity operator is compressible. The ideal $\mathcal{M}(E)$ can also be quite small, as the next result shows.

PROPOSITION 3. *Let E be a Banach space such that $\mathcal{S}(E)$, the ideal algebra of singular operators on E, has codimension one in $\mathcal{B}(E)$. Then $\mathcal{M}(E)$ is the ideal of finite rank operators on E.*

Proof. The codimension of $\mathcal{S}(E^n)$ in $\mathcal{B}(E^n)$ is n^2 and $\mathcal{S}(E^n)$ is a two-sided ideal in $\mathcal{B}(E^n)$ such that $\mathcal{B}(E^n)/\mathcal{S}(E^n)$ is isomorphic to M_n, the algebra of $n \times n$ matrices. Hence, if $\{Q_k\}_{k=1}^\infty$ is a sequence of orthogonal idempotents in $\mathcal{B}(E^n)$, then Q_k belongs to $\mathcal{S}(E^n)$ for some k. Since a singular projection has finite rank, any operator factoring through Q_k has finite rank.

It is clear that each finite rank operator on E is contained in $\mathcal{M}(E)$ if $\dim(E) = \infty$. ■

Any space E such that $\mathcal{S}(E)$ has codimension one must be indecomposable and examples of indecomposable spaces have recently been found, see [GM1]. It is not clear though that $\mathcal{M}(E)$ consists only of finite rank operators whenever E is indecomposable. Is there an indecomposable, or even hereditarily indecomposable, space E such that the identity operator on E is compressible? More explicitly, is there a hereditarily indecomposable space E such that E^2 contains a complemented copy of E^3?

NOTATION. (a) The ideal of approximable operators on the Banach space E will be denoted by $\mathcal{F}(E)$. Recall that an operator is *approximable* if it is a norm limit of finite rank operators.

(b) The ideal of all operators on E which factor through ℓ_p, where $1 \le p \le \infty$, will be denoted by $\mathcal{Q}_p(E)$.

(c) C_p, where $1 \le p \le \infty$, will denote the Banach spaces introduced by W. B. Johnson in [WJo1] and [WJo2]. These spaces have the property that each approximable operator on each Banach space E factors through C_p for each p.

That ℓ_p and C_p are cartesian spaces allows us to conclude that many approximable or ℓ_p-factoring operators are compressible.

PROPOSITION 4. *For each Banach space E,*

(i) $\mathcal{F}(E \oplus C_p) \subseteq \mathcal{M}(E \oplus C_p)$, $1 \le p \le \infty$; *and*
(ii) $\mathcal{Q}_p(E \oplus \ell_p) \subseteq \mathcal{M}(E \oplus \ell_p)$, $1 \le p \le \infty$.

In the first case, the ideal of compressible operators is always strictly bigger than the ideal of approximable operators because it also contains the projection onto C_p.

Recall that a Banach space is said to be *quasi-reflexive* if, when E is embedded in the natural way into E^{**}, it has finite codimension in this space. It is well known that quasi-reflexive, non-reflexive spaces are not cartesian.

PROPOSITION 5. *If E is quasi-reflexive but not reflexive, then $\mathcal{M}(E)$ is properly contained in $\mathcal{B}(E)$.*

Proof. Let E be quasi-reflexive and suppose that $\mathcal{M}(E) = \mathcal{B}(E)$. Then, by Proposition 2, $E^n \cong V \oplus E^{n+1}$ for some positive integer n and Banach space V. Hence

$$\dim((E^n)^{**}/E^n) = \dim(V^{**}/V) + \dim((E^{n+1})^{**}/E^{n+1})$$

and so

$$n \dim(E^{**}/E) \ge (n+1) \dim l(E^{**}/E).$$

Since $\dim(E^{**}/E)$ is a positive integer, this is impossible. Therefore $\mathcal{M}(E) \ne \mathcal{B}(E)$. ■

The first quasi-reflexive space to be constructed was the James space, see [Ja]. The James space, J, is the space of sequences of complex numbers, $x = (x_j)_{j=0}^{\infty}$, which converge to zero and satisfy

$$\|x\|^2 = \sup\Big\{\sum_{j=1}^{l} |x_{r_j} - x_{r_{j+1}}|^2 : 0 \le r_1 < \ldots < r_{l+1}\Big\} < \infty.$$

The dual spaces J^* and J^{**} may also be identified with sequence spaces but we shall not require any formula for the norms on these spaces.

In fact J^{**} may be identified with $J + \mathbb{C}e^{**}$, where e^{**} is the constant sequence with value 1, and so J has codimension one in J^{**}.

When J, J^* and J^{**} are identified with sequence spaces duality is implemented by summation, that is,

$$\langle x^*, x\rangle = \sum_{j=1}^{\infty} x_j^* x_j \quad \text{and} \quad \langle x^{**}, x^*\rangle = \sum_{j=1}^{\infty} x_j^{**} x_j^*.$$

We have

PROPOSITION 6. $\mathcal{M}(J) = \mathcal{W}(J) \equiv$ *ideal of weakly compact operators on J.*

For the proof, all notation will be as in [L&W].

We require the following

LEMMA. *Let $\{y_k^*\}_{k=1}^{\infty}$ be a sequence in J^* such that*

(i) $\|y_k^*\| \le 1$ *for all k;*

(ii) *there is a sequence, $\{I_k\}_{k=1}^{\infty}$, of mutually disjoint intervals in $\mathbb{N}$ such that* $\operatorname{supp}(y_k^*) \subseteq I_k$ *for each k; and*

(iii) $\langle e^{**}, y_k^*\rangle = 0$ *for each k.*

Then for each x in J, $\sum_{k=1}^{\infty} |\langle y_k^*, x\rangle|^2 \le \|x\|^2$.

Proof. Suppose for each k that $I_k = [p_k, q_k]$ where $p_{k+1} > q_k$ for each k. Let x be in J and let k be fixed for now. Define x^{**} in J^{**} by

$$x^{**} = \begin{cases} x_j & \text{if } j \in I_k, \\ x_{p_k} & \text{if } j < p_k, \text{ and} \\ x_{q_k} & \text{if } j > q_k. \end{cases}$$

Then, since $\operatorname{supp}(y_k^*) \subseteq I_k$,

$$\langle x^{**}, y_k^*\rangle = \langle y_k^*, x\rangle.$$

Now let $z^{**} = x^{**} - x_{q_k} e^{**}$. Then, since $\langle e^{**}, y_k^* \rangle = 0$,

$$\langle z^{**}, y_k^* \rangle = \langle x^{**}, y_k^* \rangle$$

and so

$$\begin{aligned} |\langle y_k^*, x\rangle| &= |\langle z^{**}, y_k^*\rangle| \le \|z^{**}\| \quad \text{because } \|y_k^*\| \le 1,\\ &\le \sup\Big\{\sum_{j=1}^{l} |z^{**}_{r_j} - z^{**}_{r_{j+1}}|^2 : r_1 < \ldots < r_{l+1}\Big\}^{1/2}\\ &= \sup\Big\{\sum_{j=1}^{l} |x_{r_j} - x_{r_{j+1}}|^2 : p_k \le r_1 < \ldots < r_{l+1} \le q_k\Big\}^{1/2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{\infty} |\langle y_k^*, x\rangle|^2 &\le \sum_{k=1}^{\infty} \sup\Big\{\sum_{j=1}^{l} |x_{r_j} - x_{r_{j+1}}|^2 : p_k \le r_1 < \ldots < r_{l+1} \le q_k\Big\}\\ &\le \|x\|^2 \quad \text{because } p_{k+1} > q_k \text{ for all } k. \ \blacksquare \end{aligned}$$

Proof of Proposition 6. Let T be in $\mathcal{W}(J)$. Then, by [L&W], Lemma 2.1, $T = R + K$, where K is a compact operator on J and R has a matrix $[R_{ij}]$ with respect to the standard basis in J, such that there are at most finitely many non-zero entries in each row and column of $[R_{ij}]$ and $\langle e^{**}, r_i^* \rangle = 0$ for each row, r_i^*, of $[R_{ij}]$. (Note that the rows of $[R_{ij}]$ may be identified with elements of J^*.)

We will show separately that K and R belong to $\mathcal{M}(J)$ and the result will follow.

Proof that K is compressible. Let k_i^* denote the ith row of K. Since K is compact, $\|k_i^*\| \to 0$ as $i \to \infty$. Choose a sequence $i_1 < i_2 < \ldots$ such that $\sum_{n=1}^{\infty} \|k_{i_n}^*\| < \infty$. Then the matrix which agrees with $[K_{ij}]$ on its i_nth row and is zero elsewhere is the matrix of a rank one operator with norm $\|k_{i_n}^*\|$. Hence the sum of these rank one operators converges absolutely to a compact operator, L, whose matrix agrees with that of K on the i_nth rows, $n = 1, 2, \ldots$, and is zero elsewhere.

Now, for any sequence of positive integers $j_1 < j_2 < \ldots$, let $S[j_n]$ denote the projection on J which in [L&W] is denoted by $S([j_n], [j_n])$ (see the paragraph before Lemma 2.5 in [L&W]). Then $S[j_n]$ has matrix

$$(S[j_n])_{ij} = \begin{cases} 1, & j_{n-1} < i \le j_n,\ j = j_n,\\ 0 & \text{otherwise}, \end{cases}$$

where we suppose that $j_0 = 0$. It is easily seen that $S[j_n]J \cong J$ and so, if $\ker(S[j_n]) = D$, then $J \cong J \oplus D \cong J \oplus D \oplus D \cong \ldots$ Hence any operator which factors through the projection $I - S[j_n]$ is compressible. Now put $S = S[i_n]$ where $\{i_n\}_{n=1}^{\infty}$ is the sequence chosen above when defining the compact operator L. Then, by the choice of this sequence, $(I - S)(K - L) = K - L$.

Therefore $K - L$ is compressible. Next, put $S = S[i_n + 1]$. Then supposing, as we clearly may, that $i_{n+1} > i_n + 1$, $(I - S)L = L$. Therefore L is compressible and thus so is K.

Proof that R is compressible. Let r_i^* denote the ith row of R. Each r_i^* is finitely supported and $\langle e^{**}, r_i^* \rangle = 0$ for each i. Choose a sequence $i_1 < i_2 < \ldots$ such that the $r_{i_n}^*$'s have disjoint support. (This is possible because each column in $[R_{ij}]$ has only finitely many non-zero entries.) Then the matrix which agrees with $[R_{ij}]$ in its i_nth rows and is zero elsewhere is the matrix of a bounded operator, $V : J \to \ell_2$, by the lemma. Since the set of ℓ_2 sequences is a subspace of J, we may suppose that V belongs to $\mathcal{B}(J)$. Now put $S = S[i_n]$. Then $(I - S)(R - V) = R - V$ and so $R - V$ is in $\mathcal{M}(J)$ as before. Similarly, V belongs to $\mathcal{M}(J)$. $\blacksquare$

The continuity of homomorphisms. The results of the previous section and refinements of standard automatic continuity techniques show that homomorphisms from various algebras of operators are continuous. The automatic continuity techniques required are presented in the book [Da], which is to appear soon. We shall use the notion of the continuity ideal of an algebra homomorphism $\vartheta : \mathcal{A} \to \mathcal{C}$, where $\mathcal{A}$ and $\mathcal{C}$ are Banach algebras. The *continuity ideal*, $\mathcal{I}$, of ϑ is defined by

$$\mathcal{I} = \{a \in \mathcal{A} : b \mapsto \vartheta(ab) \text{ and } b \mapsto \vartheta(ba) \text{ are continuous}\}.$$

PROPOSITION 7. *Let $\mathcal{A}$ be a Banach algebra which is an ideal in $\mathcal{B}(E)$ and let $\vartheta : \mathcal{A} \to \mathcal{C}$ be a homomorphism. Then $\mathcal{A}\mathcal{M}(E)\mathcal{A} \subseteq \mathcal{I}$.*

Proof. Note first of all that for each n, $\mathcal{B}(E^n) \cong \mathcal{B}(E) \otimes M_n$, that $\mathcal{A} \otimes M_n$ is an ideal in $\mathcal{B}(E) \otimes M_n$ and that $\vartheta \otimes I_n : \mathcal{A} \otimes M_n \to \mathcal{C} \otimes M_n$ is a homomorphism. Furthermore, the continuity ideal of $\vartheta \otimes I_n$ is $\mathcal{I} \otimes M_n$. For each n, let E_n be the $n \times n$ matrix

$$E_n = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & 0 & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots \end{pmatrix}$$

Then the map $T \mapsto T \otimes E_n$ is a homomorphism from $\mathcal{B}(E)$ into $\mathcal{B}(E) \otimes M_n$.

Now let T be in $\mathcal{M}(E)$. Then there is an integer n and there are orthogonal projections $Q_1, Q_2, \ldots$ on E^n such that T factors through each Q_k. Thus, for each k there are R_k and S_k in $\mathcal{B}(E) \otimes M_n$ such that $T \otimes E_n = R_k Q_k S_k$.

Define $\mathcal{K} = \{S \in \mathcal{B}(E) \otimes M_n : (\mathcal{A} \otimes M_n) S (\mathcal{A} \otimes M_n) \subseteq \mathcal{I} \otimes M_n\}$. Then, by Theorem 4.4.4 in [Da], Q_k belongs to $\mathcal{K}$ for some k and so, since $\mathcal{K}$ is an ideal, $T \otimes E_n$ belongs to $\mathcal{K}$. Hence, in particular,

$$(\mathcal{A} \otimes E_n)(T \otimes E_n)(\mathcal{A} \otimes E_n) \subseteq \mathcal{I} \otimes M_n$$

and it follows that $\mathcal{A} T \mathcal{A} \subseteq \mathcal{I}$. $\blacksquare$

PROPOSITION 8. *Let J be the James space. Then every homomorphism from $\mathcal{B}(J)$ is continuous.*

Proof. Let $\vartheta : \mathcal{B}(J) \to \mathcal{C}$ be a homomorphism. Then, by Propositions 6 and 7, $\mathcal{W}(J) \subseteq \mathcal{I}$, the continuity ideal of ϑ. Now $\mathcal{W}(J)$ has a right bounded approximate identity, see [L&W], Corollary 2.4. Hence, by Cohen's factorisation theorem, if $\{T_k\}_{k=1}^{\infty}$ is a sequence in $\mathcal{W}(J)$ such that $\|T_k\| \xrightarrow{k} 0$, then there are T, and $T'_k, k = 1, 2, \ldots$, in $\mathcal{W}(J)$ with $T_k = T'_k T$ for each k and $\|T'_k\| \xrightarrow{k} 0$. Since $\mathcal{W}(J)$ is contained in the continuity ideal, the restriction of ϑ to $\mathcal{W}(J)$ is continuous. Therefore, since $\mathcal{W}(J)$ has codimension one in $\mathcal{B}(J)$, ϑ is continuous. ∎

PROPOSITION 9. *Let E be any Banach space. Then every homomorphism from $\mathcal{F}(E \oplus C_p)$ is continuous.*

Proof. Let $\vartheta : \mathcal{F}(E \oplus C_p) \to \mathcal{C}$ be a homomorphism. Note that, since every approximable operator factors, as a product of approximable operators, through C_p, we have $\mathcal{F}(E \oplus C_p)^2 = \mathcal{F}(E \oplus C_p)$. Hence, by Propositions 4 and 7, $\mathcal{F}(E \oplus C_p) \subseteq \mathcal{I}$. Now if $\{T_k\}_{k=1}^{\infty}$ is a sequence in $\mathcal{F}(E \oplus C_p)$ with $\|T_k\| \xrightarrow{k} 0$, then there are T, and T'_k, $k = 1, 2, \ldots$, in $\mathcal{F}(E \oplus C_p)$ with $T_k = TT'_k$ for each k and $\|T'_k\| \xrightarrow{k} 0$. (This follows from properties of C_p, see [WJo1].) Therefore $\vartheta(T_k) = \vartheta(TT'_k)$, which converges to 0 because T belongs to the continuity ideal. ∎

Note that, if $\mathcal{F}(E \oplus C_p)$ has either a left or right bounded approximate identity, then E has the bounded approximation property. Hence some of the algebras for which we have shown continuity of homomorphisms do not have bounded approximate identities.

Every element in an algebra $\mathcal{Q}_p(E \oplus l_p)$, $1 \le p < \infty$, is a product in this algebra. Hence Propositions 4 and 7 imply that, if ϑ is a homomorphism from $\mathcal{Q}_p(E \oplus \ell_p)$, then its continuity ideal is the whole algebra. However, it is not known (to the author) whether null sequences in $\mathcal{Q}(E \oplus \ell_p)$ factor in the way they do in the last two propositions and so we cannot say whether ϑ is continuous.

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