STUDIA MATHEMATICA 115 (3) (1995)

# Reproducing properties and $\mathbf{L}^p$-estimates for Bergman projections in Siegel domains of type II

by

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**Abstract.** On homogeneous Siegel domains of type II, we prove that under certain conditions, the subspace of a weighted $\mathbf{L}^p$-space $(0 < p < \infty)$ consisting of holomorphic functions is reproduced by a weighted Bergman kernel. We also obtain some $\mathbf{L}^p$-estimates for weighted Bergman projections. The proofs rely on a generalization of the Plancherel–Gindikin formula for the Bergman space $\mathbf{A}^2$.

**I. Introduction.** Let $\mathbf{D}$ be an affine-homogeneous Siegel domain of type II. Let $dv$ denote the Lebesgue measure on $\mathbf{D}$ and let $\mathbf{H}(\mathbf{D})$ denote the space of holomorphic functions in $\mathbf{D}$. The *Bergman projection* $P$ of $\mathbf{D}$ is the orthogonal projection of $\mathbf{L}^2(\mathbf{D}, dv)$ onto its subspace $\mathbf{A}^2(\mathbf{D})$ consisting of holomorphic functions. Moreover, $P$ is the integral operator defined on $\mathbf{L}^2(\mathbf{D}, dv)$ by the Bergman kernel $B(\zeta, z)$ and for $\mathbf{D}$, this kernel was computed in [G].

Let $\varepsilon$ be a real number. For $p \in (0, \infty)$, we set $\mathbf{L}^{p,\varepsilon}(\mathbf{D}) = \mathbf{L}^p(\mathbf{D}, B^{-\varepsilon}(z,z)dv(z))$ and we define the weighted Bergman space $\mathbf{A}^{p,\varepsilon}(\mathbf{D})$ by $\mathbf{A}^{p,\varepsilon}(\mathbf{D}) = \mathbf{L}^{p,\varepsilon}(\mathbf{D}) \cap \mathbf{H}(\mathbf{D})$. There exists $\varepsilon_0 < 0$ such that $\mathbf{A}^{2,\varepsilon}(\mathbf{D}) = \{0\}$ whenever $\varepsilon \leq \varepsilon_0$; for $\varepsilon > \varepsilon_0$, the corresponding weighted Bergman projection $P_\varepsilon$ is the orthogonal projection of $\mathbf{L}^{2,\varepsilon}(\mathbf{D})$ onto $\mathbf{A}^{2,\varepsilon}(\mathbf{D})$.

The first purpose of this paper is to generalize to the weighted Bergman spaces $\mathbf{A}^{2,\varepsilon}(\mathbf{D})$ the Plancherel–Gindikin formula proved for $\mathbf{A}^2(\mathbf{D})$ by S. G. Gindikin [G] and by A. Korányi and E. M. Stein [KoS]. Our proof is an extension of that of Korányi and Stein. More precisely, assume that the Siegel domain $\mathbf{D}$ is associated with a homogeneous cone $\mathbf{V} \subset \mathbb{R}^n$, $n \geq 3$, and with a $\mathbf{V}$-Hermitian homogeneous form $\mathbf{F} : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^n$, and let $\mathbf{V}^*$ denote the conjugate cone of $\mathbf{V}$. Thus, $\mathbf{D}$ is contained in $\mathbb{C}^n \times \mathbb{C}^m$. For $\varepsilon > \varepsilon_0$, we prove that a function $f \in \mathbf{H}(\mathbf{D})$ belongs to $\mathbf{A}^{2,\varepsilon}(\mathbf{D})$ if and only if there exists a function $\widehat{f} : \mathbf{V}^* \times \mathbb{C}^m \to \mathbb{C}$ belonging to a weighted $\mathbf{L}^2$-space

1991 *Mathematics Subject Classification*: 32A07, 32A25, 32H10, 32M15.

such that

$$f(z,u)=\int_{\mathbf{V}^*}\exp(i\langle\lambda,z\rangle)\widehat{f}(\lambda,u)\,d\lambda$$

and the map $f\mapsto\widehat{f}$ is an isometry. Here, $\langle\ ,\ \rangle$ denotes the inner product with respect to which $\mathbf{V}^*$ is the conjugate cone of $\mathbf{V}$. This statement will be made more precise and more general in Section II, where a useful estimate for the Bergman kernel will be given as a corollary. J. Peetre [P] also proved a Plancherel–Gindikin formula for more general weights; his proof is different from ours.

Our second goal is to give conditions on real numbers $r>\varepsilon_0$ and $p\in(0,\infty)$ under which there exists $\varepsilon>\varepsilon_0$ such that the weighted Bergman projection $P_\varepsilon$ reproduces functions in $\mathbf{A}^{p,r}(\mathbf{D})$. We first deduce from the Plancherel–Gindikin formula that $P_\varepsilon$ is the integral operator defined on $\mathbf{L}^{2,\varepsilon}(\mathbf{D})$ by the kernel $c_\varepsilon B^{1+\varepsilon}(\zeta,z)$. Some of our reproducing formulae are based upon the density of $\mathbf{A}^{p,r}(\mathbf{D})\cap\mathbf{A}^{2,\varepsilon}(\mathbf{D})$ in $\mathbf{A}^{p,r}(\mathbf{D})$. These formulae are an ingredient in the proof of the atomic decomposition theorem for functions in $\mathbf{A}^{p,r}(\mathbf{D})$ [CR]. In a subsequent paper, we shall deal with the atomic decomposition theorem for not necessarily symmetric Siegel domains of type II.

Our last goal is to give sufficient conditions on $p\in[1,\infty)$ and real $r$ and $\varepsilon>\varepsilon_0$ under which $P_\varepsilon$ is bounded on $\mathbf{L}^{p,r}(\mathbf{D})$. Our results are far better than those obtained by M. M. Dzhrbashyan and Karapetyan [DK] for the tube over the cone of Hermitian positive definite matrices of order $n$. We also give values of $p$ and $\varepsilon$ for which $P_\varepsilon$ is not bounded on $\mathbf{L}^{p,\varepsilon}(\mathbf{D})$. In particular, let us point out that for $\varepsilon=r=0$, there are two intervals $I_1$ and $I_2$ in $[1,\infty)$ where we are unable to conclude whether $P$ is $\mathbf{L}^p$-bounded on $\mathbf{L}^p(\mathbf{D})$ or not for $p\in I_1\cup I_2$. Our results extend to general Siegel domains of type II those obtained in [B] and [BeBo] for the tube over the spherical cone of $\mathbb{R}^n$, $n\ge 3$.

The plan of this paper is as follows. In Section II, we recall some preliminary results about affine-homogeneous Siegel domains of type II and we give precise statements of our results. In Section III, we prove the Plancherel–Gindikin formula for $\mathbf{A}^{2,\varepsilon}(\mathbf{D})$ (Theorem II.2) and its useful corollary (Corollary II.4). The reproducing formulae (Theorem II.6) are proved in Section IV, while weighted $\mathbf{L}^p$-estimates for weighted Bergman projections are proved in Section V.

All the results stated below were first presented in [T]. In the sequel, as usual, the same letter $C$ will denote constants that may be different from each other.

**II. Statements of results.** Let $\mathbf{V}\subset\mathbb{R}^n$, $n\ge 3$, be an irreducible, open, convex and homogeneous cone which contains no straight line. We first recall the canonical decomposition of $\mathbf{V}$ as stated in [G].



NOTATIONS. (i) At the $j$th step, $j=1,2,\ldots$, the real line will be denoted by $R_{jj}$; at the $k$th step, $k=2,3,\ldots$, $R_k$ will stand for the $n_k$-dimensional euclidean space.

(ii) Let $\Gamma\subset\mathbb{R}^\sigma$ be a convex homogeneous cone which contains no straight line and let $\varphi$ be a homogeneous $\Gamma$-bilinear symmetric form defined on $\mathbb{R}^\tau\times\mathbb{R}^\tau$. Recall that the associated real homogeneous *Siegel domain* $\mathbf{P}=\mathbf{P}(\Gamma,\varphi)$ is defined by

$$\mathbf{P}=\mathbf{P}(\Gamma,\varphi)=\{(y,t)\in\mathbb{R}^\sigma\times\mathbb{R}^\tau: y-\varphi(t,t)\in\Gamma\}.$$

We shall denote by $\mathbf{V}(\mathbf{P})$ the homogeneous cone defined by

$$\mathbf{V}(\mathbf{P})=\{(y,t,r)\in\mathbb{R}^\sigma\times\mathbb{R}^\tau\times\mathbb{R}: r>0 \text{ and } (ry,t)\in\mathbf{P}\}.$$

In order to describe the canonical decomposition of the irreducible homogeneous cone $\mathbf{V}$, we consider at the first step the cone $\mathbf{V}^{(1)}=(0,\infty)\subset R_{11}$. At the second step, we associate with $\mathbf{V}^{(1)}$ and with a homogeneous $\mathbf{V}^{(1)}$-bilinear symmetric form $\varphi^{(2)}$ defined on $R_2$, the real Siegel domain $\mathbf{P}^{(2)}=\mathbf{P}(\mathbf{V}^{(1)},\varphi^{(2)})$ contained in $R_{11}\times R_2$ and then the irreducible cone $\mathbf{V}^{(2)}=\mathbf{V}(\mathbf{P}^{(2)})\subset R_{11}\times R_2\times R_{22}$. At the $k$th step, we associate with the cone $\mathbf{V}^{(k-1)}$ and with a homogeneous $\mathbf{V}^{(k-1)}$-bilinear symmetric form $\varphi^{(k)}$ defined on $R_k$, a real Siegel domain $\mathbf{P}^{(k)}=\mathbf{P}(\mathbf{V}^{(k-1)},\varphi^{(k)})\subset R_{11}\times R_2\times R_{22}\times\ldots\times R_{k-1,k-1}\times R_k$ and then the irreducible cone $\mathbf{V}^{(k)}\subset R_{11}\times R_2\times R_{22}\times\ldots\times R_{k-1,k-1}\times R_k\times R_{kk}$.

It follows from the results of [G] that every homogeneous irreducible cone containing no straight line can be decomposed in the form $\mathbf{V}=\mathbf{V}^{(i)}$ (up to affine isomorphism). The required number of steps to obtain $\mathbf{V}$ in this form is called the *rank* $l$ of $\mathbf{V}$, $\mathbf{V}=\mathbf{V}^{(l)}$. Hence this yields the following decomposition of $\mathbb{R}^n$ that contains $\mathbf{V}$:

$$(1)\qquad \mathbb{R}^n=R_{11}\times R_2\times R_{22}\times\ldots\times R_{l-1,l-1}\times R_l\times R_{ll},\qquad l+\sum_{i=1}^{l}n_i=n.$$

Furthermore, the projection $\varphi_{ii}^{(k)}$ of $\varphi^{(k)}$ onto $R_{ii}$ is a non-negative form. Then $\varphi_{ii}^{(k)}$ is positive definite on a subspace $R_{ik}$ of $R_k$ with $\dim R_{ik}=n_{ik}$. We have

$$(2)\qquad R_k=\prod_{i=1}^{k-1}R_{ik},\quad\text{and so}\quad n_k=\sum_{i=1}^{k-1}n_{ik}.$$

On the other hand, the projection $\varphi_{ij}^{(k)}$ of $\varphi^{(k)}$ onto $R_{ij}$ $(i<j<k\le l)$ is concentrated on $R_{ik}\times R_{jk}$.

We denote by $\mathbf{G}(\mathbf{V})$ the simply transitive group of affine automorphisms of $\mathbf{V}$ described in [G]. With respect to its canonical decomposition, the cone $\mathbf{V}$ can be described in the following quantitative manner: let $x$ be in

$\mathbf{V}$ and let $x_j$, $j = 2,\ldots,l$ (resp. $x_{ii}$, $i = 1,\ldots,l$) denote the projection of $x$ onto $R_j$ (resp. $R_{ii}$). Then there exists a unique transformation $h \in \mathbf{G}(\mathbf{V})$ such that $(h(x))_j = 0$ for all $j = 2,\ldots,l$. We set $\widetilde{x} = h(x)$. The $l$ functions $\chi_j$ defined for $j = 1,\ldots,l$ by $\chi_j(x) = \widetilde{x}_{jj}$, $j = 1,\ldots,l$, define the cone $\mathbf{V}$ in the following sense: a point $x \in \mathbb{R}^n$ belongs to $\mathbf{V}$ if and only if $\chi_j(x) > 0$, $j = 1,\ldots,l$. More explicitly, set $x^{(0)} = x$; then there exists a unique transformation in $\mathbf{G}(\mathbf{V})$ which maps $x^{(0)}$ into $x^{(1)} \in \mathbf{V}$ satisfying $x_l^{(1)} = 0$. A second automorphism maps $x^{(1)}$ to $x^{(2)}$ satisfying $x_{l-1}^{(2)} = x_l^{(2)} = 0,\ldots,$ and finally, the $(l-1)$th automorphism assigns to $x^{(l-2)}$ satisfying $x_3^{(l-2)} = \ldots = x_l^{(l-2)} = 0$ the point $x^{(l-1)} = \widetilde{x}$. Moreover, the defining functions $\chi_j$ of $\mathbf{V}$ are given by

$$\chi_l(x) = x_{ll},$$

$$\chi_j(x) = x_{jj} - \sum_{i=j+1}^{l} \frac{\varphi_{jj}^{(i)}(x_i^{(l-i)}, x_i^{(l-i)})}{\chi_i(x)} \quad \text{for } j = l, l-1,\ldots,1.$$

Since the decomposition (1) and (2) of $\mathbb{R}^n$ yields in a natural way the following decomposition of $\mathbb{C}^n$:

$$\mathbb{C}^n = \prod_{i=1}^{l} \mathbb{C}_{ii} \times \prod_{i<j} \mathbb{C}_{ij}, \tag{3}$$

the functions $\chi_j$, $j = 1,\ldots,l$, can naturally be extended as rational functions on $\mathbb{C}^n$.

Let $\varrho \in \mathbb{C}^l$. We define the function $(z)^\varrho$ by

$$(z)^\varrho = \prod_{i=1}^{l} \chi_i^{\varrho_i}(z), \quad z \in \mathbb{C}^n, \ \varrho = (\varrho_1,\ldots,\varrho_l).$$

For $i = 1,\ldots,l$ we set

$$m_i = \sum_{j>i} n_{ij} \quad \text{and} \quad d_i = -(1 + (n_i + m_i)/2)$$

and $d$ will denote the vector of $\mathbb{R}^l$ whose components are $d_i$. Also, in the sequel, $e$ will denote the point of $\mathbf{V}$ whose components are $e_{ii} = 1$ for all $i = 1,\ldots,l$ and $e_j = 0$ for all $j = 2,\ldots,l$.

Let us now recall the definition of the conjugate cone $\mathbf{V}^*$ of $\mathbf{V}$. Consider the inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$ defined with respect to the canonical decomposition of $\mathbb{R}^n$ by

$$\langle x, y\rangle = \sum_{i=1}^{l} x_{ii}y_{ii} + 2\sum_{j<i} \varphi_{jj}^{(i)}(x_i, y_i).$$



The *conjugate cone* $\mathbf{V}^*$ of $\mathbf{V}$ with respect to $\langle\ ,\ \rangle$ is defined by

$$\mathbf{V}^* = \{x \in \mathbb{R}^n : \langle x, y\rangle > 0 \text{ for all } y \in \overline{\mathbf{V}} - \{0\}\}.$$

The *adjoint group* $\mathbf{G}^*(\mathbf{V})$ of $\mathbf{G}(\mathbf{V})$ with respect to $\langle\ ,\ \rangle$ is a simply transitive group of affine automorphisms of $\mathbf{V}^*$. The cone $\mathbf{V}^*$ is an irreducible, convex homogeneous cone of rank $l$ in $\mathbb{R}^n$. We shall denote by $\chi_j^*$ the defining functions of $\mathbf{V}^*$. Moreover, we have the following:

$$n_{ij}^* = n_{ij}(\mathbf{V}^*) = n_{l-i+1,l-j+1}, \quad 1 \le i < j \le l,$$
$$n_i^* = m_{l-i+1}, \quad m_i^* = n_{l-i+1}, \quad d_i^* = d_{l-i+1}, \quad 1 \le i \le l.$$

For $\varrho \in \mathbb{C}^l$, we define $\varrho^*$ by $\varrho_i^* = \varrho_{l-i+1}$, and we define the function $(z)_*^{\varrho^*}$ on $\mathbb{C}^n$ by

$$(z)_*^{\varrho^*} = \prod_{i=1}^{l} (\chi_i^*)^{\varrho_i^*}(z), \quad z \in \mathbb{C}^n.$$

The *Siegel domain of type II* associated with the cone $\mathbf{V} \subset \mathbb{R}^n$ and a $\mathbf{V}$-Hermitian, homogeneous form $\mathbf{F} : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^n$ is defined by

$$\mathbf{D} = \mathbf{D}(\mathbf{V},\mathbf{F}) = \left\{(z,u) \in \mathbb{C}^n \times \mathbb{C}^m : \frac{z - \overline{z}}{2i} - \mathbf{F}(u,u) \in \mathbf{V}\right\}.$$

This domain is affine-homogeneous. Now, let $\mathbf{F}_{ii}$ denote the projection of $\mathbf{F}$ onto the complex plane $\mathbb{C}_{ii}$ and let $\mathbb{C}_i$ denote the complex subspace of $\mathbb{C}^m$ where $\mathbf{F}_{ii}$ is positive definite. Set $q_i = \dim_{\mathbb{C}} \mathbb{C}_i$ (the complex dimension); then $\mathbb{C}^m = \prod_{i=1}^{l} \mathbb{C}_i$ and $m = \sum_{i=1}^{l} q_i$. Hence, by (2) and (3), the space $\mathbb{C}^n \times \mathbb{C}^m$ containing $\mathbf{D}$ is decomposed as follows:

$$\mathbb{C}^n \times \mathbb{C}^m = \prod_{i=1}^{l} \mathbb{C}_{ii} \times \prod_{i<j} \mathbb{C}_{ij} \times \prod_{i=1}^{l} \mathbb{C}_i.$$

We shall denote by $q$ the vector of $\mathbb{N}^l$ whose components are $q_i$.

We first recall the following two expressions of the Bergman kernel of $\mathbf{D} = \mathbf{D}(\mathbf{V},\mathbf{F})$.

II.1. Proposition [G]. *The Bergman kernel $B((\zeta,v),(z,u))$ of $\mathbf{D}$ is given by*

$$B((\zeta,v),(z,u)) = c\left(\frac{\zeta - \overline{z}}{2i} - \mathbf{F}(v,u)\right)^{2d-q}$$
$$= c\int_{\mathbf{V}^*} \exp\left(-\left\langle \lambda, \frac{\zeta - \overline{z}}{2i} - \mathbf{F}(v,u)\right\rangle\right)(\lambda)_*^{-d^*+q^*}\, d\lambda.$$

Notations. For $\varrho = (\varrho_1,\ldots,\varrho_l) \in \mathbb{C}^l$, the notation $1 + \varrho$ stands for the vector $(1 + \varrho_1,\ldots,1 + \varrho_l)$. Let $\varrho' = (\varrho_1',\ldots,\varrho_l') \in \mathbb{C}^l$; we set $\varrho\varrho' =$

$(\varrho_1\varrho'_1, \ldots, \varrho_l\varrho'_l)$. For two points $(\zeta, v)$ and $(z, u)$ in $\mathbf{D} \subset \mathbb{C}^n \times \mathbb{C}^m$, we let $b((\zeta, v), (z, u))$ denote the kernel

$$b((\zeta,v),(z,u)) = \left(\frac{\zeta - \overline{z}}{2i} - \mathbf{F}(v,u)\right)^{2d-q}$$

Notice that $B = cb$. Moreover, we let $b^\varrho$ and $b^{1+\varrho}$ denote the expressions

$$b^\varrho((\zeta,v),(z,u)) = \left(\frac{\zeta - \overline{z}}{2i} - \mathbf{F}(v,u)\right)^{(2d-q)\varrho}$$

and

$$b^{1+\varrho}((\zeta,v),(z,u)) = \left(\frac{\zeta - \overline{z}}{2i} - \mathbf{F}(v,u)\right)^{2d-q+(2d-q)\varrho}$$

Let $\varepsilon$ be a vector in $\mathbb{R}^l$. For $p \in (0, \infty)$, we set $\mathbf{L}^{p,\varepsilon}(\mathbf{D}) = \mathbf{L}^p(\mathbf{D}, b^{-\varepsilon}(z,z)dv(z))$ and define the *weighted Bergman space* $\mathbf{A}^{p,\varepsilon}(\mathbf{D})$ by $\mathbf{A}^{p,\varepsilon}(\mathbf{D}) = \mathbf{L}^{p,\varepsilon}(\mathbf{D}) \cap \mathbf{H}(\mathbf{D})$. The "norm" of $\mathbf{A}^{p,\varepsilon}(\mathbf{D})$ is the $\mathbf{L}^{p,\varepsilon}$-"norm" $\|\cdot\|_{p,\varepsilon}$. The corresponding *weighted Bergman projection* $P_\varepsilon$ is the orthogonal projection of $\mathbf{L}^{2,\varepsilon}(\mathbf{D})$ onto $\mathbf{A}^{2,\varepsilon}(\mathbf{D})$.

In order to state the Plancherel–Gindikin formula for $\mathbf{A}^{2,\varepsilon}(\mathbf{D})$, let $\widehat{\mathbf{L}}^2(\mathbf{V}^* \times \mathbb{C}^m, \varepsilon)$ stand for the Hilbert space consisting of functions $f : \mathbf{V}^* \times \mathbb{C}^m \to \mathbb{C}$ such that

(i) for all compact subsets $\mathbf{K}_1$ and $\mathbf{K}_2$ of $\mathbf{V}^*$ and $\mathbb{C}^m$ respectively, the map $u \mapsto f(\cdot, u)$ is holomorphic on $\mathbf{K}_2$ with values in $\mathbf{L}^2(\mathbf{K}_1, \varepsilon)$, where

$$\mathbf{L}^2(\mathbf{K}_1,\varepsilon) = \Big\{g : \mathbf{K}_1 \to \mathbb{C} : \int_{\mathbf{K}_1} |g(\lambda)|^2 (\lambda)_*^{(2d-q)^*\varepsilon^* + d^*}\, d\lambda < \infty\Big\},$$

(ii) the quantity

$$\|f\|_{\widehat{\mathbf{L}}^2(\mathbf{V}^*\times\mathbb{C}^m,\varepsilon)} = \Big(\int_{\mathbf{V}^*\times\mathbb{C}^m} \exp(-2\langle\lambda, \mathbf{F}(u,u)\rangle)|f(\lambda,u)|^2(\lambda)_*^{(2d-q)^*\varepsilon^*+d^*}\, d\lambda\, dv(u)\Big)^{1/2}$$

is finite and is the norm of $f$ in $\widehat{\mathbf{L}}^2(\mathbf{V}^* \times \mathbb{C}^m, \varepsilon)$.

II.2. Theorem (Plancherel–Gindikin formula). *Let $\varepsilon \in \mathbb{R}^l$ satisfy*

$$\varepsilon_i > \frac{n_i + 2}{2(2d-q)_i} \quad (i = 1, \ldots, l).$$

1) *For all $f \in \mathbf{A}^{2,\varepsilon}(\mathbf{D})$, there exists a function $\widehat{f} \in \widehat{\mathbf{L}}^2(\mathbf{V}^* \times \mathbb{C}^m, \varepsilon)$ such that*

(i) $$f(z,u) = \int_{\mathbf{V}^*} \exp(i\langle\lambda, z\rangle)\widehat{f}(\lambda,u)\, d\lambda$$



*with the estimate*

(ii) $$\|f\|_{2,\varepsilon} = c_\varepsilon \|\widehat{f}\|_{\widehat{\mathbf{L}}^2(\mathbf{V}^*\times\mathbb{C}^m,\varepsilon)}.$$

2) *Conversely, for all $\widehat{f} \in \widehat{\mathbf{L}}^2(\mathbf{V}^* \times \mathbb{C}^m, \varepsilon)$, there exists $f \in \mathbf{A}^{2,\varepsilon}(\mathbf{D})$ such that* (i) *and* (ii) *hold.*

The condition on $\varepsilon$ is justified by the following consequence of the proof of Theorem II.2:

II.3. Corollary. *Let $\varepsilon \in \mathbb{R}^l$ be such that there exists $i \in \{1, \ldots, l\}$ for which*

$$\varepsilon_i \le \frac{n_i + 2}{2(2d-q)_i}.$$

*Then $\mathbf{A}^{2,\varepsilon}(\mathbf{D}) = \{0\}$.*

From Theorem II.2 and Proposition II.1, we shall also deduce the following useful corollary:

II.4. Corollary. *Let $\alpha$ and $\varepsilon$ be two vectors in $\mathbb{R}^l$ and $(\zeta, v)$ a point of $\mathbf{D}$. Then the quantity*

$$\int_{\mathbf{D}} |b^{1+\alpha}((\zeta,v),(z,u))| b^{-\varepsilon}((z,u),(z,u))\, dv(z,u)$$

*is finite if and only if*

$$\varepsilon_i > \frac{n_i + 2}{2(2d-q)_i} \quad \text{and} \quad \alpha_i - \varepsilon_i > \frac{n_i}{-2(2d-q)_i} \quad \text{for all } i = 1, \ldots, l.$$

*In that case, the following identity holds:*

$$\int_{\mathbf{D}} |b^{1+\alpha}((\zeta,v),(z,u))| b^{-\varepsilon}((z,u),(z,u))\, dv(z,u) = c_{\alpha,\varepsilon} b^{\alpha-\varepsilon}((\zeta,v),(\zeta,v)).$$

Theorem II.2 and its corollaries are proved in Section III.

Before stating our reproducing formulae, let us give the following description of $P_\varepsilon$:

II.5. Proposition. *The weighted Bergman projection $P_\varepsilon$, where $\varepsilon \in \mathbb{R}^l$ satisfies*

$$\varepsilon_i > \frac{n_i + 2}{2(2d-q)_i} \quad (i = 1, \ldots, l),$$

*is equal to the integral operator defined on $\mathbf{L}^{2,\varepsilon}(\mathbf{D})$ associated with the kernel $c_\varepsilon b^{1+\varepsilon}((\zeta,v),(z,u))$.*

Our reproducing formulae read as follows:

II.6. Theorem. 1) *Let $r \in \mathbb{R}^l$ and $p \in \mathbb{R}_+$ be such that $r_i \ge 0$ and*

$$1 \le p < \frac{n_i - 2(2d-q)_i(1 + r_i)}{n_i} \quad (i = 1, \ldots, l).$$

*Then for all* $\varepsilon \in \mathbb{R}^l$ *satisfying*

$$\varepsilon_i > \frac{n_i+2}{2(2d-q)_i} \cdot \frac{p-1}{p} + \frac{r_i}{p} \quad (i=1,\ldots,l)$$

*when* $p>1$ *(resp. for all* $\varepsilon \in \mathbb{R}^l$ *satisfying* $\varepsilon_i \geq r_i$ $(i=1,\ldots,l)$ *when* $p=1$*), the reproducing formula* $P_\varepsilon f = f$ *holds for all* $f \in \mathbf{A}^{p,r}(\mathbf{D})$.

2) *Let* $r \in \mathbb{R}^l$ *and* $p \in (0,\infty)$ *satisfy*

$$r_i > \frac{n_i+2}{2(2d-q)_i} \quad \text{and} \quad 0 < p < \frac{-2(2d-q)_i(1+r_i)}{n_i} \quad (i=1,\ldots,l).$$

*Then for all* $\varepsilon \in \mathbb{R}^l$ *satisfying*

$$\varepsilon_i > \frac{n_i+2}{2(2d-q)_i} + \frac{r_i+1}{p} \quad (i=1,\ldots,l),$$

*the reproducing formula* $P_\varepsilon f = f$ *holds for all* $f \in \mathbf{A}^{p,r}(\mathbf{D})$.

In particular, for $r=0$, we are able to prove that $\mathbf{A}^p(\mathbf{D})$ has a reproducing operator $P_\varepsilon$ just when $p \in (0,p_0)$ where

$$p_0 = \min_{i=1,\ldots,l} \left\{ \frac{n_i - 2(2d-q)_i}{n_i} \right\}.$$

These reproducing formulae will be proved in Section IV.

Our last results are weighted $\mathbf{L}^p$-estimates ($p \in [1,\infty)$) for the weighted Bergman projections $P_\varepsilon$. We let $P_\varepsilon^*$ denote the integral operator on $\mathbf{L}^{2,\varepsilon}(\mathbf{D})$ associated with the positive kernel $|b^{1+\varepsilon}((\zeta,v),(z,u))|$.

II.7. THEOREM. 1) *Let* $\varepsilon$ *and* $r$ *in* $\mathbb{R}^l$ *satisfy*

$$\varepsilon_i > \frac{n_i+2}{2(2d-q)_i}, \quad r_i > \frac{n_i+2}{2(2d-q)_i}$$

*and*

$$\varepsilon_i - r_i > \frac{n_i}{-2(2d-q)_i} \quad (i=1,\ldots,l).$$

*Then* $P_\varepsilon^*$ *is bounded on* $\mathbf{L}^{1,r}(\mathbf{D})$.

2) *Let* $r$ *and* $\varepsilon$ *in* $\mathbb{R}^l$ *be such that*

$$\varepsilon_i > \frac{1}{(2d-q)_i} \quad \text{and} \quad r_i > \frac{n_i+2}{2(2d-q)_i} \quad (i=1,\ldots,l).$$

*Let* $p \in (1,\infty)$ *satisfy*

$$\max_{i=1,\ldots,l}\left(1, \frac{2n_i+2-2(2d-q)_i r_i}{n_i+2-2(2d-q)_i\varepsilon_i}\right) < p < \min_{i=1,\ldots,l} \frac{2n_i+2-2(2d-q)_i r_i}{n_i}.$$

*Then* $P_\varepsilon^*$ *is bounded on* $\mathbf{L}^{p,r}(\mathbf{D})$.

Of course, the $\mathbf{L}^{p,r}$-boundedness of $P_\varepsilon^*$ implies that of $P_\varepsilon$. We also prove the following negative result for $P_\varepsilon$.



II.8. THEOREM. *Let* $\varepsilon \in \mathbb{R}^l$ *satisfy*

$$\varepsilon_i > \frac{n_i+2}{2(2d-q)_i} \quad (i=1,\ldots,l)$$

*and set*

$$p_0(\varepsilon) = \max_{i=1,\ldots,l}\left\{ \frac{n_i}{-2(2d-q)_i(1+\varepsilon_i)} + 1 \right\}.$$

*Then* $P_\varepsilon$ *is not bounded on* $\mathbf{L}^{p,\varepsilon}(\mathbf{D})$ *when* $p \in [1, p_0(\varepsilon)]$.

In particular, in view of Theorems II.7 and II.8, for $r=\varepsilon=0$, $P$ is bounded on $\mathbf{L}^p(\mathbf{D})$ if $p \in (p_1, p_1')$ where

$$p_1 = \max\left\{\frac{2n_i+2}{n_i+2}\right\}$$

and $p_1'$ denotes the conjugate exponent of $p_1$. On the other hand, $P$ is not bounded on $\mathbf{L}^p(\mathbf{D})$ when $p \in [1,p_0]$, with

$$p_0 = \left\{\frac{n_i}{-2(2d-q)_i} + 1\right\} < p_1,$$

and $P$ is not well defined on $\mathbf{L}^p(\mathbf{D})$ when $p \in (p_0', \infty)$. There are two gaps, $p \in (p_0, p_1)$ and $p \in (p_1', p_0')$, where we are unable to decide whether $P$ is bounded on $\mathbf{L}^p(\mathbf{D})$ or not. Theorems II.7 and II.8 will be proved in Section V.

## III. Proof of Theorem II.2 and its corollaries

Proof of Theorem II.2. The proof relies heavily on ideas of [KoS]. In particular, we use the following lemma:

III.1. LEMMA [KoS]. *Let* $\mathbf{U}$ *be a subdomain of* $\mathbb{C}^N$. *Let* $\mathbf{M}$ *be a measure space and let* $f : \mathbf{U} \to \mathbf{L}^2(\mathbf{M})$ *be holomorphic. Then for each* $z \in \mathbf{U}$, *one can define* $f(z)(p)$ *for almost all* $p \in \mathbf{M}$ *so that*

(i) *for almost all* $p \in \mathbf{M}$, *the function* $z \mapsto f(z)(p)$ *is holomorphic on* $\mathbf{U}$;
(ii) *the map* $(z,p) \mapsto f(z)(p)$ *is jointly measurable on* $\mathbf{U} \times \mathbf{M}$;
(iii) *for each subdomain* $\mathbf{U}_0$ *whose closure is compact in* $\mathbf{U}$, *there exists* $\varphi \in \mathbf{L}^2(\mathbf{M})$ *so that* $|f(z)(p)| \leq \varphi(p)$ *for all* $z \in \mathbf{U}_0$ *and almost all* $p \in \mathbf{M}$.

We first prove assertion 2) of Theorem II.2. Let $\widehat{f} \in \widehat{\mathbf{L}}^2(\mathbf{V}^* \times \mathbb{C}^m, \varepsilon)$. By Lemma III.1, for almost all $\lambda \in \mathbf{V}^*$ the function $u \mapsto \widehat{f}(\lambda,u)$ is an entire function on $\mathbb{C}^m$. Then, by the Fubini theorem, for all $u \in \mathbb{C}^m$, we get

$$(4) \qquad \int_{\mathbf{V}^*} \exp(-2\langle\lambda, \mathbf{F}(u,u)\rangle)|\widehat{f}(\lambda,u)|^2(\lambda)_*^{(2d-q)^*\varepsilon^*+d^*}\,d\lambda < \infty.$$

We can now define $f$ on $\mathbf{D}$ by

$$f(z,u)=\int_{\mathbf{V}^*}\exp(i\langle\lambda,z\rangle)\widehat{f}(\lambda,u)\,d\lambda\quad((z,u)\in\mathbf{D}).$$

For this definition, let us prove that the integral on the right hand side is absolutely convergent. By the Hölder inequality, for $z=x+iy$, we have

$$\int_{\mathbf{V}^*}\exp(-\langle\lambda,y\rangle)|\widehat{f}(\lambda,u)|\,d\lambda \le \Big(\int_{\mathbf{V}^*}\exp(-2\langle\lambda,\mathbf{F}(u,u)\rangle)|\widehat{f}(\lambda,u)|^2(\lambda)_*^{(2d-q)^*\varepsilon^*+d^*}\,d\lambda\Big)^{1/2} \times\Big(\int_{\mathbf{V}^*}\exp(-2\langle\lambda,y-\mathbf{F}(u,u)\rangle)(\lambda)_*^{-(2d-q)^*\varepsilon^*-d^*}\,d\lambda\Big)^{1/2}.$$

The first integral on the right converges by (4) while the second also converges under the hypothesis $\varepsilon_i>(n_i+2)/(2(2d-q)_i)$ $(i=1,\ldots,l)$ by virtue of the following proposition (assertion 2)):

III.2. Proposition [G]. A) *Let* $z\in\mathbb{C}^n$ *and* $\varrho\in\mathbb{C}^l$.

1) *The integral* $\int_{\mathbf{V}}\exp(-\langle\lambda,z\rangle)(\lambda)^{\varrho+d}d\lambda$ *is absolutely convergent if and only if* $\operatorname{Re}z\in\mathbf{V}^*$ *and* $\operatorname{Re}\varrho_i>m_i/2$ $(i=1,\ldots,l)$. *In this case, this integral is equal to* $c(\varrho)(z)_*^{-\varrho^*}$.

2) *The integral* $\int_{\mathbf{V}^*}\exp(-\langle\lambda,z\rangle)(\lambda)_*^{\varrho^*+d^*}d\lambda$ *is absolutely convergent if and only if* $\operatorname{Re}z\in\mathbf{V}$ *and* $\operatorname{Re}\varrho_i>n_i/2$ $(i=1,\ldots,l)$. *In this case, this integral is equal to* $c(\varrho)(z)^{-\varrho}$.

B) *Let* $\mathbf{F}$ *be a* $\mathbf{V}$*-Hermitian homogeneous form on* $\mathbb{C}^m$. *Then for all* $\lambda\in\mathbf{V}^*$,

$$\int_{\mathbb{C}^m}\exp(-\langle\lambda,\mathbf{F}(u,u)\rangle)\,dv(u)=c(\lambda)_*^{-q^*}.$$

Let us show next that $f$ is holomorphic on $\mathbf{D}$. It suffices to show that for each compact subset $\mathbf{K}$ of $\mathbf{D}$, there is a function $h$ in $\mathbf{L}^1(\mathbf{V}^*)$ such that for all $(z,u)\in\mathbf{K}$,

$$\exp(-\langle\lambda,y\rangle)|\widehat{f}(\lambda,u)|\le h(\lambda)\quad(\lambda\in\mathbf{V}^*).\tag{5}$$

Take $\mathbf{K}=\mathbf{K}_1\times\mathbf{K}_2$, where $\mathbf{K}_1$ and $\mathbf{K}_2$ are compact subsets of $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively (recall that $\mathbf{D}\subset\mathbb{C}^n\times\mathbb{C}^m$). Moreover, we can suppose that there exists $y_0\in\mathbf{V}$ such that $y-y_0\in\mathbf{V}$ for all $y\in\operatorname{Im}\mathbf{K}_1$ and $y_0-\mathbf{F}(u,u)\in\mathbf{V}$ for all $u\in\mathbf{K}_2$. It is well known that there exists a constant $\varrho$ such that for all $\lambda\in\mathbf{V}^*$ and for all $y$ lying in the compact set $\operatorname{Im}\mathbf{K}_1$ (note that $\operatorname{Im}\mathbf{K}_1-y_0\subset\mathbf{V}$), we have $\langle\lambda,y-y_0\rangle\ge\varrho|\lambda|$.



Thus, if we set $\delta=\varrho/|y_0|$, then the left hand side of (5) is less than $\exp(-\langle\lambda,y_0(1+\delta)\rangle)|\widehat{f}(\lambda,u)|$.

Let $\widehat{f}_{y_0}$ be the function defined on $\mathbf{V}^*\times\mathbb{C}^m$ by

$$\widehat{f}_{y_0}(\lambda,u)=\exp(-\langle\lambda,y_0\rangle)\widehat{f}(\lambda,u).$$

For almost all $\lambda\in\mathbf{V}^*$, the function $\widehat{f}_{y_0}(\lambda,\cdot)$ is entire. In the sequel, $\mathbf{L}^2(\mathbf{V}^*,\varepsilon)$ will stand for the space $\mathbf{L}^2(\mathbf{V}^*,(\lambda)_*^{(2d-q)^*\varepsilon^*+d^*}d\lambda)$. We next prove that $\widehat{f}_{y_0}(\cdot,u)\in\mathbf{L}^2(\mathbf{V}^*,\varepsilon)$. Since $y_0-\mathbf{F}(u,u)\in\mathbf{V}$ for all $u\in\mathbf{K}_2$, we obtain

$$\|\widehat{f}_{y_0}(\cdot,u)\|^2_{\mathbf{L}^2(\mathbf{V}^*,\varepsilon)}\le\int_{\mathbf{V}^*}\exp(-2\langle\lambda,\mathbf{F}(u,u)\rangle)|\widehat{f}(\lambda,u)|^2(\lambda)_*^{(2d-q)^*\varepsilon^*+d^*}\,d\lambda.$$

The integral on the right is finite by (4), because $\widehat{f}\in\widehat{\mathbf{L}}^2(\mathbf{V}^*\times\mathbb{C}^m,\varepsilon)$. It follows that $\widehat{f}_{y_0}$ satisfies the hypotheses of Lemma III.1 with $\mathbf{M}=(\mathbf{V}^*,(\lambda)_*^{(2d-q)^*\varepsilon^*+d^*}d\lambda)$ and $\mathbf{U}=\mathbb{C}^m$. Then by assertion (iii) of this lemma, using an open relatively compact neighbourhood of $\mathbf{K}_2$, there exists $g\in\mathbf{L}^2(\mathbf{V}^*,\varepsilon)$ such that $|\widehat{f}_{y_0}(\lambda,u)|\le g(\lambda)$ for all $u\in\mathbf{K}_2$ and almost all $\lambda\in\mathbf{V}^*$.

Inequality (5) is of course satisfied by the function $h(\lambda)=g(\lambda)\times\exp(-\langle\lambda,\delta y_0\rangle)$. By the Hölder inequality, we obtain

$$\|h\|_{\mathbf{L}^1(\mathbf{V}^*)}\le\|g\|_{\mathbf{L}^2(\mathbf{V}^*,\varepsilon)}\Big(\int_{\mathbf{V}^*}\exp(-2\langle\lambda,y_0\delta\rangle)(\lambda)_*^{-(2d-q)^*\varepsilon^*-d^*}\,d\lambda\Big)^{1/2}$$

and the integral on the right converges by Proposition III.2.2 because $\varepsilon_i>(n_i+2)/(2(2d-q)_i)$ $(i=1,\ldots,l)$. This concludes the proof of identity (i).

Let us next prove estimate (ii). Write (i) as

$$f(x+iy,u)=\int_{\mathbf{V}^*}\exp(-\langle\lambda,y\rangle)\widehat{f}(\lambda,u)\exp(i\langle\lambda,x\rangle)\,d\lambda.$$

By the classical Plancherel formula on $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n}|f(x+iy,u)|^2\,dx=\int_{\mathbf{V}^*}\exp(-2\langle\lambda,y\rangle)|\widehat{f}(\lambda,u)|^2\,d\lambda$$

and hence

$$\begin{aligned}\|f\|^2_{2,\varepsilon}&=\int_{\mathbf{V}^*}d\lambda\int_{\mathbb{C}^m}dv(u)|\widehat{f}(\lambda,u)|^2\exp(-2\langle\lambda,\mathbf{F}(u,u)\rangle)\\&\quad\times\int_{\mathbf{V}+\mathbf{F}(u,u)}\exp(-2\langle\lambda,y\rangle)(y-\mathbf{F}(u,u))^{-(2d-q)\varepsilon}dy\\&=c_\varepsilon\int_{\mathbf{V}^*}d\lambda\int_{\mathbb{C}^m}|\widehat{f}(\lambda,u)|^2\exp(-2\langle\lambda,\mathbf{F}(u,u)\rangle)\\&\quad\times(\lambda)_*^{(2d-q)^*\varepsilon^*+d^*}\,dv(u),\end{aligned}\tag{6}$$

where the last equality follows from the application of Proposition III.2 to the integral with respect to $y$: the convergence of this integral requires $\varepsilon_i > (n_i+2)/(2(2d-q)_i)$ $(i=1,\ldots,l)$.

Now, let us prove assertion 1). Let $f \in \mathbf{A}^{2,\varepsilon}(\mathbf{D})$. Then by the Fubini theorem, for all $u \in \mathbb{C}^m$ and $y \in \mathbb{R}^n$ such that $y - \mathbf{F}(u,u) \in \mathbf{V}$ we have

$$\int_{\mathbb{R}^n} |f(x+iy,u)|^2\,dx < \infty. \tag{7}$$

For each $y \in \mathbf{V}$, define the function $f_y$ on $\mathbb{R}^n \times \{u \in \mathbb{C}^m : y - \mathbf{F}(u,u) \in \mathbf{V}\}$ by $f_y(x,u) = f(x+iy,u)$. Then by (7), the function $x \mapsto f_y(x,u)$ admits a Fourier transform $\lambda \mapsto \widehat{f}_y(\lambda,u)$, i.e.

$$f_y(x,u) = \int_{\mathbb{R}^n} \exp(i\langle\lambda,x\rangle)\widehat{f}_y(\lambda,u)\,d\lambda \tag{8}$$

and $\widehat{f}_y(\cdot,u) \in \mathbf{L}^2(\mathbb{R}^n)$.

Let us prove that for all $\lambda \in \mathbb{R}^n$ and $u \in \mathbb{C}^m$, the quantity $\widehat{f}_y(\lambda,u)/\exp(-\langle\lambda,y\rangle)$ does not depend on $y \in \mathbf{V}+\mathbf{F}(u,u)$. Let $y$ and $y'$ be two points in $\mathbf{V}+\mathbf{F}(u,u)$. Notice that since $\mathbf{V}$ is an open cone, there exists $r \in \mathbb{N}$, $r \geq 2$, such that $ry - y' \in \mathbf{V}$. Set $t = ry - y'$; then

$$\begin{aligned}\widehat{f}_{ry}(\lambda,u) &= \int_{\mathbb{R}^n} \exp(-i\langle\lambda,x\rangle)f(x+iry,u)\,dx\\ &= \int_{\mathbb{R}^n} \exp(-i\langle\lambda,x\rangle)f(x+i(t+y'),u)\,dx\\ &= \int_{\mathbb{R}^n+it} \exp(-i\langle\lambda,x\rangle)f(x+iy',u)\exp(-\langle\lambda,t\rangle)\,dx.\end{aligned}$$

The analyticity of $f$ on $\mathbf{D}$ and (7) now yield the equality $\widehat{f}_{ry}(\lambda,u) = \exp(-\langle\lambda,t\rangle)\widehat{f}_{y'}(\lambda,u)$, which is equivalent to

$$\frac{\widehat{f}_{ry}(\lambda,u)}{\exp(-\langle\lambda,ry\rangle)} = \frac{\widehat{f}_{y'}(\lambda,u)}{\exp(-\langle\lambda,y'\rangle)}. \tag{9}$$

In particular, since $ry - y \in \mathbf{V}$, (9) also gives

$$\frac{\widehat{f}_{ry}(\lambda,u)}{\exp(-\langle\lambda,ry\rangle)} = \frac{\widehat{f}_{y}(\lambda,u)}{\exp(-\langle\lambda,y\rangle)}. \tag{10}$$

Combining (9) and (10) then leads to the desired conclusion.

We can now define $\widehat{f}$ on $\mathbb{R}^n \times \mathbb{C}^m$ by $\widehat{f}(\lambda,u) = \exp(\langle\lambda,y\rangle)\widehat{f}_y(\lambda,u)$, where $y$ is an arbitrary point of $\mathbf{V}+\mathbf{F}(u,u)$. Replacing in (8) gives

$$f(z,u) = \int_{\mathbb{R}^n} \exp(i\langle\lambda,z\rangle)\widehat{f}(\lambda,u)\,d\lambda. \tag{11}$$



We next prove that $\widehat{f}(\lambda,u)$ is concentrated on $\mathbf{V}^*$, for all $u \in \mathbb{C}^m$. By a contradiction argument, assume that there exists an open subset $N$ of $\mathbb{R}^n \setminus \mathbf{V}^*$ where $\widehat{f}(\cdot,u)$ is never zero. Then for all $\lambda_0 \in N$, there exist $\delta > 0$, $y \in \mathbf{V}$ and a neighbourhood $N(\lambda_0)$ of $\lambda_0$ contained in $N$ such that for all $\lambda \in N(\lambda_0)$, we have $\langle\lambda,y\rangle < -\delta < 0$. By the classical Plancherel theorem, we deduce from (11) that for all $t \geq 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x+ity,u)|^2\,dx &= \int_{\mathbb{R}^n} \exp(-2\langle\lambda,ty\rangle)|\widehat{f}(\lambda,u)|^2\,d\lambda\\ &\geq \exp(2\delta t)\int_{N(\lambda_0)} |\widehat{f}(\lambda,u)|^2\,d\lambda.\end{aligned}$$

We let $t$ tend to $\infty$. Then the left hand side tends to 0. To see this, it is enough to prove that for all $u \in \mathbb{C}^m$, $f(\cdot,u) \in \mathbf{L}^2(\mathbb{R}^n + i(\mathbf{V}+\mathbf{F}(u,u)))$; since $f \in \mathbf{A}^{2,\varepsilon}(\mathbf{D})$, this follows by the Fubini theorem. Thus, the latter inequality implies that $\widehat{f}(\lambda,u) = 0$ for almost all $\lambda \in N(\lambda_0)$. This contradicts the assumption that $\widehat{f}(\cdot,u)$ is never zero on $N$. Hence (11) becomes

$$f(z,u) = \int_{\mathbf{V}^*} \exp(i\langle\lambda,z\rangle)\widehat{f}(\lambda,u)\,d\lambda,$$

and this ends the proof of assertion 1). Thus Theorem II.2 is entirely proved.

Proof of Corollary II.3. Let $\varepsilon \in \mathbb{R}^l$ be arbitrary and let $f \in \mathbf{A}^{2,\varepsilon}(\mathbf{D})$. When we go back carefully to the proof of Theorem II.2, we notice that identity (11) holds for $f$ and the associated function $\widehat{f}$ is concentrated on $\mathbf{V}^* \times \mathbb{C}^m$. Moreover, the first identity of (6) holds:

$$\begin{aligned}&\int_{\mathbf{D}} |f(z,u)|^2 b^{-\varepsilon}((z,u),(z,u))\,dv(z,u)\\ &\quad= \int_{\mathbf{V}^*} d\lambda \int_{\mathbb{C}^m} dv(u)|\widehat{f}(\lambda,u)|^2\exp(-2\langle\lambda,\mathbf{F}(u,u)\rangle)\\ &\qquad\times\int_{\mathbf{V}} \exp(-2\langle\lambda,y\rangle)(y)^{-(2d-q)\varepsilon}\,dy.\end{aligned}$$

Assume now that $\varepsilon$ satisfies

$$\varepsilon_i \leq \frac{n_i+2}{2(2d-q)_i} \quad \text{for some } i \in \{1,\ldots,l\}.$$

The integral over $\mathbf{V}$ is infinite for all $\lambda \in \mathbf{V}^*$ by Proposition III.2.1. Since $f \in \mathbf{A}^{2,\varepsilon}(\mathbf{D})$, we conclude that $\exp(-2\langle\lambda,\mathbf{F}(u,u)\rangle)|\widehat{f}(\lambda,u)|^2 = 0$ for all $u \in \mathbb{C}^m$ and almost all $\lambda \in \mathbf{V}^*$. Hence $\widehat{f}$ is identically zero and this implies $f = 0$ on $\mathbf{D}$.

Proof of Corollary II.4. The two conditions

$$\varepsilon_i > \frac{n_i+2}{(2d-q)_i} \quad \text{and} \quad \alpha_i - \varepsilon_i > \frac{n_i}{-2(2d-q)_i} \quad (i=1,\ldots,l)$$

imply that

$$\alpha_i > \frac{1}{(2d-q)_i} \quad (i=1,\ldots,l).$$

Then, by Propositions II.1 and III.2.2, we obtain

$$b^{(1+\alpha)/2}((\zeta,v),(z,u)) = c_\alpha \int_{\mathbf{V}^*} \exp(-i\langle\lambda,\overline{z}\rangle)\exp(\langle\lambda, i\zeta + 2\mathbf{F}(u,v)\rangle)(\lambda)_*^{-(2d-q)^*(1+\alpha)^*/2+d^*}\,d\lambda.$$

By Theorem II.2, if we set $\zeta = s+it$, we have

$$\|b^{(1+\alpha)/2}(\cdot,(\zeta,v))\|^2_{2,\varepsilon} = c_{\alpha,\varepsilon}\int_{\mathbf{V}^*\times\mathbb{C}^m} \exp(-2\langle\lambda, t-\mathbf{F}(v,v)\rangle)\exp(-2\langle\lambda,\mathbf{F}(u-v,u-v)\rangle) \times (\lambda)_*^{-(2d-q)^*(1+\alpha-\varepsilon)^*+3d^*}\,d\lambda\,dv(u).$$

By Proposition III.2.B, the integral over $\mathbb{C}^m$ is equal to $c(\lambda)_*^{-q^*}$. Hence

$$\|b^{(1+\alpha)/2}(\cdot,(\zeta,v))\|^2_{2,\varepsilon} = c_{\alpha,\varepsilon}\int_{\mathbf{V}^*} \exp(-2\langle\lambda, t-\mathbf{F}(v,v)\rangle)(\lambda)^{-(2d-q)^*(\alpha-\varepsilon)^*+d^*}\,d\lambda;$$

this integral converges by Proposition III.2.2 because

$$\alpha_i - \varepsilon_i > n_i/(-2(2d-q)_i) \quad (i=1,\ldots,l)$$

and its value is $c_{\alpha,\varepsilon} b^{\alpha-\varepsilon}((\zeta,v),(\zeta,v))$.

## IV. Proofs of Proposition II.5 and Theorem II.6

Proof of Proposition II.5. It suffices to prove that the kernel $b^{1+\varepsilon}((\zeta,v),(z,u))$ is in $\mathbf{A}^{2,\varepsilon}(\mathbf{D})$, is conjugate symmetric and reproduces $\mathbf{A}^{2,\varepsilon}(\mathbf{D})$. The first two properties are satisfied in view of the definition of $b^{1+\varepsilon}$ (cf. notations adopted after Proposition II.1) and Corollary II.4.

Let us prove that this kernel reproduces $\mathbf{A}^{2,\varepsilon}(\mathbf{D})$. Let $f\in\mathbf{A}^{2,\varepsilon}(\mathbf{D})$; then by Theorem II.2, there exists $\widehat{f}\in\widehat{\mathbf{L}}^2(\mathbf{V}^*\times\mathbb{C}^m,\varepsilon)$ such that

$$f(z,u) = \int_{\mathbf{V}^*}\exp(i\langle\lambda,z\rangle)\widehat{f}(\lambda,u)\,d\lambda \quad ((z,u)\in\mathbf{D}). \tag{12}$$

On the other hand, since $\varepsilon_i > (n_i+2)/(2d-q)_i$ $(i=1,\ldots,l)$, we have by Propositions II.1 and III.2,



$$b^{1+\varepsilon}((z,u),(\zeta,v)) = c_\varepsilon\int_{\mathbf{V}^*}\exp(i\langle\lambda,z\rangle)\exp(\langle\lambda,-i\overline{\zeta}+2\mathbf{F}(u,v)\rangle)(\lambda)_*^{-(2d-q)^*(1+\varepsilon)^*+d^*}\,d\lambda.$$

Hence by Theorem II.2 (Plancherel–Gindikin formula), we get

$$\begin{aligned}\langle f, b^{1+\varepsilon}(\cdot,(\zeta,v))\rangle_{2,\varepsilon} &= c_\varepsilon\langle\widehat{f},(b^{1+\varepsilon})^\wedge(\cdot,(\zeta,v))\rangle_{\widehat{\mathbf{L}}^2(\mathbf{V}^*\times\mathbb{C}^m,\varepsilon)}\\ &= c_\varepsilon\int_{\mathbf{V}^*}\exp(\langle\lambda,i\zeta+2\mathbf{F}(v,v)\rangle)\\ &\quad\times\Big(\int_{\mathbb{C}^m}\widehat{f}(\lambda,u)\exp(-2\langle\lambda,\mathbf{F}(u,v)\rangle)\\ &\quad\times\exp(-2\langle\lambda,\mathbf{F}(u-v,u-v)\rangle)\,dv(u)\Big)\,d\lambda.\end{aligned} \tag{13}$$

We use the following lemma:

IV.1. Lemma. *Let $G$ be an entire function on $\mathbb{C}^m$ and let $\varphi$ be a continuous function with circular symmetry: $\varphi(u_1e^{i\psi},\ldots,u_me^{i\psi}) = \varphi(u_1,\ldots,u_m)$ for all $\psi\in\mathbb{R}$. Assume that the integral $I=\int_{\mathbb{C}^m}G(u)\varphi(u)\,dv(u)$ converges absolutely. Then $I = G(0)\int_{\mathbb{C}^m}\varphi(u)\,dv(u)$.*

By Lemma IV.1 (use the change of variables $u' = u-v$ and take $G(u') = \widehat{f}(\lambda,u'+v)\exp(-2\langle\lambda,\mathbf{F}(u'+v,v)\rangle)$ and $\varphi(u') = \exp(-2\langle\lambda,\mathbf{F}(u',u')\rangle)$) and by Proposition III.2, the integral over $\mathbb{C}^m$ is equal to

$$c_\varepsilon\int_{\mathbf{V}^*}\exp(i\langle\lambda,\zeta\rangle)\widehat{f}(\lambda,v)\,d\lambda = c_\varepsilon f(\zeta,v),$$

where the last equality is (12). This completes the proof of Proposition II.5.

In the sequel, to simplify the notations, a point of $\mathbf{D}$ will be denoted by $z$ or $\zeta$, instead of $(z,u)$ or $(\zeta,v)$, while $ie$ will stand for $(ie,0)$. Also, we will write $z/n$, $ie/n$, $z+ie/n$ instead of $(z/n,u/\sqrt{n})$, $(ie/n,0)$ and $(z+ie/n,u)$ respectively $(n\in\mathbb{N})$.

Proof of Theorem II.6. We shall use the following density lemma:

IV.2. Lemma. *Let $r\in\mathbb{R}^l$ satisfy $r_i\ge 0$ $(i=1,\ldots,l)$ and let $p\in[1,\infty)$. Then for all $\varepsilon\in\mathbb{R}^l$ such that*

$$\varepsilon_i > \frac{n_i+2}{(2d-q)_i} \quad (i=1,\ldots,l),$$

*the subspace $\mathbf{A}^{p,r}(\mathbf{D})\cap\mathbf{A}^{2,\varepsilon}(\mathbf{D})$ is dense in $\mathbf{A}^{p,r}(\mathbf{D})$.*

The proof of Lemma IV.2 is somewhat lengthy: we shall give it in an appendix. It also uses the following lemmas:

IV.3. LEMMA [R]. *Let* $r \in \mathbb{R}^l$ *and let* $p \in (0,\infty)$. *Then for all* $f \in \mathbf{H}(\mathbf{D})$,

$$|f(z)|^p \le cB^{1+r}(z,z)\|f\|_{p,r}^p \quad (z \in \mathbf{D}).$$

IV.4. LEMMA. *Let* $\beta \in \mathbb{R}^l$ *be such that* $\beta_i \ge 0$ $(i = 1,\ldots,l)$. *Then for all* $z$ *and* $\zeta$ *in* $\mathbf{D}$, *we have*

$$b^\beta(z+\zeta, z+\zeta) \le b^\beta(z,z) \tag{14}$$

*and*

$$|b^\beta(\zeta,z)| \le b^\beta(z,z). \tag{15}$$

Lemma IV.4 is an easy consequence of Proposition III.2.2.

1) Let us first prove assertion 1) of Theorem II.6. For $\zeta \in \mathbf{D}$, consider the linear functional $\varphi$ on $\mathbf{A}^{p,r}(\mathbf{D})$ defined by

$$\varphi(f) = f(\zeta) - c_\varepsilon \int_{\mathbf{D}} b^{1+\varepsilon}(\zeta,z) f(z) b^{-\varepsilon}(z,z)\, dv(z).$$

The integral on the right converges absolutely. For $p \in (1,\infty)$, this follows by the Hölder inequality and Corollary II.4 since

$$\varepsilon_i > \frac{n_i+2}{2(2d-q)_i} \cdot \frac{p-1}{p} + \frac{r_i}{p}$$

and

$$p < \frac{n_i - 2(2d-q)_i(1+r_i)}{n_i} \quad (i = 1,\ldots,l).$$

For $p = 1$, Lemma IV.4 implies

$$|b^{1+\varepsilon}(\zeta,z)|b^{-\varepsilon}(z,z)b^r(z,z) \le |b^{1+r}(\zeta,z)| \le |b^{1+r}(\zeta,\zeta)|$$

when $\varepsilon_i - r_i \ge 0$ and $1 + r_i \ge 0$ $(i = 1,\ldots,l)$. The conclusion follows and its proof also yields that $\varphi$ is a bounded linear functional on $\mathbf{A}^{p,r}(\mathbf{D})$.

The functional $\varphi$ is identically zero on $\mathbf{A}^{p,r}(\mathbf{D}) \cap \mathbf{A}^{2,\varepsilon}(\mathbf{D})$. Hence, by the Hahn–Banach theorem and Lemma IV.2, $\varphi$ is identically zero on $\mathbf{A}^{p,r}(\mathbf{D})$. This proves assertion 1).

2) Let us next prove assertion 2). Let $f \in \mathbf{A}^{p,r}(\mathbf{D})$. For $\alpha \in \mathbb{R}^l$ such that $\alpha_i \ge 0$ $(i = 1,\ldots,l)$, define the sequence $f_n(z) = b^\alpha(z/n, ie) f(z + ie/n)$, $n \in \mathbb{N}$ and $z \in \mathbf{D}$. Since the function $z \mapsto f(z + ie/n)$ is bounded in $\mathbf{D}$ by Lemmas IV.3 and IV.4 since $1 + r_i \ge 0$, we deduce by Corollary II.4, for $\alpha_i$ sufficiently large, that $f_n \in \mathbf{A}^{2,\varepsilon}(\mathbf{D})$. Then by Proposition II.5, we have

$$f_n(\zeta) = c_\varepsilon \int_{\mathbf{D}} b^{1+\varepsilon}(\zeta,z) f_n(z) b^{-\varepsilon}(z,z)\, dv(z). \tag{16}$$

On the other hand, by Lemmas IV.3 and IV.4, since $1 + r_i \ge 0$, we get

$$|f(z+ie/n)|^p \le b^{1+r}(z+ie/n, z+ie/n)\|f\|_{p,r}^p \le b^{1+r}(z,z)\|f\|_{p,r}^p$$



and $|b^\alpha(z/n, ie)| \le b^\alpha(ie, ie)$. Thus the function $|b^{1+\varepsilon}(\zeta,z)|b^{(1+r)/p-\varepsilon}(z,z)$ dominates the integrand in (16) and it belongs to $\mathbf{L}^1(\mathbf{D}, dv(z))$ by Corollary II.4, under our assumptions on $\varepsilon$ and $r$. Hence by the dominated convergence theorem, we obtain the equality

$$f(z) = c_\varepsilon \int_{\mathbf{D}} b^{1+\varepsilon}(z,\zeta) f(\zeta) b^{-\varepsilon}(\zeta,\zeta)\, dv(\zeta).$$

This proves Theorem II.6.

**V. Proof of Theorem II.7.** 1) Let $f \in \mathbf{L}^{1,r}(\mathbf{D})$. Then by the Fubini theorem, we obtain

$$\begin{aligned}\int_{\mathbf{D}} P_\varepsilon^* f(\zeta) b^{-r}(\zeta,\zeta)\, dv(\zeta) &\le c_\varepsilon \int_{\mathbf{D}} \Big(\int_{\mathbf{D}} |b^{1+\varepsilon}(\zeta,z)| b^{-r}(\zeta,\zeta)\, dv(\zeta)\Big) |f(z)| b^{-\varepsilon}(z,z)\, dv(z)\\ &\le c_\varepsilon \|f\|_{1,r},\end{aligned}$$

where the last inequality follows from Corollary II.4 since $\varepsilon_i - r_i > n_i/(-2(2d-q)_i)$ and $r_i > (n_i+2)/(2(2d-q)_i)$ $(i = 1,\ldots,l)$.

2) By the Schur lemma [FRu], it suffices to prove the existence of a positive function $g$ on $\mathbf{D}$ and of two positive constants $c_1$ and $c_2$ such that

(i) for all $\zeta \in \mathbf{D}$,

$$\int |b^{1+\varepsilon}(\zeta,z)| g^{p'}(z) b^{-\varepsilon}(z,z)\, dv(z) \le c_1 g^{p'}(\zeta),$$

where $p'$ is the conjugate exponent of $p$, and

(ii) for all $z \in \mathbf{D}$,

$$\int_{\mathbf{D}} |b^{1+\varepsilon}(\zeta,z)| g^p(\zeta) b^{-r}(\zeta,\zeta)\, dv(\zeta) \le c_2 g^p(z) b^{\varepsilon-r}(z,z).$$

We take $g(z) = b^\delta(z,z)$, with $\delta \in \mathbb{R}^l$. By Corollary II.4, estimates (i) and (ii) hold if and only if

$$\frac{n_i}{-2(2d-q)_i}(p-1) < \delta_i p < \frac{n_i + 2 - 2(2d-q)_i\varepsilon_i}{-2(2d-q)_i}(p-1)$$

and

$$\varepsilon_i > \frac{1}{(2d-q)_i} \quad (i = 1,\ldots,l)$$

on the one hand, and on the other hand,

$$\frac{n_i - 2(2d-q)_i(r-\varepsilon)_i}{-2(2d-q)_i} < \delta_i p < \frac{n_i + 2 - 2(2d-q)_i r_i}{-2(2d-q)_i}$$

and

$$r_i > \frac{n_i+2}{2(2d-q)_i} \quad (i=1,\ldots,l).$$

A suitable exponent $\delta$ exists when these conditions are compatible. A simple calculation gives that this is the case when for all $i=1,\ldots,l$,

$$n_i(p-1) < n_i + 2 - 2(2d-q)_i r_i$$

and

$$n_i - 2(2d-q)_i(r-\varepsilon)_i < (n_i + 2 - 2(2d-q)_i\varepsilon_i)(p-1).$$

This completes the proof of Theorem II.7.

Remark. Quite recently, M. M. Dzhrbashyan and A. O. Karapetyan [DK] also studied the $\mathbf{L}^{p,r}$-boundedness for weighted Bergman projections $P_\varepsilon$ in the tube over the cone of Hermitian positive definite matrices of order $n$ for $\varepsilon_i = \varepsilon_0$ and $r_i = r_0$ $(i=1,\ldots,l)$. They proved a positive result when

$$\max\left\{1, \frac{n(2r_0+1)}{2n\varepsilon_0+1}\right\} < p < \frac{n(2r_0+1)}{n-1}.$$

According to Theorem II.7, our sufficient condition is

$$\max_{i=1,\ldots,n}\left\{1, \frac{2i-1+4nr_0}{4n\varepsilon_0+i}\right\} < p < \min_{i=1,\ldots,n}\left\{\frac{2i-1+4nr_0}{i-1}\right\},$$

$\varepsilon_0 > -1/(4n)$ and $r_0 > -1/(4n)$. More particularly, in the case $\varepsilon_0 = r_0 = 0$, those two authors obtain no value of $p$ for which the Bergman projection $P$ is bounded on $\mathbf{L}^p(\mathbf{D})$. We prove that $P$ has this property when

$$\frac{2n-1}{n} < p < \frac{2n-1}{n-1}.$$

Our method of proof is more efficient because we allow the exponent $\delta$ of the Schur test function $g(z) = b^\delta(z,z)$ to be a vector instead of a real number.

Proof of Theorem II.8. Notice that under our hypotheses, $P_\varepsilon f$ is well defined for all $f \in \mathbf{L}^{p,\varepsilon}(\mathbf{D})$. Let $B(ie)$ be a Euclidean ball centered at $ie$ whose closure is contained in $\mathbf{D}$. Let $f$ denote the function defined on $\mathbf{D}$ by

$$f(z) = \begin{cases} B^\varepsilon(z,z) & \text{if } z \in B(ie), \\ 0 & \text{if } z \in \mathbf{D} \setminus B(ie). \end{cases}$$

Of course, $f \in \mathbf{L}^{p,\varepsilon}(\mathbf{D})$. Moreover, by the mean value property, $P_\varepsilon f(\zeta) = c_\varepsilon b^{1+\varepsilon}(\zeta, ie)$ for all $\zeta \in \mathbf{D}$. Hence,

$$\|P_\varepsilon f\|^p_{\mathbf{L}^{p,\varepsilon}(\mathbf{D})} = c\int_{\mathbf{D}} |b^{1+\varepsilon}(\zeta, ie)|^p b^{-\varepsilon}(\zeta,\zeta)\,dv(\zeta).$$



By Corollary II.4, the integral on the right is infinite if $p \le p_0(\varepsilon)$. This concludes the proof.

Remark. The same proof can be used to show that for each $\varepsilon \in \mathbb{R}^l$ such that $\varepsilon_i > (n_i+2)/(2(2d-q)_i)$ $(i=1,\ldots,l)$, $P_\varepsilon$ is not bounded on $\mathbf{L}^{1,r}(\mathbf{D})$ when the conditions of Theorem II.7.1 are not satisfied, i.e. when either $r_i \le (n_i+2)/(2(2d-q)_i)$ or $\varepsilon_i - r_i \le n_i/(-2(2d-q)_i)$ for some $i \in \{1,\ldots,l\}$.

## Appendix

Proof of Lemma IV.2. Let $f \in \mathbf{A}^{p,r}(\mathbf{D})$. Let $\alpha \in \mathbb{R}^l$ satisfy $\alpha_i \ge 0$ $(i=1,\ldots,l)$. Define the sequence $\{f_n\}$ of holomorphic functions on $\mathbf{D}$ by $f_n(z) = c_\alpha f(z+ie/n)b^\alpha(z/n, ie)$, $n \in \mathbb{N}$, where $c_\alpha$ is a complex number such that $\lim_{n\to\infty} c_\alpha b^\alpha(z/n, ie) = 1$. We are going to prove that if the numbers $\alpha_i$ are large enough, then $f_n \in \mathbf{A}^{p,r}(\mathbf{D}) \cap \mathbf{A}^{2,\varepsilon}(\mathbf{D})$ for all $n$ and $\lim_{n\to\infty}\|f_n - f\|_{p,r} = 0$.

By Lemma IV.3 and inequality (14) of Lemma IV.4, since $(1+r_i)/p \ge 0$ $(i=1,\ldots,l)$, we get

$$\begin{aligned}|f(z+ie/n)| &\le c_{p,r} b^{(1+r)/p}(z+ie/n, z+ie/n)\|f\|_{p,r}\\ &\le c_{p,r} b^{(1+r)/p}(ie/n, ie/n)\|f\|_{p,r}.\end{aligned}$$

Then each function $z \mapsto f(z+ie/n)$ is bounded on $\mathbf{D}$ and hence by Corollary II.4, $f_n \in \mathbf{A}^{2,\varepsilon}(\mathbf{D})$ for all $n$ if $\alpha$ and $\varepsilon$ satisfy

$$2\alpha_i - 1 - \varepsilon_i > \frac{n_i}{-2(2d-q)_i} \quad \text{and} \quad \varepsilon_i > \frac{n_i+2}{2(2d-q)_i} \quad (i=1,\ldots,l).$$

Moreover, $f_n \in \mathbf{A}^{p,r}(\mathbf{D})$ because by Lemma IV.4 (inequality (15)), condition $\alpha_i \ge 0$ $(i=1,\ldots,l)$ implies

$$|b^\alpha(z/n, ie)| \le b^\alpha(ie, ie) \tag{17}$$

on the one hand, while on the other hand, the function $z \mapsto f(z+ie/n)$ belongs to $\mathbf{A}^{p,r}(\mathbf{D})$.

Let us next prove that $(f_n)$ converges to $f$ in $\mathbf{A}^{p,r}(\mathbf{D})$. By the Minkowski inequality and (17), we obtain $\|f_n - f\|_{p,r} \le I_1 + I_2$, where

$$I_1 = \Big(\int_{\mathbf{D}} |c_\alpha b^\alpha(z/n, ie) - 1|^p |f(z)|^p b^{-r}(z,z)\,dv(z)\Big)^{1/p},$$

$$I_2 = \Big(\int_{\mathbf{D}} |f(z+ie/n) - f(z)|^p b^{-r}(z,z)\,dv(z)\Big)^{1/p}.$$

The integral $I_1$ tends to zero by the Lebesgue dominated convergence theorem. Let us prove that $I_2$ also tends to zero.

For $s \in \mathbb{N}$, denote by $\mathbf{K}_s$ the compact set defined by $\mathbf{K}_s = \{z \in \mathbf{D} : d(z, \partial D) \geq 1/s,\ |z| \leq s\}$. Of course $\bigcup_s \mathbf{K}_s = \mathbf{D}$ and $\mathbf{K}_s \subset \mathbf{K}_{s+1}$. For all $s \in \mathbb{N}$, we have $I_2(s) \leq I_3(s) + I_4(s)$, where (with $\mathbf{K}_s^c = \mathbf{D} \setminus \mathbf{K}_s$)

$$I_3(s) = \Big( \int_{\mathbf{K}_s} |f(z + ie/n) - f(z)|^p b^{-r}(z,z)\, dv(z) \Big)^{1/p},$$

$$I_4(s) = \Big( \int_{\mathbf{K}_s^c} |f(z + ie/n) - f(z)|^p b^{-r}(z,z)\, dv(z) \Big)^{1/p}$$

Moreover, $I_4(s) \leq I_5(s) + I_6(s)$, where

$$I_5(s) = \Big( \int_{\mathbf{K}_s^c} |f(z)|^p b^{-r}(z,z)\, dv(z) \Big)^{1/p},$$

$$I_6(s) = \Big( \int_{\mathbf{K}_s^c} |f(z + ie/n)|^p b^{-r}(z,z)\, dv(z) \Big)^{1/p}$$

Let $\eta$ be an arbitrary positive number. Then there exists $N \in \mathbb{N}$ such that

$$I_5(s) < \eta/3 \quad \text{for all } s > N. \tag{18}$$

Fix $s$ such that $s > 2N$. Since by estimate (14) of Lemma IV.4, we have $b^{-r}(z,z) \leq b^{-r}(z + ie/n, z + ie/n)$ (because $r_i \geq 0$ $(i = 1, \ldots, l)$), it follows that whenever $|e/n| < \min(1/s, s/2)$, then

$$\begin{aligned} I_6(s) &\leq \Big( \int_{\mathbf{K}_s^c} |f(z + ie/n)|^p b^{-r}(z + ie/n, z + ie/n)\, dv(z) \Big)^{1/p} \\ &\leq \Big( \int_{\mathbf{K}_{s/2}^c} |f(z)|^p b^{-r}(z,z)\, dv(z) \Big)^{1/p} < \eta/3, \end{aligned} \tag{19}$$

where the last inequality follows from (18) because $s/2 > N$.

Lastly, fix $s$ such that $s > 2N$. Since $f$ is uniformly continuous on $\mathbf{K}_s$, there exists $\delta = \delta(\mathbf{K}_s) > 0$ such that $|f(z + ie/n) - f(z)| < \eta/(3(|\mathbf{K}_s|_r)^{1/p})$ whenever $|e/n| < \delta$. Here $|\mathbf{K}_s|_r$ stands for $\int_{\mathbf{K}_s} b^{-r}(z,z)\, dv(z)$. Hence,

$$I_3(s) < \eta/3. \tag{20}$$

Combining (18)–(20) yields that $I_2 < \eta$ whenever $n > \delta/|e|$. This completes the proof of Lemma IV.2.

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*Received May 26, 1994* (3281)