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INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
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Symbol calculus on the affine group "$ax + b$"

by

QIHONG FAN (Beijing)

Abstract. The symbol calculus on the upper half plane is studied from the viewpoint of the Kirillov theory of orbits. The main result is the L^p-estimates for Fuchs type pseudodifferential operators.

1. Introduction. The Weyl calculus on the linear phase space $\mathbb{R}^n \times \mathbb{R}^n$ has a natural group theoretic interpretation in terms of the Kirillov method of orbits (cf. [H]). Let H be the Heisenberg group. In the coordinates (p, q, t), $p, q \in \mathbb{R}^n$, $t \in \mathbb{R}$, the group law is given by

$$(p,q,t)(p',q',t') = \big(p + p', q + q', t + t' + \tfrac{1}{2}(pq' - qp')\big).$$

The phase space $\mathbb{R}^n \times \mathbb{R}^n$ can be considered as an orbit O under the coadjoint representation of the Heisenberg group H. According to Kirillov's theory (cf. [K]), the irreducible unitary representation which corresponds to this orbit is given by

$$\varrho(p,q,t) = e^{2\pi i t} e^{2\pi i (pD + qX)}$$

i.e.

$$\varrho(p,q,t) f(x) = e^{2\pi i t + 2\pi i q x + \pi i p q} f(x + p),$$

for $f \in L^2(\mathbb{R}^n)$.

Let $e_0 \in O$. Then for any $X \in O$, there exists a $g \in H$ such that $X = ge_0$, where the action is given by the coadjoint representation. Modulo the center of H, the element g is unique. We identify an element $X = (x, \xi) \in O$ with the element $\widetilde{X} = (x, \xi, 0) \in H$. For a given $e_0 \in O$ and a given unitary operator I_0, we define

$$\sigma_{e_0} = I_0,$$

and for any $X \in O$, put

$$\sigma_X = \varrho(\widetilde{X}) I_0 \varrho(\widetilde{X}^{-1}).$$

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For a function $F(X)$, we define the corresponding operator $Q(F)$ as

$$Q(F) = \int_O F(X)\sigma_X \, d\mu(X),$$

where the measure $d\mu(X)$ is the H-invariant measure on the orbit O. If we take $e_0 = (0,0)$ and $I_0 f(x) = f(-x)$ for $f \in L^2(\mathbb{R}^n)$, then the correspondence $F \to Q(F)$ is just the Weyl correspondence on the phase space $\mathbb{R}^n \times \mathbb{R}^n$.

In this paper, we shall study the symbol calculus on the upper half plane $\mathbb{R}^2_+$. This phase space is an orbit of the affine group "$ax+b$" under the coadjoint representation. It turns out that the correspondence is just the Fuchs calculus, which has been developed by Unterberger in [U]. We give an introduction to this calculus in Section 2. We construct the above-mentioned correspondence from the Kirillov theory of orbits. This gives a natural group theoretic interpretation of the Fuchs calculus. From this point of view, one can easily generalize the Fuchs calculus to tube type affine homogeneous Siegel domains. In the case of symmetric domains, this has been studied by Unterberger and Upmeier in [UU]. In Section 3, we study the L^p-continuity of the corresponding operators. In the case of the Weyl correspondence on the linear phase space $\mathbb{R}^n \times \mathbb{R}^n$, the L^p-estimates of the corresponding operators were studied by many authors (cf. [B], [CM]).

2. The symbol calculus. Let G be the affine group "$ax+b$". In the coordinates (a,b), with $a \in \mathbb{R}^+$, $b \in \mathbb{R}$, the group law is given by

$$(a,b)(a',b') = (aa', ab'+b).$$

The left and right Haar measures are given by

$$d\mu_l(a,b) = \frac{dadb}{a^2}, \qquad d\mu_r(a,b) = \frac{dadb}{a},$$

respectively. The modular function is given by $\Delta(a,b) = 1/a$. We can identify the group G with a subgroup of $GL(2,\mathbb{R})$ by the map

$$(a,b) \to \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

Under this map, the Lie algebra g of G has a basis A, B, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

One has $[A,B] = B$. Let $X = xA + yB \in g$. Then the exponential map is given by

$$\exp X = \begin{pmatrix} e^x & (e^x-1)y/x \\ 0 & 1 \end{pmatrix}.$$

The adjoint action of G on g is

$$\gamma(x,y) = (x, -bx+ay) \quad \text{for } \gamma = (a,b) \in G.$$

Let $X^* = \xi A^* + \eta B^* \in g^*$, the dual of g. The coadjoint action of G on g^* is given by

$$\gamma X^* = \gamma(\xi,\eta) = \left(\xi + \frac{b}{a}\eta\right)A^* + \frac{\eta}{a}B^*. \tag{2.1}$$

Under this action, the group G has two non-trivial coadjoint orbits: $O_\pm = \{\xi A^* + \eta B^* : \xi \in \mathbb{R},\ \eta \in \mathbb{R}^\pm\}$. Under the action (2.1) the G-invariant measure on $O_\pm$ is given by $d\mu(X^*) = d\xi d\eta/|\eta|$. The corresponding irreducible unitary representation of G can be realized on the Hilbert space $L^2(\mathbb{R}^+, dt)$. For $(a,b) \in G$,

$$\pi^\pm(a,b)u(t) = \sqrt{a}u(at)e^{\mp 2\pi ibt},$$

where $u \in L^2(\mathbb{R}^+, dt)$. Since the orbits O_+ and O_- have the same structure, we only study the orbit O_+ and the corresponding representation π^+, and in the following, we denote them by O and π.

For $(\xi,\eta) \in O$, one can find a unique $\gamma \in G$ such that $(\xi,\eta) = \gamma(0,1)$. In fact, $\gamma = (1/\eta, \xi/\eta)$. For $u(t) \in L^2(\mathbb{R}^+, dt)$, put

$$\theta u(t) = u\left(\frac{1}{t}\right)\frac{1}{t}.$$

We have $\theta^2 = I$ and $\theta^* = \theta$. For $X = (\xi,\eta) \in O$, put

$$\sigma_X = \pi(\gamma)\theta\pi(\gamma^{-1}), \tag{2.2}$$

where γ is determined by $X = \gamma(0,1)$. For $u(t) \in L^2(\mathbb{R}^+, dt)$, we have

$$\sigma_X u(t) = \frac{\eta}{t}u\left(\frac{\eta^2}{t}\right)e^{2\pi i\xi(\eta/t - t/\eta)}.$$

One may check that $\sigma_X = \sigma_X^* = \sigma_X^{-1}$. Let f be an integrable function defined on O. We define the operator $Q(f)$ by

$$Q(f) = 2\int_O f(X)\sigma_X \, d\mu(X). \tag{2.3}$$

That is, for $u \in L^2(\mathbb{R}^+, dt)$,

$$Q(f)u(t) = \frac{2}{t}\int_O f(\xi,\eta)e^{2\pi i\xi(\eta/t - t/\eta)}u(\eta^2/t)\, d\xi\, d\eta. \tag{2.4}$$

Following Unterberger (cf. [U]), the operator $Q(f)$ will be called the *Fuchs type pseudodifferential operator with active symbol* f. For a trace class operator A in $L^2(\mathbb{R}^+, dt)$, we call the function $h(X) = 2\,\mathrm{Tr}(A\sigma_X)$ the *passive symbol* of A. The passive and active symbols can be generalized to distributions in $S'(O)$, and the formula (2.4) holds for $u \in S(O)$ in the sense of distributions (cf. [U]). By (2.2) and (2.3) we have the following covariance properties for the correspondence $f \to Q(f)$:

$$\pi(\gamma)Q(f)\pi(\gamma^{-1}) = Q(f\circ\gamma^{-1}) \tag{2.5}$$

for any $\gamma \in G$, and

$$\theta Q(f)\theta^{-1} = Q(\widetilde{f}), \tag{2.6}$$

where $\widetilde{f}(\xi,\eta) = f(-\xi,\eta^{-1})$. We note that if A has active symbol f and B has passive symbol h, then

$$\operatorname{Tr}(AB) = \int_O f(X)h(X)\,d\mu(X). \tag{2.7}$$

Let

$$F = \left(1 + \frac{1}{4}\left(\frac{1}{2\pi i}\frac{\partial}{\partial\xi}\right)^2\right)^{-1/2}$$

Then the passive symbol of the operator $Q(f)$ is given by Ff (cf. [U]). By (2.7) we have the following proposition.

PROPOSITION 2.1. *The operator $Q(f)$ is a Hilbert–Schmidt operator if and only if $(Ff,f) < \infty$.*

Let $e_0 = (0,1) \in O$, $X_1 = (\xi_1,\eta_1)$, $X_2 = (\xi_2,\eta_2)$. Put

$$\omega_{e_0}(X_1,X_2) = \xi_2(\eta_1 - \eta_1^{-1}) + \xi_1(\eta_2^{-1} - \eta_2).$$

For $X = \gamma e_0 \in O$, we define

$$\omega_X(X_1,X_2) = \omega_{e_0}(\gamma^{-1}X_1,\gamma^{-1}X_2). \tag{2.8}$$

By a trivial computation, we have the following result.

PROPOSITION 2.2. *If $A = Q(f)$ and $B = Q(g)$, then the passive symbol h of AB is given by*

$$h(Z) = 4\int\int f(X)g(Y)e^{2\pi i\omega_Z(X,Y)}\,d\mu(X)\,d\mu(Y). \tag{2.9}$$

If the symbols f, g are in some good class, then the equality (2.9) has an asymptotic expansion. For details, see [U].

3. L^p-estimates of pseudodifferential operators. In [U], Unterberger characterized a class of symbols which correspond to bounded operators on $L^2(\mathbb{R}^+,dt)$. Our purpose in this section is to study the L^p-boundedness of pseudodifferential operators. We shall prove the following theorem.

THEOREM 3.1. *Let $f \in C^\infty(O)$, and for any $\alpha,\beta \ge 0$,*

$$|(\eta\partial_\eta)^\beta\partial_\xi^\alpha f| \le C_{\alpha\beta}(1+|\xi|)^{-|\alpha|}. \tag{3.1}$$

Let K be the operator of multiplication by t, i.e. $Ku(t) = tu(t)$, and $T = K^{1/2}Q(f)K^{-1/2}$. Then

(1) *for $1 < p < \infty$, the operator T is bounded in $L^p(\mathbb{R}^+,dt/t)$.*
(2) *T is of weak type $(1,1)$.*

To prove the theorem, we introduce a distance on $\mathbb{R}^+$ such that when equipped with the measure $t^{-1}dt$, $\mathbb{R}^+$ become a space of homogeneous type, and then we prove that if f satisfies the estimate (3.1), then the operator $Q(f)$ is a Calderón–Zygmund singular integral operator.

For $s,\eta \in \mathbb{R}^+$, let $\widetilde{d}(s,\eta) = |s-\eta|(s\eta)^{-1/2}$. We define $d(s,\eta) = \ln(1+\widetilde{d}(s,\eta))$. One can check that $d(s,\eta)$ has the following properties:

$$d(s,\eta) \le d(s,t) + d(t,\eta) \quad \text{for any } s,\eta \in \mathbb{R}^+, \tag{3.2}$$

$$d(\lambda s,\lambda\eta) = d(s,\eta) \quad \text{for any } \lambda,s,\eta \in \mathbb{R}^+. \tag{3.3}$$

The formula (3.3) is obvious. (3.2) follows from (3.3) and the inequality

$$d(s,1) \le d(s,t) + d(t,1), \tag{3.2$'$}$$

which follows from a trivial computation of the minimal value of the function $h(t) = d(s,t) + d(t,1)$.

Let $B(1,r) = \{\eta : d(1,\eta) < r\}$ and $B(s,r) = \{\eta : d(s,\eta) < r\}$. By a trivial computation,

$$B(1,r) = \left(1 + \frac{\widetilde{r} - \sqrt{4\widetilde{r}+\widetilde{r}^2}}{2}, 1 + \frac{\widetilde{r} + \sqrt{4\widetilde{r}+\widetilde{r}^2}}{2}\right),$$

$$B(s,r) = \left(s + \frac{\widetilde{r} - \sqrt{4\widetilde{r}+\widetilde{r}^2}}{2}s, s + \frac{\widetilde{r} + \sqrt{4\widetilde{r}+\widetilde{r}^2}}{2}s\right),$$

where $\widetilde{r} = (e^r - 1)^2$. Hence for $d\mu(\eta) = d\eta/\eta$,

$$\mu(B(1,r)) \sim \begin{cases} 2r & \text{as } r \to 0, \\ 4r & \text{as } r \to \infty. \end{cases} \tag{3.4}$$

So there is a constant A such that

$$0 < \mu(B(s,2r)) \le A\mu(B(s,r)) < \infty. \tag{3.5}$$

By (3.2) and (3.5), $(\mathbb{R}^+,d,d\mu)$ is a space of homogeneous type in the sense of Coifman and Weiss (cf. [CW]). We have the following proposition.

PROPOSITION 3.2. *Let $u \in L^1(\mathbb{R}^+,d\mu)$ and $\alpha > 0$. Then u may be decomposed as $u = v + b$, where*

(i) $\|v\|_{L^2(d\mu)} \le C\alpha\|u\|_{L^1(d\mu)}$.
(ii) *$b = \sum b_j$, where each b_j is supported on some ball $B(s_j,r_j)$.*
(iii) $\int b_j\,d\mu = 0$.
(iv) $\|b_j\|_{L^1} \le C\alpha\mu(B(s_j,r_j))$.
(v) $\sum_j \mu(B(s_j,r_j)) \le C\alpha^{-1}\|u\|_{L^1}$.

The proof of this proposition can be found in [CW]. By (2.4), the integral kernel $k(s,\eta)$ of the operator $T = K^{1/2}Q(f)K^{-1/2}$ with respect to the measure $d\mu$ is given by

$$k(s,\eta) = \int_{\mathbb{R}} f(\xi,(s\eta)^{1/2})e^{2\pi i((s/\eta)^{1/2}-(\eta/s)^{1/2})\xi}\,d\xi. \tag{3.6}$$

That is,

$$Tu(s) = \int_{\mathbb{R}^+} k(s,\eta)u(\eta)\,\frac{d\eta}{\eta}.$$

LEMMA 3.3. *Assume f satisfies the estimate* (3.1). *Then for any $\alpha \ge 2$, there exists a constant C_α such that*

$$|\partial_\eta k(s,\eta)| \le C_\alpha \widetilde{d}(s,\eta)^{-\alpha}(s^{1/2}\eta^{-3/2} + s^{-1/2}\eta^{-1/2}). \tag{3.7}$$

Proof. We use a partition of unity

$$1 = \sum_{j=1}^{\infty}\psi_j(\xi), \qquad \psi_j \text{ supported on } |\xi| \sim 2^j,$$

such that $\psi_j(\xi) = \psi_1(2^{1-j}\xi)$ for $j \ge 2$. Put

$$L_j = (1+\widetilde{d}(s,\eta)^2 2^{2j\delta_1})^{-1}(1-(4\pi^2)^{-1}2^{2j\delta_1}\partial_\xi^2).$$

Then

$$L_j e^{2\pi i((s/\eta)^{1/2}-(\eta/s)^{1/2})\xi} = e^{2\pi i((s/\eta)^{1/2}-(\eta/s)^{1/2})\xi}.$$

Let

$$k_j(s,\eta) = \int_{\mathbb{R}} f_j(\xi,(s\eta)^{1/2})e^{2\pi i((s/\eta)^{1/2}-(\eta/s)^{1/2})\xi}\,d\xi,$$

where $f_j(\xi,\eta) = \psi_j(\xi)f(\xi,\eta)$. Then

$$\partial_\eta k_j(s,\eta) = \int_{\mathbb{R}} \bigg(\frac{1}{2}\frac{s^{1/2}}{\eta^{1/2}}\partial_\eta f_j(\xi,(s\eta)^{1/2}) - \pi i\xi(s^{1/2}\eta^{-3/2}+s^{-1/2}\eta^{-1/2}) \times f_j(\xi,(s\eta)^{1/2})\bigg)e^{2\pi i((s/\eta)^{1/2}-(\eta/s)^{1/2})\xi}\,d\xi.$$

If $\delta_1 \le 1$, by (3.1) we obtain $\|(1-(4\pi^2)^{-1}2^{2j\delta_1}\partial_\xi^2)^N f_j\|_{L^\infty} \le C$. By partial integration, we have

$$|\partial_\eta k_j(s,\eta)| \le C(1+\widetilde{d}(s,\eta)^2 2^{2j\delta_1})^{-N}2^{2j}(s^{1/2}\eta^{-3/2}+s^{-1/2}\eta^{-1/2}).$$

Consequently,

$$\begin{aligned}|\partial_\eta k(s,\eta)| &\le \sum_j |\partial_\eta k_j(s,\eta)|\\ &\le C(s^{1/2}\eta^{-3/2}+s^{-1/2}\eta^{-1/2})\sum_j 2^{2j}(1+\widetilde{d}(s,\eta)^2 2^{2j\delta_1})^{-N}\end{aligned}$$

$$\begin{aligned}&\le C(s^{1/2}\eta^{-3/2}+s^{-1/2}\eta^{-1/2})\int_0^\infty (1+(\widetilde{d}(s,\eta)^{1/\delta_1}t)^{2\delta_1})^{-N}t\,dt\\ &\le C\widetilde{d}(s,\eta)^{-2/\delta_1}(s^{1/2}\eta^{-3/2}+s^{-1/2}\eta^{-1/2}).\end{aligned}$$

If we choose $\delta_1 = 2/\alpha$, then $\delta_1 \le 1$, so we get the estimate (3.7). This completes the proof. ∎

LEMMA 3.4. *For any $\eta,\eta_0 \in \mathbb{R}^+$, the following estimate holds:*

$$\int_{d(s,\eta)>2d(\eta,\eta_0)} |k(s,\eta)-k(s,\eta_0)|\,\frac{ds}{s} < C. \tag{3.8}$$

Proof. Let $r = d(\eta,\eta_0)$ and $\widetilde{r} = (e^r-1)^2$. We have

$$k(s,\eta)-k(s,\eta_0) = (\eta-\eta_0)\partial_\eta k(s,\widetilde{\eta})$$

with $d(\widetilde{\eta},\eta) \le d(\eta,\eta_0)$. Since $d(s,\widetilde{\eta}) \ge d(s,\eta)-d(\eta,\widetilde{\eta})$, we have $1+\widetilde{d}(s,\widetilde{\eta}) \ge (1+\widetilde{d}(s,\eta))e^{-r}$. Note that if $d(s,\eta) > 2r$, then $d(s,\widetilde{\eta}) \ge \frac{1}{2}e^{-r}d(\eta,\eta_0)$. We get

$$\begin{aligned}&\int_{d(s,\eta)>2r} |k(s,\eta)-k(s,\eta_0)|\,\frac{ds}{s}\\ &\quad\le \int_{d(s,\eta)>2r} |\eta-\eta_0|\cdot|\partial_\eta k(s,\widetilde{\eta})|\,\frac{ds}{s}\\ &\quad\le C\int_{d(s,\eta)>2r} |\eta-\eta_0|\widetilde{d}(s,\widetilde{\eta})^{-\alpha}(s^{1/2}\widetilde{\eta}^{-3/2}+s^{-1/2}\widetilde{\eta}^{-1/2})\,\frac{ds}{s}\\ &\quad\le Ce^{\alpha r}\int_{d(s,\eta)>2r} |\eta-\eta_0|\widetilde{d}(s,\eta)^{-\alpha}(s^{1/2}\widetilde{\eta}^{-3/2}+s^{-1/2}\widetilde{\eta}^{-1/2})\,\frac{ds}{s}\\ &\quad\le Ce^{\alpha r}\int_{d(s,1)>2r} \widetilde{d}(s,1)^{-\alpha}(s^{1/2}(\eta/\widetilde{\eta})^{1/2}+s^{-1/2}(\widetilde{\eta}/\eta)^{1/2})\left|\frac{\eta-\eta_0}{\widetilde{\eta}}\right|\frac{ds}{s}.\end{aligned}$$

We note that

$$\frac{|\eta-\eta_0|}{\widetilde{\eta}} \le (\widetilde{r}+\sqrt{4\widetilde{r}+\widetilde{r}^2})\bigg(1+\frac{\widetilde{r}+\sqrt{4\widetilde{r}+\widetilde{r}^2}}{2}\bigg),$$

and

$$\frac{\eta}{\widetilde{\eta}} \le \bigg(1+\frac{\widetilde{r}+\sqrt{4\widetilde{r}+\widetilde{r}^2}}{2}\bigg)^2, \qquad \frac{\widetilde{\eta}}{\eta} \le \bigg(1+\frac{\widetilde{r}+\sqrt{4\widetilde{r}+\widetilde{r}^2}}{2}\bigg)^2.$$

Thus

$$(3.9)\qquad \int\limits_{d(s,\eta)>2r} |k(s,\eta)-k(s,\eta_0)|\,\frac{ds}{s}$$
$$\le Ce^{\alpha r}(\widetilde r+\sqrt{4\widetilde r+\widetilde r^2})\Big(1+\frac{\widetilde r+\sqrt{4\widetilde r+\widetilde r^2}}{2}\Big)^2 \times \int\limits_{d(s,1)>2r} \widetilde d(s,1)^{-\alpha}(s^{1/2}+s^{-1/2})\,\frac{ds}{s}.$$

To estimate the last integral, we note that

$$(3.10)\qquad \int\limits_{d(s,1)>2r} \widetilde d(s,1)^{-\alpha}(s^{1/2}+s^{-1/2})\,\frac{ds}{s} = 2\int\limits_{s(r)}^{\infty} (s^{1/2}-s^{-1/2})^{-\alpha}(s^{1/2}+s^{-1/2})\,\frac{ds}{s},$$

where

$$s(r)=1+\frac{r'+\sqrt{4r'+r'^2}}{2}\quad \text{with } r'=(e^{2r}-1)^2.$$

Since $s(r)=1+2r+O(r^2)$ as $r\to 0$, for $r_0>0$ sufficiently small and $r<r_0$ we have

$$\int\limits_{s(r)}^{\infty} (s^{1/2}-s^{-1/2})^{-\alpha}(s^{1/2}+s^{-1/2})\,\frac{ds}{s} \le \int\limits_{s(r)}^{\infty} (s^{1/2}-1)^{-\alpha}(s^{1/2}+s^{-1/2})\,\frac{ds}{s} \le C\int\limits_{s(r)^{1/2}}^{\infty} (s-1)^{-\alpha}\,ds \le Cr^{-\alpha+1}.$$

If we take $\alpha=2$, then by (3.9), for $0<r\le r_0$, we obtain

$$\int\limits_{d(s,\eta)>2r} |k(s,\eta)-k(s,\eta_0)|\,\frac{ds}{s}\le C.$$

Since $s(r)=e^{4r}(1+o(1))$ as $r\to\infty$, for $r_1>0$ sufficiently large and $r>r_1$ we have

$$\int\limits_{s(r)}^{\infty} (s^{1/2}-s^{-1/2})^{-\alpha}(s^{1/2}+s^{-1/2})\,\frac{ds}{s}\le Ce^{(-2\alpha+2)r}.$$

In this case, if we take $\alpha=8$, then by (3.9) and $\widetilde r=e^{2r}(1+o(1))$, for $r_1\le r<\infty$ we have

$$\int\limits_{d(s,\eta)>2r} |k(s,\eta)-k(s,\eta_0)|\,\frac{ds}{s}\le Ce^{6r}e^{\alpha r}e^{(-2\alpha+2)r}\le C.$$

Obviously there is a constant C such that for $r_0\le r\le r_1$,

$$\int\limits_{d(s,\eta)>2r} |k(s,\eta)-k(s,\eta_0)|\,\frac{ds}{s}\le C.$$

Consequently,

$$\int\limits_{d(s,\eta)>2d(\eta,\eta_0)} |k(s,\eta)-k(s,\eta_0)|\,\frac{ds}{s}\le C.$$

This completes the proof. ■

The proof of Theorem 3.1 now follows from the Calderón–Zygmund theory. We sketch the argument for completeness.

Proof of Theorem 3.1. We note that $k(s,\eta)=\overline{k}(\eta,s)$. By duality, it is enough to prove the result when $1<p\le 2$. By (3.7) and the result in [U], T is a bounded operator in $L^2(\mathbb{R}^+,d\mu)$. By the Marcinkiewicz interpolation theorem it is enough to show that T is of weak type $(1,1)$. That is, for any $\alpha>0$ and $u\in L^1(\mathbb{R}^+,d\mu)$,

$$(3.10)\qquad \mu(\{s:|Tu(s)|>2\alpha\})\le C\alpha^{-1}\int|u|\,d\mu.$$

By Proposition 3.2, u can be decomposed as $u=v+b$, where $b=\sum b_j$ and v, b_j have the properties (i)–(v) in Proposition 3.2. We have

$$\mu(\{s:|Tu(s)|>2\alpha\})\le\mu(\{s:|Tv(s)|>\alpha\})+\mu(\{s:|Tb(s)|>\alpha\})$$

and

$$\mu(\{s:|Tv(s)|>\alpha\})\le\frac{1}{\alpha^2}\int|Tv(s)|^2\,d\mu(s)\le C\frac{1}{\alpha}\|u\|_{L^1(d\mu)}.$$

Since $\mu(\bigcup_j B(s_j,2r_j))\le C\alpha^{-1}\|u\|_{L^1(d\mu)}$, it is enough to estimate $\mu(\{s\in\mathbb{R}^+\setminus\bigcup_j B(s_j,2r_j):\sum|Tb_j(s)|>\alpha\})$. Now,

$$\mu\Big(\Big\{s\in\mathbb{R}^+\setminus\bigcup_j B(s_j,2r_j):\sum|Tb_j(s)|>\alpha\Big\}\Big)\le\frac{1}{\alpha}\sum_j\int\limits_{\mathbb{R}^+\setminus B(s_j,2r_j)}|Tb_j(s)|\,d\mu(s).$$

By the property (iii) and (3.8),

$$\int\limits_{\mathbb{R}^+\setminus B(s_j,2r_j)}|Tb_j(s)|\,d\mu(s) = \int\limits_{d(s,s_j)>2r_j}\Big|\int(k(s,\eta)-k(s,s_j))b_j(\eta)\,d\mu(\eta)\Big|\,d\mu(s)$$
$$\le\int\Big(\int\limits_{d(s,s_j)>2r_j}|k(s,\eta)-k(s,s_j)|\,d\mu(s)\Big)|b_j(\eta)|\,d\mu(\eta)$$
$$\le C\int|b_j(\eta)|\,d\mu(\eta)\le C\alpha\mu(B(s_j,r_j)).$$

Consequently,

$$\mu\Big(\Big\{s\in\mathbb{R}^+\backslash\bigcup_j B(s_j,2r_j):\sum|Tb_j(s)|>\alpha\Big\}\Big)\le C\alpha^{-1}\|u\|_{L^1(d\mu)}.$$

This finishes the proof of (3.10). ∎

Remark 1. In [U], Unterberger proved the L^2-boundedness for a very general class of symbols. Only a subclass of these symbols satisfy the estimates (3.1).

Remark 2. Using the same technique as in the proof of Lemma 3.3, one can get the following estimate on the kernel:

$$|k(s,\eta)|\le C_\alpha\widetilde{d}(s,\eta)^{-\alpha},$$

where C_α is a constant and $\alpha\ge 1$. Moreover, the estimate near $s=\eta$ cannot be improved. In fact, one can find a symbol f which satisfies the estimate (3.1) and the corresponding kernel $k(s,\eta)$ has the singularity $\widetilde{d}(s,\eta)^{-1}$ near the set $\{(s,\eta):s=\eta\}$. From this estimate we know that the corresponding operator is not bounded in L^1. Therefore we can only get a weak type $(1,1)$ estimates. For strong type estimates, as in the usual case, we have the following theorem.

THEOREM 3.5. *With the notations as in Theorem* 3.1, *suppose* $f\in C^\infty(O)$ *satisfies the estimate*

$$|(\eta\partial_\eta)^\alpha\partial_\xi^\beta f(\xi,\eta)|\le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}, \tag{3.11}$$

with $m<0$. *Then the corresponding operator* $T=K^{1/2}Q(f)K^{-1/2}$ *is bounded in* $L^p(\mathbb{R}^+,t^{-1}dt)$ *for* $1\le p\le\infty$.

Proof. Since the symbol f satisfies the estimates (3.11) with $m<0$, as in the proof of Lemma 3.3, we can find a constant $\varepsilon>0$ (depending on m) such that

$$|k(s,\eta)|\le C_\alpha\widetilde{d}(s,\eta)^{-\alpha} \tag{3.12}$$

for $\alpha\ge 1-\varepsilon$. By (3.12), we have

$$\int_0^\infty |k(s,\eta)|\,\frac{d\eta}{\eta}<\infty,\qquad \int_0^\infty |k(s,\eta)|\,\frac{ds}{s}<\infty.$$

By the Schur Lemma, T is a bounded operator in L^p for $1\le p\le\infty$. This completes the proof. ∎

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DEPARTMENT OF MATHEMATICS
PEKING UNIVERSITY
BEIJING 100871, P.R. CHINA

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