STUDIA MATHEMATICA 115 (3) (1995)

Generic smooth cocycles of degree zero over irrational rotations

by

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Abstract. If a rotation α of $\mathbb{T}$ has unbounded partial quotients then "most" of its skew-product diffeomorphic extensions to the 2-torus $\mathbb{T}\times\mathbb{T}$ defined by C^1 cocycles of topological degree zero enjoy nontrivial ergodic properties. In fact they admit a cyclic approximation with speed $o(1/n)$ and have nondiscrete (simple) spectrum. Similar results are obtained for C^r cocycles if α admits a sufficiently good approximation by rationals. For a.e. α and generic C^1 cocycles the speed can be improved to $o(1/(n\log n))$. For generic α and generic C^r cocycles ($r=1,\ldots,\infty$) the spectral measure of the skew product has a continuous component and Hausdorff dimension zero.

Introduction. On the 2-torus, we study skew-product diffeomorphisms of the form

$$T_\phi(x,y)=(x+\alpha,y+\phi(x))$$

where the addition is mod 1, the number α is irrational, and $\phi:\mathbb{T}\to\mathbb{T}$ is a C^r function ($r=1,\ldots,\infty$). It is well known that if ϕ has nonzero topological degree then T_ϕ is ergodic and the only eigenvalues are the numbers $\exp(2\pi in\alpha)$. Moreover, the maximal continuous spectral type is a Rajchman measure, i.e. its Fourier coefficients converge to zero. This is actually true even if ϕ is only absolutely continuous; if ϕ is C^2, then, as proved in [ILR], the continuous component of the spectral measure is Lebesgue with infinite multiplicity.

The situation is quite different if ϕ has degree zero. Now T_ϕ may not even be ergodic (as for certain constant functions) and the spectral measure is always singular. Besides, if α has bounded partial quotients in its continued fraction expansion then the spectrum is purely discrete.

Our aim is to study typical properties of zero degree skew products over rotations with unbounded partial quotients. Here, by "typical" or generic we mean for ϕ from a residual set of smooth functions.

1991 *Mathematics Subject Classification*: Primary 28D05.

Key words and phrases: Anzai skew product, weakly mixing cocycle, cyclic approximation, simple spectrum.

In Section 2 we prove that generically T_ϕ admits a good cyclic approximation, in particular it is rank-1. The rate of cyclic approximation of "most" C^r skew products is related to the speed of diophantine approximation of the number α. The result is independent of r and holds also for more general spaces (see Remark). Although our method borrows from [R] and [IS], the argument is different as the C^r norm requires keeping track of the derivatives. We apply an idea of Katok to control the diameters of approximating partitions (cf. [CFS], 16.3). The obtained rate of cyclic approximation will allow us to majorize the size of spectral measure, at least for certain α's.

In Section 3 we apply another result of Katok to show that typical skew-product diffeomorphisms have nondiscrete spectrum. The analogous problem of ergodicity for real-valued cocycles was studied in Baggett and Merill [BM2].

The author thanks the referee for simplifying the second part of the proof of Theorem 1.

1. Definitions and basic facts. Let T be an automorphism (invertible measure preserving transformation) of a standard Lebesgue probability space (X,μ) and let $v(n)$ be a sequence of positive numbers converging to zero. We denote by ϵ the partition of X into singletons. According to [KS] (see also [CFS]), T admits a *cyclic approximation* (*c.a.*) with speed $v(n)$ if there exist a sequence of finite measurable partitions

$$\xi_n = \{C_0,\ldots,C_{h_n-1}\} \to \epsilon$$

and automorphisms T_n permuting cyclically the elements of ξ_n such that

$$\sum_{j=0}^{h_n-1} \mu(TC_j \Delta T_n C_j) < v(h_n).$$

If T admits c.a. with speed $o(1/n)$ then T is rank-1 and rigid so it has singular simple spectrum ([KS], [CFS], [IS]). As in [KS], we let

$$d(T) = \sup\{t : T \text{ admits c.a. with speed } 1/n^t\}.$$

Let A be a nonempty set of real numbers. We will say that A admits a *simultaneous diophantine approximation* with speed $v(n)$ if there exist integers $q_n \to \infty$ such that for every $x \in A$ one can find integers $p_n(x)$ and a number $n(x)$ with $|x - p_n(x)/q_n| < v(q_n)$ for all $n \geq n(x)$. We write $d(A)$ for the supremum of the real numbers t such that A admits a simultaneous diophantine approximation with speed $1/n^t$. Then the Hausdorff dimension of A does not exceed $1/d(A)$ (see [I3]). For $A = \{\alpha\}$ we always have $d(\alpha) \geq 2$. Clearly $d(\alpha) = \infty$ means that α is a Liouville number.

It follows from [I1] (see also [I3]) that if T is an irrational rotation $x \to x + \alpha \bmod 1$, then $d(T) = d(\alpha)$. For an arbitrary automorphism T

it is proved in [I3] that if T admits c.a. with speed $o(v(n))$ then its spectral measure is concentrated on a set that admits a simultaneous diophantine approximation also with speed $o(v(n))$. In particular, the Hausdorff dimension of the spectral measure is always $\leq 1/d(T)$.

We will consider Anzai skew products over irrational rotations (see [A]). Let α be an irrational number and $\phi : \mathbb{T} \to \mathbb{T}$ be a measurable function (a *cocycle*). In the additive notation we identify the 1-torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with the unit interval and put $T_\phi(x,y) = (x+\alpha,\ y+\phi(x))$. This (Anzai) skew product preserves Lebesgue measure of the 2-torus $\mathbb{T}^2$ and is a homeomorphism or a diffeomorphism if ϕ is continuous or C^1, respectively. Two cocycles ϕ,ψ are called *cohomologous* if there exists another cocycle γ such that

$$\phi(x) = \psi(x) + \gamma(x+\alpha) - \gamma(x),$$

in which case we write $\phi \sim \psi$. If $\phi \sim 0$ (or $\phi \sim 1$ in the multiplicative notation) then ϕ is called a *coboundary*. It is easy to see that cohomologous cocycles give rise to isomorphic skew products. We say that ϕ is a *weakly mixing cocycle* over α if $m\phi \sim c$ cannot hold for $m \in \mathbb{Z} \setminus \{0\}, c \in \mathbb{T}$. It is well known (see [A]) that T_ϕ is then ergodic with nondiscrete spectrum and its only eigenvalues are the numbers $\exp(2\pi i n\alpha)$.

We denote by Φ the set of measurable cocycles, where functions that are equal a.e. are identified; Φ with convergence in measure becomes a Polish space. Some ergodic properties of generic cocycles from Φ are known. It was proved in [R], in a more general setup, that if α has a sufficiently good diophantine approximation then there is a residual (i.e. co-meager) set of cocycles such that the corresponding skew product has simple spectrum. In fact, from [IS] and [I1] we know that for any irrational α the generic cocycle ϕ is weakly mixing and $d(T_\phi) = d(\alpha)$, which implies a nondiscrete simple singular spectrum and in fact rank-1. Concrete examples of such step function cocycles have been constructed in [I2].

Now let Φ_0 denote the set of continuous cocycles ϕ with zero topological degree. This means that ϕ, viewed as a complex-valued function on the unit interval, can be represented as $\phi(x) = \exp(2\pi i f(x))$, where f is a continuous real-valued 1-periodic function on $\mathbb{R}$. It is shown in [IS] that for the uniform topology the generic ϕ in Φ_0 is weakly mixing and if α has unbounded partial quotients then T_ϕ admits c.a. with speed $o(1/n)$, so it is rank-1. By the same proof we easily deduce that $d(T_\phi) \geq d(\alpha) - 1$ generically in Φ_0.

In the sequel we will only study smooth skew products. We denote by Φ^r the set of all cocycles of class C^r $(r = 1,\ldots,\infty)$ and by Φ_0^r its subset consisting of the C^r cocycles of topological degree zero. For $\phi \in \Phi^r$ there exists a C^r real-valued function $f(x)$ on $\mathbb{R}$ such that $f(x+1) - f(x) \in \mathbb{Z}$ is the topological degree of ϕ and $\phi(x) = \exp(2\pi i f(x))$. If $\phi \in \Phi_0^r$ then

f is 1-periodic. We choose the following C^r distance as the metric in Φ^r $(1 \le r \le \infty)$:

$$\varrho^r(\phi_1, \phi_2) = \|f_1 - f_2\| + \sum_{i=1}^{r} \frac{2^{-i}\|D^i f_1 - D^i f_2\|}{1 + \|D^i f_1 - D^i f_2\|},$$

where D^i denotes the ith derivative, $\|\ \|$ is the uniform norm, and the functions f_1, f_2 with $\phi_j = \exp(2\pi i f_j)$ are chosen to minimize the right hand side. With this metric Φ^r becomes a Polish space so the notion of genericity is meaningful in Φ^r and Φ_0^r.

If $\phi \in \Phi_0^1$, then we know from [GLL] that the spectral measure must be singular. The spectrum of T_ϕ becomes "too small" if α has bounded partial quotients, because ϕ is then cohomologous to a constant (see e.g. [BM1], [BM2]), which forces the spectrum to be purely discrete. By the same token the latter holds true for any given α if ϕ is "too smooth". For a.e. α this is the case for any degree zero C^2 cocycle, which then becomes cohomologous to a constant. From this statistical point of view the class Φ_0^1 seems to be the most interesting one (Corollaries 1 and 3 below).

2. Cyclic approximation. Throughout the paper we fix an irrational number α and a sequence of rationals $p_n/q_n \to \alpha$ with q_n positive and p_n, q_n relatively prime. For a fixed n, if $f : \mathbb{R} \to \mathbb{R}$ and $k = 1, 2, \ldots$, we write

$$f^{(k)}(x) = f(x) + f(x + p_n/q_n) + \ldots + f(x + (k-1)p_n/q_n).$$

We will need the following simple lemma.

LEMMA. *Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and 1-periodic. Then there exist continuous 1-periodic functions g_n such that*

(1) $\|g_n - g\| \to 0$,
(2) *g_n is linear on $[(j-1)/q_n, j/q_n]$, $j \in \mathbb{Z}$,*
(3) $g_n^{(q_n)} = \text{const}$,
(4) $\int_0^1 g_n = \int_0^1 g$.

Proof. First let $\widetilde{g}_n(j/q_n) = g(j/q_n)$ and then extend $\widetilde{g}_n$ to a piecewise linear function on $\mathbb{R}$. It is clear that $\widetilde{g}_n^{(q_n)}$ is $1/q_n$-periodic and linear on the intervals specified in (2). Consequently, $\widetilde{g}_n^{(q_n)} = \text{const}$. On the other hand, by uniform continuity $\|\widetilde{g}_n - g\| \to 0$. Note that

$$\Big|\int_0^1 \widetilde{g}_n - \int_0^1 g\Big| \le \|\widetilde{g}_n - g\|,$$

so by adding $a_n = \int_0^1 g - \int_0^1 \widetilde{g}_n \to 0$ we obtain (1)–(4) for $g_n = \widetilde{g}_n + a_n$.

THEOREM 1. *Suppose $|\alpha - p_n/q_n| = o(\varepsilon(q_n)/q_n^2)$, where $\varepsilon(n)$ is a bounded sequence of positive numbers satisfying $\sup_n \varepsilon(n)/\varepsilon(2n) < \infty$. Then the set of cocycles $\phi \in \Phi_0^r$ such that T_ϕ admits c.a. with speed $o(\varepsilon(n)/n)$ is residual in Φ_0^r $(r = 1, \ldots, \infty)$.*

Proof. First we choose a positive sequence δ_n satisfying the condition

$$|\alpha - p_n/q_n| \le \delta_n = o(\varepsilon(q_n)/q_n^2)$$

and fix an auxiliary sequence of positive integers $s_n \to \infty$ such that

$$s_n\delta_n = o(\varepsilon(s_n q_n)/q_n^2)$$

(the assumption $\sup_n \varepsilon(n)/\varepsilon(2n) < \infty$ is used to allow the change of argument in ε). Let $\phi = \exp(2\pi i f) \in \Phi_0^r$. Now the proof splits into two parts.

First assume $r < \infty$. We apply the lemma to $g = D^r f$. Let $f_{r,n} = g_n$. If $r > 1$, define

$$f_{r-1,n}(x) = c_{r-1,n} + \int_0^x g_n,$$

where $c_{r-1,n}$ is determined by the condition

$$\int_0^1 f_{r-1,n} = 0.$$

It is now easy to see that $f_{r-1,n}$ is a smooth 1-periodic function. Since

$$(f_{r-1,n}^{(q_n)})' = g_n^{(q_n)} = \text{const} = q_n \int_0^1 g_n = 0,$$

the function $f_{r-1,n}^{(q_n)}$ must be constant; in fact it must vanish because its integral does. We iterate the procedure to define functions $f_{i,n}$, $i = r-1, \ldots, 1$, such that $f_{i,n}' = f_{i+1,n}$, $\int_0^1 f_{i,n} = 0$, and $f_{i,n}^{(q_n)} = 0$. We have $\|f_{i,n} - D^i f\| \to 0$. Finally, we let

$$f_n(x) = c_n + \int_0^x f_{1,n},$$

where the constant c_n is chosen to ensure

$$q_n \int_0^1 f_n = 1/s_n \bmod 1$$

along with the convergence $f_n \to f$ in C^r (this is always possible since $q_n \to \infty$). As before, $f_n^{(q_n)} = \text{const}$. More precisely,

$$f_n^{(q_n)}(x) = \int_0^1 f_n^{(q_n)} = q_n \int_0^1 f_n = 1/s_n \bmod 1.$$

Now for every $\phi = \exp(2\pi i f) \in \Phi_0^r$ we select a sequence f_n as above and for every n denote by $\Phi(n)$ the set of all the cocycles $\phi_n = \exp(2\pi i f_n)$. Clearly the union $\bigcup_{n\ge N}\Phi(n)$ is dense in Φ_0^r for all N. We choose positive numbers

$$\varrho_n = o\left(\frac{\varepsilon(s_n q_n)}{s_n^2 q_n}\right)$$

and for each of the selected $f_n\in\Phi(n)$ take the ball of radius ϱ_n centered at f_n. For a given n denote by $\Psi(n)$ the union of all such balls where $f_n\in\Phi(n)$. The set $\bigcup_{n\ge N}\Psi(n)$ being open and dense, we find that the intersection

$$\Psi=\bigcap_N\bigcup_{n\ge N}\Psi(n)$$

is residual.

It remains to show that for any $\psi=\exp(2\pi i f)\in\Psi$ the skew product T_ψ admits c.a. with required speed. First note that for infinitely many n we have $\varrho^r(\psi,\phi_n)<\varrho_n$. For these n's we are going to define ξ_n and T_n according to the definition of cyclic approximation. It will be convenient to identify the 2-torus with the unit square. We begin by letting

$$T_n(x,y)=(x+p_n/q_n, y+\phi_n(x))$$

(in the additive notation) and

$$C_0=[0,1/q_n)\times[0,1/s_n).$$

Now we put $C_j=T_n^jC_0$ for $j=1,\dots,Q_n-1$, where $Q_n=s_nq_n$. Since $f_n^{(q_n)}(x)=1/s_n \bmod 1$, it is clear that

$$C_{iq_n}=[0,1/q_n)\times[i/s_n,(i+1)/s_n)$$

for $i=0,1,\dots,s_n-1$. As $T_nC_{Q_n-1}=C_0$, we have a cycle of length Q_n. It is clear that the sets C_{iq_n} $(0\le i<s_n)$ are rectangles and $\xi_n=\{C_0,\dots,C_{Q_n-1}\}$ is a partition. In order to prove $\xi_n\to\epsilon$ we have to control the distortion of the sets $T_n^kC_{iq_n}$. In fact the diameter of C_j does not exceed

$$1/q_n+1/s_n+\sup|f_n^{(k)}(x)-f_n^{(k)}(y)|,$$

where the sup is taken over the points $x,y\in[0,1/q_n)$ and over all $1\le k<q_n$. First we compare the value of $f_n^{(k)}(x)$ with the sum $\sum_{j=0}^{k-1}f_n(x+j\alpha)$ corresponding to the irrational rotation. We have

$$f_n^{(k)}(x)-\sum_{j=0}^{k-1}f_n(x+j\alpha)=\sum_{j=0}^{k-1}\int_{I_j}f_n',$$

where I_j is an interval of length $|jp_n/q_n-j\alpha|<q_n\delta_n$. Therefore,

$$\Big|f_n^{(k)}(x)-\sum_{j=0}^{k-1}f_n(x+j\alpha)\Big|\le q_n^2\delta_n\sup_n\|f_n'\|\to 0.$$

Next, since $\varrho^r(\phi_n,\psi)<\varrho_n$ implies $\|f_n-f\|<\varrho_n$, we get

$$\sup\Big|\sum_{j=0}^{k-1}f_n(x+j\alpha)-\sum_{j=0}^{k-1}f(x+j\alpha)\Big|<q_n\varrho_n\to 0.$$

Finally, it is well known that if f is a 1-periodic C^1 function then for any irrational number α,

$$\Big|\sum_{j=0}^{k-1}f(x+j\alpha)-\sum_{j=0}^{k-1}f(y+j\alpha)\Big|\to 0$$

uniformly in k and x,y such that $1\le k\le q_n$, $|x-y|\le 1/q_n$ (see [CFS], 16.3, Lemma 2). Consequently, the diameters of the sets C_j tend to zero uniformly in j as $n\to\infty$. This clearly implies $\xi_n\to\epsilon$. To estimate the approximation error we note that it has two different parts, $\Delta=\Delta_1+\Delta_2$. The first, Δ_1, is due to the fact that $\alpha\ne p_n/q_n$. It is easy to see that each of the q_n columns contributes to Δ_1 at most $2\delta_n$, so

$$\Delta_1\le 2\delta_nq_n=2s_n\delta_nq_n^2/Q_n=o(\varepsilon(Q_n)/Q_n).$$

The second part, Δ_2, is due to $\psi\ne\phi_n$. Since, however, the distance is less than ϱ_n, we get

$$\Delta_2<Q_n\frac{2\varrho_n}{q_n}=2s_n\varrho_n=o(\varepsilon(Q_n)/Q_n).$$

This gives the required speed of approximation and ends the proof of the case $r<\infty$.

The proof for Φ_0^∞ requires a modification. Let f be in C^∞ and, for a fixed $1\le r<\infty$, let f_n be the sequence constructed before in C^r (i.e., $\|f_n-f\|_{C^r}\to 0$ and $f_n^{(q_n)}=1/s_n \bmod 1$.) To smooth the f_n's we use a nonnegative C^∞ function u_r such that $\int_0^1u_r=1$ and $u_r(x)=0$ for $x\in(\gamma_r,1]$, where $\gamma_r\to 0$. Now define $\widetilde f_n=f_n*u_r$. These are 1-periodic C^∞ functions and $\|\widetilde f_n-f*u_r\|_{C^r}\to 0$, while

$$\widetilde f_n^{(q_n)}=1/s_n \bmod 1.$$

We do this for $r=1,2,\dots$, which produces for each r a sequence of C^∞ functions approximating $f*u_r$ in C^r. Now select from the rth sequence an approximating function $\widetilde f_{n_r}$ in such a way that $n_r\to\infty$ and $\|\widetilde f_{n_r}-f\|_{C^r}\to 0$ as $r\to\infty$. Now we simply write f_n for $\widetilde f_{n_r}$, let $\phi_n=\exp(2\pi i f_n)$, and repeat in C^∞ the argument from the first part of the proof.

If follows readily from Theorem 1 that, generically in Φ_0^r, we have

$$d(T_\phi)\ge d(\alpha)-1.$$

In particular, the Hausdorff dimension of the spectral measure is then $\le 1/(d(\alpha)-1)$ (see [I3]). The following two corollaries are for "most" α in the sense of measure or category.

COROLLARY 1. *For a.e. α the set of cocycles ϕ such that T_ϕ admits c.a. with speed $o(1/(n\log n))$ is residual in Φ_0^1.*

Proof. We know from [Kh] that almost every α admits a rational approximation

$$\left|\alpha - \frac{p_n}{q_n}\right| = o\left(\frac{1}{q_n^2\log q_n}\right).$$

It suffices to let $\varepsilon(n) = 1/\log n$ in Theorem 1.

COROLLARY 2. *There exists a residual subset $A\subset\mathbb{T}$ such that for every $\alpha\in A$ the set of cocycles ϕ satisfying $d(T_\phi)=\infty$ is residual in Φ_0^r.*

Proof. Let $A=\{\alpha : d(\alpha)=\infty\}$. Given $\alpha\in A$, $k>2$, we put $\varepsilon(n) = n^{-k+2}$ and obtain a residual set Ψ_k of cocycles ϕ such that $d(T_\phi)\ge k-1$. Now $d(T_\phi)=\infty$ for every $\phi\in\bigcap_{k=1}^{\infty}\Psi_k$.

Remark. In the second part of the proof of Theorem 1, the case $r=\infty$, we may define $\Psi(n)$ as the ϱ_n-neighborhood of $\Phi(n)$ in Φ_0^1 for the C^1 distance (actually it is only the uniform distance that is used to estimate Δ_2 but we need C^1 convergence to prove $\xi_n\to\epsilon$). Now Ψ becomes a G_δ set in Φ_0^1 containing a dense subset of Φ_0^∞ and consisting solely of cocycles which admit c.a. with speed $o(\varepsilon(n)/n)$. This implies that $\Psi\cap\Phi_0^r$ is residual in Φ_0^r for $r=1,\dots,\infty$. More generally, if E is any topological space of cocycles such that

$$\Phi_0^\infty\subset E\subset\Phi_0^1$$

with continuous identity imbeddings $\Phi_0^\infty\to E\to\Phi_0^1$ then $\Psi\cap E$ is a dense G_δ subset of the space E and the theorem remains valid for E. In particular, this applies to the spaces of $C^{r+\delta}$ cocycles of degree zero ($0<\delta<1$, $r=1,2,\dots$).

3. Weakly mixing cocycles. In this section we show that for any α with unbounded partial quotients most cocycles in Φ_0^1 are weakly mixing. More generally, if α admits approximation of the type $|\alpha-p_n/q_n| = o(1/q_n^{r+1})$ then the weakly mixing cocycles over α form a residual set in Φ_0^r. A similar result for ergodic $\mathbb{R}$-valued cocycles was obtained by Baggett and Merill [BM2]. Our proof is based on the following result of Katok (see [K], Theorem 12.7):

Suppose $\sum_{n\in\mathbb{Z}}|na_n|<\infty$, where $a_{-n}=\overline{a}_n$. If $|\alpha-p_n/q_n|q_n = o(|a_{q_n}|)$ and $\inf_n(|a_{q_n}|/\sum_{k\ge1}|a_{kq_n}|)>0$ then the cocycle

$$\psi(x) = \exp\Big(2\pi i\sum_n a_n\exp(2\pi i n x)\Big)$$

is weakly mixing.

We note that although the original statement in [K] requires that the trigonometric series be a C^2 function, the proof uses only $\sum|na_n|<\infty$.

THEOREM 2. *Suppose $|\alpha-p_n/q_n| = o(1/q_n^{r+1})$. Then the weakly mixing C^r cocycles form a dense G_δ set in Φ_0^r ($1\le r<\infty$). If $d(\alpha)=\infty$, the same holds true for $r=\infty$.*

Proof. First we show that there is at least one such cocycle. Assume $r<\infty$. Without loss of generality

$$|\alpha-p_n/q_n|\le\varepsilon_n^2/q_n^{1+r},$$

where the ε_n are positive numbers decreasing to 0. By passing to a subsequence we may as well assume $\sum\varepsilon_n<\infty$ and $q_n/q_{n+1}\le 1/2$. Now let

$$a_k = a_{-k} = \varepsilon_n/q_n^r$$

if $|k|=q_n$ ($n=1,2,\dots$), and $a_k=0$ otherwise. The assumptions of Katok's result are satisfied. In fact,

$$\sum_{n\ge1}q_n^r a_{q_n} = \sum_{n\ge1}\varepsilon_n<\infty,\qquad \frac{|\alpha-p_n/q_n|q_n}{a_{q_n}}\le\frac{\varepsilon_n^2}{q_n^r a_{q_n}} = \varepsilon_n\to0,$$

$$\frac{a_{q_n}}{\sum_{k\ge1}a_{kq_n}}\ge\frac{a_{q_n}}{\sum_{i\ge n}a_{q_i}}>\frac{1}{1+q_n/q_{n+1}+q_n/q_{n+2}+\cdots}\ge1/2.$$

As a result we obtain a weakly mixing cocycle $\psi=\exp(2\pi i f)$, where

$$f(x) = 2\sum_{n=1}^{\infty}\frac{\varepsilon_n}{q_n^r}\cos(2\pi q_n x).$$

Since $\sum\varepsilon_n<\infty$, we have $f\in C^r$ so $\psi\in\Phi_0^r$. The same construction works for Φ_0^∞ if r is replaced with a sequence $r_n\to\infty$.

The rest of the proof is rather standard (cf. [IS], Theorem 4). Every cocycle $\phi(x)=\exp(2\pi i p(x))$, where p is a real trigonometric polynomial with $\int_0^1 p=0$, is a coboundary (see [B]). The constant cocycles of the form $\exp(2\pi i n\alpha)$ are also coboundaries, so we deduce by the Weierstrass theorem that the coboundaries are dense in Φ_0^r ($r=1,\dots,\infty$). Therefore the cocycles $\phi\psi$, where ϕ is a coboundary, are dense. Since they are all weakly mixing, we get the denseness.

To show that the set is G_δ we just recall Theorem 4 of [IS], where it is shown that in Φ_0 the weakly mixing cocycles form a G_δ set for the uniform topology. Since the topology of Φ_0^r is stronger, the result follows by intersecting with the latter space.

It was observed in [BM2] that if the above approximation condition for α is not satisfied then every cocycle in Φ_0^r is cohomologous to a constant. It follows that the condition in the theorem is necessary and sufficient.

The following is an immediate corollary of Theorems 1 and 2.

COROLLARY 3. *If α has unbounded partial quotients then a generic C^1 cocycle of degree zero is weakly mixing and the associated skew-product diffeomorphism admits c.a. with speed $o(1/n)$. In particular it is rank-1 and has a nondiscrete singular simple spectrum with the only eigenvalues of the form* $\exp(2\pi in\alpha)$.

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Received July 4, 1994
Revised version April 11, 1995 (3308)

STUDIA MATHEMATICA 115 (3) (1995)

Compressible operators and the continuity of homomorphisms from algebras of operators

by

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Abstract. The notion of a compressible operator on a Banach space, E, derives from automatic continuity arguments. It is related to the notion of a cartesian Banach space. The compressible operators on E form an ideal in $\mathcal{B}(E)$ and the automatic continuity proofs depend on showing that this ideal is large. In particular, it is shown that each weakly compact operator on the James' space, J, is compressible, whence it follows that all homomorphisms from $\mathcal{B}(J)$ are continuous.

Introduction. It is shown by B. E. Johnson in [BJo] that all homomorphisms and derivations from $\mathcal{B}(E)$, the algebra of bounded, linear operators on the Banach space E, are continuous if E satisfies one of two decomposability conditions. The first of these conditions is that E be *cartesian*, that is, satisfy $E \cong F \oplus E \oplus E$ for some Banach space F, and the second that E have *continued bisection*, that is, $E \cong E_2 \oplus E_2$, where $E_2 \cong E_3 \oplus E_3$ and so on.

The continuity of derivations can be proved for some spaces which do not satisfy either of these decomposability conditions and this is done in [L&W] for the James space, J, and for $C(\Omega)$, where Ω is the space of ordinals less than or equal to the first uncountable ordinal. (However, it is not resolved in [L&W] whether all homomorphisms from $\mathcal{B}(J)$ and $\mathcal{B}(C(\Omega))$ are continuous.) There are spaces E, whose construction is based on that of J, such that all derivations from $\mathcal{B}(E)$ are continuous but for which there exist discontinuous homomorphisms from $\mathcal{B}(E)$, see [D,L&W]. There are also spaces for which there are discontinuous derivations from $\mathcal{B}(E)$, see [Re]. The existence of discontinuous derivations implies the existence of discontinuous homomorphisms.

The question of the continuity of homomorphisms from $\mathcal{B}(C(\Omega))$ has been treated by C. Ogden, who has shown that they are all continuous, see [Og].

1991 *Mathematics Subject Classification*: Primary 46H40; Secondary 46B03, 47D30.