Topologies on the space of ideals of a Banach algebra

by

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Abstract. Some topologies on the space \( \text{Id}(A) \) of two-sided and closed ideals of a Banach algebra are introduced and investigated. One of the topologies, namely \( \tau_{\infty} \), coincides with the so-called strong topology if \( A \) is a \( C^* \)-algebra. We prove that for a separable Banach algebra \( \tau_{w} \) coincides with a weaker topology when restricted to the space \( \text{Min-Primal}(A) \) of minimal closed primal ideals and that \( \text{Min-Primal}(A) \) is a Polish space if \( \tau_{w} \) is Hausdorff; this generalizes results from [1] and [5]. All subspaces of \( \text{Id}(A) \) with the relative hull kernel topology turn out to be separable Lindelöf spaces if \( A \) is separable, which improves results from [14].

1. Introduction. In [14] D. W. B. Somerset has started the investigation of the space \( \text{Min-Primal}(A) \) of minimal primal closed ideals of a general Banach algebra. He proved, among other things, that if \( A \) is a separable Banach algebra, then the space \( \text{Prime}(A) \) of closed prime ideals with the hull kernel topology (or weak topology \( \tau_{w} \)) is separable, and if additionally \( A \) is topologically semiprimal, then \( \text{Min-Primal}(A) \) is also separable. We will prove the much stronger result that all subspaces of the space \( \text{Id}(A) \) of closed two-sided ideals of \( A \) are separable Lindelöf spaces if \( A \) is separable. On page 50 of [14] the author left open the question whether \( \text{Min-Primal}(A) \) is second countable if \( A \) is separable. We will give an example of a unital separable subalgebra \( A \) of a commutative \( C^* \)-algebra such that \( \text{Min-Primal}(A) \) is not even first countable.

The methods used here are rather different from those of [14]. Here we try to generalize the idea of the strong topology \( \tau_{s} \) in \( \text{Id}(A) \) (see [1] for this or next paragraph). If \( A \) is a \( C^* \)-algebra, then this topology makes \( \text{Id}(A) \) a compact Hausdorff space. This useful topology has been investigated and applied in the theory of \( C^* \)-algebras (see e.g. [1]–[6]).

If \( (A, \| \cdot \|) \) is a Banach algebra, then \( \tau_{s} \) is by definition the weak topology of all maps

\[
\text{Id}(A) \to \mathbb{R}_{0}^{+}, \quad I \mapsto \| x + I \|, \quad x \in A.
\]

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For a general Banach algebra $\tau_\alpha$ need not be compact, and obviously $\tau_\alpha$ depends on the special norm on $A$. Indeed, it is easy to find equivalent algebra norms on the $C^*$-algebra of convergent sequences such that the $\tau_\alpha$-topology defined by this new norm is not compact. We will define another topology $\tau_\infty$ on $\text{Id}(A)$ with the following properties:

(i) $(\text{Id}(A), \tau_\infty)$ is compact (not Hausdorff in general).

(ii) $\tau_w \subset \tau_\infty \subset \tau_\alpha$.

(iii) $\tau_\infty$ only depends on the norm topology of $A$, not on the special norm.

(iv) $\tau_\infty = \tau_\alpha$ if $A$ is a $C^*$-algebra.

(v) If $A$ is a commutative Banach algebra with a bounded approximate identity then on the Gelfand space $\tau_\infty$ coincides with the Gelfand topology.

This topology can be used to answer the above mentioned questions. Further properties and examples are given in the following sections.

2. An example

Example 1. Let $D$ be the closed unit disc in the plane, $2D$ the disc with radius 2. Let $A(D)$ be the disc algebra, i.e. the Banach algebra of continuous functions $D \to \mathbb{C}$ that are holomorphic in the interior of $D$. Let $C(2D)$ be the commutative $C^*$-algebra of continuous functions on $2D$. Define

$$A := \{f \in C(2D) : f|D \in A(D)\}.$$ 

For $M \subset 2D$ let $I_M$ be the ideal of functions in $A$ vanishing on $M$. Then it is not difficult to prove that

$$\text{Min-Prim}(A) = \{I_{\{M\}} : z \in 2D \setminus D\} \cup \{I_D\}.$$ 

Assertion. $I_D$ is in the $\tau_w$-closure of $M_0 := \{I_{\{M\}} : z \in 2D \setminus D\}$ but no sequence in $M_0$ $\tau_w$-converges to $I_D$.

Proof. Let $U$ be an open $\tau_w$-neighbourhood of $I_D$. Then $U$ contains a neighbourhood of the form

$$V := \{I \in \text{Min-Prim}(A) : f_1 \not\in I, \ldots, f_n \not\in I\}, \quad f_1, \ldots, f_n \in A, \quad n \in \mathbb{N}.$$ 

Since $I_D \in V$ we have $f_j|D \not\in D$. Since $A(D)$ is an integral domain we have $f_1 \ldots f_n \not= 0$, and by the maximum modulus principle there is a point $t$ in $\partial D$ such that $f_1(t) \ldots f_n(t) \not= 0$. By the continuity of the $f_j$ we can conclude that there is a point $s$ in $2D \setminus D$ such that $f_1(s) \ldots f_n(s) \not= 0$, and this means $I(s) \in V \subset U$. Hence $I_D$ is in the $\tau_w$-closure of $M_0$.

Now assume that there were a sequence $(t_n)_n$ in $2D \setminus D$ such that $I_{\{t_n\}}$ converges to $I_D$. Since $2D$ is compact we may assume that $t_n \to t \in 2D$.

Then $F := D \cup \{t_n : n \in \mathbb{N}\} \cup \{t\}$ is a closed subset of $2D$. Define

$$f : F \to \mathbb{C}, \quad f(z) := \begin{cases} z - t & \text{if } z \in D, \\ 0 & \text{if } z \in F \setminus D. \end{cases}$$

This clearly is continuous and by the Tietze extension theorem it can be extended to an element $g \in C(2D)$, and obviously $g \in A$. Since $g \not\in I_D$ we must have $g \not\in I_{t_n}$ for large $n$ by the assumed $\tau_w$-convergence, but this is not the case. This contradiction finishes the proof of the assertion. ■

Hence $I_D \in \text{Min-Prim}(A)$ cannot have a countable neighbourhood base, and so $\text{Min-Prim}(A)$ is not a first countable space in the relative $\tau_w$-topology. This answers a question on page 50 of [14] in the negative.

3. Construction of the topology and simple properties. Let $(A, \| \cdot \|)$ be a Banach algebra. For $k \in \mathbb{N}$ let $S_k(A, \| \cdot \|)$ be the set of all algebra seminorms bounded by $k$, i.e. the set of seminorms $p : A \to \mathbb{R}_+$ such that $p(ab) \leq p(a)p(b)$ and $p(a) \leq k\|a\|$ for all $a, b \in A$. Write only $S_k$ if $(A, \| \cdot \|)$ is clear. We have in the obvious manner

$$S_k(A, \| \cdot \|) \subset \bigcap_{a \in A} [0, k\|a\|].$$

Since the conditions for a real-valued function to be an element of $S_k$ are pointwise conditions, $S_k$ is closed, hence compact subspace of the product space, i.e. $S_k$ is compact with respect to the topology of pointwise convergence.

Lemma 2. If $A$ is separable, then $S_k$ is metrizable.

Proof. If $(a_n)_n$ is a dense sequence in $A$ then the injection

$$S_k \hookrightarrow \bigcap_{n \in \mathbb{N}} [0, k\|a_n\|]$$

defines the same topology on $S_k$. Indeed, if $(p_i)_i$ is a net in $S_k$, $p \in S_k$, and if $p_i(a_n) \to p(a_n)$ for all $n$, then for $a \in A$,

$$|p_i(a) - p(a)| \leq p_i(a - a_n) + |p_i(a_n) - p(a_n)| + p(a_n - a) \leq 2k\|a - a_n\| + |p_i(a_n) - p(a_n)|,$$

and this is small for large $i$ if $n$ is chosen appropriately. ■

Now define

$$\kappa_k : S_k(A, \| \cdot \|) \to \text{Id}(A), \quad p \mapsto \ker(p).$$

This map is surjective since if $I \in \text{Id}(A)$ then the corresponding quotient seminorm

$$q_I(a) := \|a + I\|, \quad a \in A,$$
obviously is in $S_1 \subset S_2$. Let $\tau_k(A, \| \cdot \|)$ be the quotient topology of this map on $\text{Id}(A)$, and finally $\tau_\infty(A, \| \cdot \|) := \cap \tau_k(A, \| \cdot \|)$. Simply write $\tau_k$ or $\tau_\infty$ if no confusion can arise.

$\tau_\infty$ may alternatively be described as follows: Equip $\bigcup_k S_k$ with the inductive topology; then $\tau_\infty$ is the quotient topology of the map $p \mapsto \ker(p)$.

**Lemma 3.** For any Banach algebra $(A, \| \cdot \|)$ the topologies $\tau_k$, $k \in \mathbb{N}_\infty$, are compact (in general not Hausdorff) and

$$
\tau_\infty \subset \tau_\infty \subset \ldots \subset \tau_{k+1} \subset \tau_k \subset \ldots \subset \tau_1 \subset \tau_0.
$$

**Proof.** Since the $S_k$ are compact, it is clear that the topologies $\tau_k$ and hence $\tau_\infty$ are also compact.

Let $I \rightarrow f \in (\text{Id}(A), \tau_k)$, i.e. $q_I \rightarrow q_I$ in $S_1$. This implies $I \rightarrow I$ with respect to $\tau_1$, thus proving the inclusion $\tau_1 \subset \tau_0$.

Since $\kappa_{k+1} : S_{k+1} \rightarrow (\text{Id}(A), \tau_{k+1})$ is continuous, so is the restriction

$$\kappa_k = \kappa_{k+1}|S_k : S_k \rightarrow (\text{Id}(A), \tau_k).$$

By definition $\tau_k$ is the finest topology on $\text{Id}(A)$ making this map continuous, and this implies $\tau_{k+1} \subset \tau_k$.

The only thing left to show is $\tau_w \subset \tau_k$ for all $k \in \mathbb{N}$. If $p_k \rightarrow p$ in $S_k$ and $x \not\in \ker(p)$, then $p_k(x) \rightarrow p(x) \neq 0$ and therefore $x \not\in \ker(p_k)$ for large $k$. This proves the $\tau_w$-convergence $\kappa_k(p_k) \rightarrow \kappa_k(p)$, and again by the definition of the quotient topology $\tau_k$ we conclude $\tau_w \subset \tau_k$.

The topologies $\tau_k$ seem to depend on the special norm chosen on $A$ although I do not know any example for this. But we have the following.

**Proposition 4.** Let $(A, \| \cdot \|)$ be a Banach algebra. The topology $\tau_\infty(A, \| \cdot \|)$ on $\text{Id}(A)$ is compact, $\tau_w \subset \tau_\infty \subset \tau_1$, and if $\| \cdot \|$ is another equivalent norm then $\tau_\infty(A, \| \cdot \|) = \tau_\infty(A, \| \cdot \|_0)$.

**Proof.** We only have to show the independence on the special norm. There are constants $\alpha, \beta > 0$ such that $\alpha \| \cdot \|_0 \leq \| \cdot \| \leq \beta \| \cdot \|_0$. For $k \in \mathbb{N}$ let $I \in \mathbb{N}$ be such that $I \geq \beta k$. Then $S_k(A, \| \cdot \|) \subset S_k(A, \| \cdot \|_0)$. Since the restriction

$$\kappa_k : S_k(A, \| \cdot \|_0) \rightarrow (\text{Id}(A), \tau_k(A, \| \cdot \|_0))$$

is continuous we have $\tau_k(A, \| \cdot \|_0) \supset \tau_k(A, \| \cdot \|) \supset \tau_\infty(A, \| \cdot \|_0)$ by the definition of the quotient topologies. As $k$ was arbitrary we see $\tau_\infty(A, \| \cdot \|) \supset \tau_\infty(A, \| \cdot \|_0)$; the reverse inclusion is similar.

**Proposition 5.** Let $\varphi : A \rightarrow B$ be a continuous homomorphism between Banach algebras and define

$$\overline{\varphi} : \text{Id}(B) \rightarrow \text{Id}(A), \quad I \mapsto \varphi^{-1}(I).$$

Then $\overline{\varphi}$ is $\tau_\infty$-continuous. If $\varphi$ is surjective then $\overline{\varphi}$ is a homeomorphism onto its image.

**Proof.** Since $\varphi$ is continuous, $\overline{\varphi}$ maps closed ideals to closed ideals. For $k \in \mathbb{N}$ let $I \in \mathbb{N}$ be such that $I \geq k(\| \cdot \|)$. Then we have a map

$$\overline{\varphi} : S_k(B) \rightarrow S_k(A), \quad p \mapsto p \circ \varphi,$$

and the diagram

$$
\begin{array}{ccc}
S_k(B) & \rightarrow & S_k(A) \\
\kappa_k & \downarrow & \kappa_k \\
\text{Id}(B) & \rightarrow & \text{Id}(A)
\end{array}
$$

is obviously commutative. Since $\varphi \circ \kappa_k = \kappa_k \circ \overline{\varphi}$ is continuous, we deduce the $\tau_\infty$-$\tau_k$-continuity of $\overline{\varphi}$, and hence the $\tau_\infty$-$\tau_\infty$-continuity. As $k$ was arbitrary the first assertion follows.

Now let $\varphi$ be surjective. By Proposition 4 we may assume that $B = A/J$ for $J = \ker(\varphi)$ and that $\varphi$ is the quotient map. Then

$$\text{im}(\overline{\varphi}) = \{ I \in \text{Id}(A) : I \supset J \} \subset \text{Id}(A)$$

is $\tau_w$-closed, hence $\tau_\infty$-closed for all $k \in \mathbb{N}$. Then the restricted topology $\tau_k(\text{im}(\overline{\varphi}))$ coincides with the quotient topology of the map

$$\kappa_k : \kappa_k^{-1}(\text{im}(\overline{\varphi})) \rightarrow \text{im}(\overline{\varphi}).$$

Therefore it is enough to show that

$$\overline{\varphi}^{-1} \circ \kappa_k : \kappa_k^{-1}(\text{im}(\overline{\varphi})) \rightarrow \text{Id}(A/J, \tau_\infty)$$

is continuous. To this end let $p_k \rightarrow p$ in $\kappa_k^{-1}(\text{im}(\overline{\varphi}))$. Then $\ker(p_k), \ker(p) \supset J$,

$$p_k(a + J) := p_k(a), \quad p(a + J) := p(a)$$

are well-defined elements of $S_k(A/J)$ and we have $p_k \rightarrow p$; moreover,

$$\overline{\varphi}^{-1} \circ \kappa_k(p_k) \rightarrow \overline{\varphi}^{-1}(\ker(p_k)) = \ker(p_k)/J = \ker(p)/J \rightarrow \ker(p) = \ldots \rightarrow \overline{\varphi}^{-1} \circ \kappa_k(p),$$

and this finishes the proof of the second assertion.

**Proposition 6.** Let $I$ be a two-sided closed ideal in a Banach algebra $A$. Then

(i) The intersection map $i : \text{Id}(A) \rightarrow \text{Id}(I), J \mapsto J \cap I$, is $\tau_\infty$-continuous.

(ii) If $I$ has an approximate identity then $\text{Id}(I) \subset \text{Id}(A)$.

(iii) If $I$ has a bounded approximate identity then $\text{Id}(I), \tau_\infty(I)$ carries the subspace topology from $(\text{Id}(A), \tau_\infty(A))$.

**Proof.** Let $r : S_k(A) \rightarrow S_k(I)$ be the restriction map. Then the diagram

$$
\begin{array}{ccc}
S_k(A) & \rightarrow & S_k(I) \\
\kappa_k & \downarrow & \kappa_k \\
\text{Id}(A) & \rightarrow & \text{Id}(I)
\end{array}
$$
is obviously commutative, and the \( \tau_k \)-continuity is easily deduced from this. Since \( k \) was arbitrary this proves (i).

(ii) is easy.

(iii) Let \( (e_j)_J \) be an approximate identity with bound \( c \geq 1 \). For \( k \in \mathbb{N} \) let \( I \subseteq \mathbb{N} \) be such that \( I \subseteq c k \). First let us prove

\[
\forall p \in \mathcal{S}_k(I) : \exists p \in \mathcal{S}_k(A) : p \leq p I \leq c k p.
\]

Define \( \bar{p}(a) := \sup \{ p(a x) : x \in I, \| x \| \leq 1 \} \). Then \( \bar{p} \) obviously is a seminorm on \( A \), and for \( a, b \in A, x \in I, \| x \| \leq 1 \) we have

\[
P(abx) = \lim_j p(a p_j b x) \leq \lim_j p(a p_j b) \leq c p(a) \bar{p}(b).
\]

Hence \( \bar{p}(a) \leq c p(a) \bar{p}(b) \) and this implies that \( \bar{p} := c \bar{p} \) is an algebra seminorm on \( A \). It is easily seen that this \( \bar{p} \) satisfies \( (**) \).

Let \( i : \text{Id}(I) \subseteq \text{Id}(A) \) be the inclusion, and let \( p_i \to p \) be any convergent net in \( \mathcal{S}_k(I) \). Given any subnet \( (p_j)_J \) choose \( p_j \) as in \( (**) \), and find a convergent subnet \( (p_{m(j)})_J \) by compactness of \( \mathcal{S}_k(A) \). Then \( x \in I \) we have \( p_{m(j)}(x) \leq \| x \| \leq c p_{m(j)}(x) \), and this yields \( p(x) \leq q(x) \leq c p(x) \). This implies \( \ker(p_{m(j)}) \cap I = \ker(p_{m}) \cap I = I = \ker(p) \). Then we see

\[
\kappa_{k}(p_{m}) \in \mathcal{S}_k(I) \cap I \Rightarrow \ker(p) \cap I \text{ by } (i) \Rightarrow \kappa_{k}(p) \in \kappa_{k}(p_{m}).
\]

Therefore \( \kappa : \text{Id}(I) \to (\text{Id}(A), \tau_{\infty}) \) is continuous for all \( k \), and this implies the \( \tau_{\infty} \)-continuity of \( \kappa \). Together with \( (i) \) this proves the claim since the intersection map \( \kappa \) is obviously the continuous inverse of \( \kappa \).

Without proof I would like to mention the following

**Proposition 7.** Let \( (A_n) \) be a sequence of Banach algebras having approximate identities and let \( A \) be the \( c_{0} \)-sum of the \( A_n \). For \( I \subseteq \text{Id}(A) \) and \( n \in \mathbb{N} \) let \( I(n) \subseteq A_n \) be the \( n \)-th projection of \( I \). Then

\[
(\text{Id}(A), \tau_{\infty}) \to \prod_{n \in \mathbb{N}} (\text{Id}(A_n), \tau_{\infty}), \quad I \mapsto (I(n))_{n \in \mathbb{N}},
\]

is a continuous bijection.

4. Comparison with other topologies

**Proposition 8.** Let \( A \) be a \( C^* \)-algebra. Then \( \tau_k = \tau_k \) for all \( k \in \mathbb{N} \), in particular \( \tau_k = \tau_k \).

**Proof.** We have to prove \( \tau_k \subseteq \tau_k \) for all positive integers \( k \). To this end, let \( p_i \to p \) in \( \mathcal{S}_k \). The claim is

\[
\| a + \ker(p_i) \| \to \| a + \ker(p) \| \text{ for all } a \in A.
\]

Since \( \| a + I \|^2 = \| a^* a + I \| \), for all \( I \subseteq \text{Id}(A) \) we may assume that \( a \) is selfadjoint. By an old theorem of Kaplansky (see [13], Th. I.2.4) we have

\[
\| a + \ker(q) \| \leq q(a) \text{ for all seminorms } q \in \mathcal{S}_k.
\]

Because \( a + \ker(q) \) generates a commutative \( C^* \)-algebra in \( A/\ker(q) \). By the definition of \( S_k \) we have

\[
\| a + \ker(q) \| \leq q(a) \leq k \| a + \ker(q) \| \text{ for all } q \in \mathcal{S}_k, a \in A \text{ selfadjoint}.
\]

Therefore for all \( n \in \mathbb{N} \) we see

\[
\| a + \ker(q) \| \leq q(a) \leq k \| a + \ker(q) \| \leq k \| a + \ker(q) \|^n,
\]

and from this by taking the \( n \)-th root

\[
\| a + \ker(q) \| \leq q(a) \leq k \| a + \ker(q) \|^n \leq k \| a + \ker(q) \|^n \text{ for all selfadjoint } a \in A \text{ and all seminorms } q \in \mathcal{S}_k.
\]

Back to our convergent net \( p_i \to p \). For \( \varepsilon > 0 \) find \( n \in \mathbb{N} \) such that \( \| a - a^\varepsilon \| < \varepsilon \). There is an index \( i_0 \) such that \( |p_i(a^\varepsilon) - p(a^\varepsilon)| < \varepsilon \) for all \( i \geq i_0 \). Then by (1),

\[
\| a + \ker(p_i) \| \to \| a + \ker(p) \| \leq \| a + \ker(p_i) \| \| p_i(a^\varepsilon) - p(a^\varepsilon) \| + |p_i(a^\varepsilon) - p(a^\varepsilon)| \| p_i(a^\varepsilon) - p(a^\varepsilon) \| + |p_i(a^\varepsilon) - p(a^\varepsilon)| \| p_i(a^\varepsilon) - p(a^\varepsilon) \| + |p_i(a^\varepsilon) - p(a^\varepsilon)| \| p_i(a^\varepsilon) - p(a^\varepsilon) \| < 3 \varepsilon
\]

for all \( i \geq i_0 \), and this proves the proposition.

Next we will compare \( \tau_{\infty} \) with the Gelfand topology. Let \( A \) be a Banach algebra. Let \( \mathcal{M} \) be the space of maximal modular ideals with codimension 1 together with the trivial ideal \( A \). Then the Gelfand topology this corresponds to the space of homomorphisms \( A \to \mathcal{C} \), and so carries the relative \( u \)-topology from the dual \( A^* \) which is known as the Gelfand topology. We will show that \( \mathcal{M} \subseteq \text{Id}(A) \) is \( \tau_{\infty} \)-closed and that the relative \( \tau_{\infty} \)-topology coincides with the Gelfand topology, provided \( A \) has a bounded approximate identity \( (e_i)_I \).

**Lemma 9.** Let \( A \) be a Banach algebra and \( \varphi : A \to \mathcal{C} \) a non-zero homomorphism. Then \( \varphi(\cdot) \).

\[
\kappa^{-1}_{k}(I) = \{ t \in |\varphi(\cdot) | : t \leq 1, k \| \varphi(\cdot) \| \}
\]

**Proof.** The inclusion \( \supseteq \) is obvious. If \( p \in \kappa^{-1}_{k}(I) \), then \( p \) induces a norm \( \| \cdot \| \) on \( A/I \) via \( \| a + I \| : = p(a) \). But \( A/I \cong \mathcal{C} \) via \( \| \cdot \| : A/I \to \mathcal{C} \), \( a + I \mapsto \varphi(a) \), hence \( \| \cdot \| \) is an algebra norm on \( A \), and this implies \( \| \cdot \| \) for some \( t \geq 1 \). But then \( p = t \| \varphi(\cdot) \| \), and from \( \kappa^{-1}_{k} \) we conclude \( t \leq k \| \varphi(\cdot) \| \).

Lemma 10. Let $A$ be a Banach algebra, and $p \in S_k(A)$. Then the following assertions are equivalent:

(i) There is a homomorphism $\varphi : A \to \mathbb{C}$ and a real number $t > 0$ such that $p = t|\varphi(\cdot)|$.

(ii) $\exists s > 0 : \forall a, b \in A : p(ab) \geq sp(a)p(b)$.

Proof. The implication from (i) to (ii) is trivial, just let $s = 1/t$. Conversely, assume that (ii) holds. Then $p$ induces a norm $\overline{p}$ on $A/\ker(p)$ with the property $\overline{p}(xy) \geq \overline{p}(x)p(y)$ for all $x, y \in A/\ker(p)$, and this property also holds in the completion $\overline{B}$ of $A/\ker(p)$. From $\overline{p}(x^2) \geq \overline{p}(x)^2$ we get by induction

$$\overline{p}(x^{2^n}) \geq s^{2^n-1}\overline{p}(x)^{2^n},$$

and then by the Beurling formula for the spectral radius $r_{\overline{B}}(x) \geq \overline{p}(x)$. By the theorem from [10] we know that $B$ is commutative. See also [9], p. 345, for this argument. Let $\overline{B}$ be a Banach algebra or $\overline{B} = B_1$, the algebra which emerges from the process of adjoining a unit $e$, otherwise. By Lemma 2 of [10] we may introduce a norm $\overline{q}$ on $B_1$ in such a way that $\overline{p}$ and $q|_B$ are equivalent (and $r_{\overline{B}}(x) \geq s^\alpha q(x)$ for all $x \in B$).

In the case where $B$ does not have a unit we have

$$q(x + \lambda e) := \sup\{\overline{p}(xz + \lambda z) : z \in B, \overline{p}(z) \leq 1\} \quad (x \in A, \lambda \in \mathbb{C}).$$

For $z_1, z_2 \in B$ with $||z_1||, ||z_2|| \leq 1$ we get

$$q((x + \lambda e)(y + \mu e)) \geq \overline{p}((x + \lambda e)(y + \mu e)z_1z_2) = \overline{p}((x + \lambda e)z_1(y + \mu e)z_2) \geq \overline{p}((x + \lambda e)z_1)(y + \mu e)z_2,$$

and so

$$q((x + \lambda e)(y + \mu e)) \geq sq(x + \lambda e)q(y + \mu e)$$

for all $x + \lambda e, y + \mu e \in B$.

Hence we may assume $q(x^2) \geq q(x)q(y)$ for all $x, y \in B$ in either case.

We now follow the argument of Theorem 10.19 of [12] to conclude that $\overline{B}$ is isomorphic to the complex numbers. We have $q(1) = q(x^{-1}) \geq sq(x)q(x^{-1})$, and hence

$$q(x^{-1}) \leq \frac{q(1)}{sq(x)}$$

for all invertible elements $x \in B$.

If $(x_n)_n$ is a sequence of invertibles converging to $x \in B \setminus \{0\}$, then

$$q(x_n^{-1} - x_m^{-1}) = q(x_n^{-1} - x_m)q(x_m^{-1}) \leq q(x_n^{-1})q(x_m^{-1}) \leq \frac{1}{q(1)^2} \frac{q(x_n - x_m)^n}{s^2q(x_n)q(x_m)} \to 0.$$

Hence the sequence $(x_n^{-1})_n$ converges to an element $y \in B$ which is easily seen to be the inverse of $x$.

So the invertible elements of $\overline{B}$ are open and closed in $\overline{B} \setminus \{0\}$. Since the latter set clearly is connected we conclude that $\overline{B}$ is a division algebra, hence $\overline{B} \cong \mathbb{C}$ by Mazur's theorem.

Of course this implies $A/\ker(p) \cong \mathbb{C}$. Therefore $\ker(p) = \ker(\varphi)$ for a homomorphism $\varphi : A \to \mathbb{C}$, and since a norm on $\mathbb{C}$ necessarily is a multiple of the absolute value, $p$ must be of the form $t|\varphi(\cdot)|$ for some $t > 0$.

Theorem 11. Let $A$ be a Banach algebra with a bounded approximate identity $(e_k)$. Then the Gelfand space $M$ is $\tau_0$-closed, and the Gelfand topology coincides with all $\tau_k$, hence with $\tau_0$.

Proof. Let $H$ be the set of homomorphisms $A \to \mathbb{C}$. Then the norms of $\varphi \in H \setminus \{0\}$ stay away from zero. For this, let $\beta$ be a bound for $(e_k)$. Since $\varphi(e_k) \to 1$ for $0 \neq \varphi \in H$, we easily see that $||\varphi|| \geq 1/\beta$.

By a combination of the lemmas above we have

$$\kappa_k^{-1}(M) = \left\{ p \in S_k : \forall a, b \in A : p(ab) \geq \frac{1}{k\beta} p(a)p(b) \right\}.$$

In particular, $\kappa_k^{-1}(M)$ is closed and this means that $M$ is $\tau_k$-closed for all $k$, hence $\tau_0$-closed.

But then the $\tau_k$-topology on $M$ coincides with the quotient topology of the map

$$\kappa_k |\kappa_k^{-1}(M) : \kappa_k^{-1}(M) \to M.$$

Let $p \to p_\varphi$ in $\kappa_k^{-1}(M)$. We have $p_{\varphi} = t_1|\varphi(\cdot)|$ and $p = t|\varphi(\cdot)|$ for $\varphi, \varphi_1 \in H$ and $t, t_1 \in [1, \beta]$ by Lemma 9. Given any subnet $(p_{\varphi_1})$ we may find a finer subnet $(p_{\varphi_2})$ such that $t_2 \to s$ in $[1, \beta]$ and $\varphi_{\varphi_2} \to \psi$ in $(\mathcal{H}, w^*)$. This implies $t|\varphi_\varphi(\cdot)| = p = \lim p_{\varphi} = \lim t_1|\varphi(\cdot)| = s|\psi(\cdot)|$.

So $\varphi$ and $\psi$ are proportional homomorphisms, hence equal. This implies $\kappa_k(p_{\varphi_1}) = \ker(\varphi_1) \to \ker(\varphi) = \kappa_k(p)$ in the Gelfand topology. Therefore the map (1) is continuous if $M$ carries the Gelfand topology, and this in turn means that $\tau_k$ is finer than the Gelfand topology $\tau_0$ for all $k$, and then $\tau_0 \supseteq \tau_0$.

Conversely, if $\varphi \to \varphi$ in $(\mathcal{H}, w^*)$ then $|\varphi_\varphi(\cdot)| \to |\varphi(\cdot)|$ in $S_1$, and this implies $\ker(\varphi_\varphi) \to \ker(\varphi)$ with respect to $\tau_1$. But this yields $\tau_0 \supseteq \tau_1$ on $M$. This finally proves the theorem.

5. Topological properties. When is $(\text{Id}(A), \tau_{\text{top}})$ a $T_\theta$-space, i.e. when are points closed? This of course is the case iff all topologies $\tau_k$ are $T_\theta$, and this is the case iff

$$\forall I \in \text{Id}(A) : \quad \{ p \in S_k : \ker(p) = I \}$$

is closed in $S_k$. 

Say that a Banach algebra \((A, \| \cdot \|)\) has the norm property iff any pointwise limit of a uniformly \(\| \cdot \|\)-bounded net of norms on \(A\) is again a norm. Since the seminorms \(p \in S_\infty(A)\) with \(\ker(p) = I\) correspond bijectively to the norms in \(S_\infty(A/I)\) (where \(A/I\) carries the quotient norm), a simple reformulation of the above consideration yields:

**Proposition 12.** A point \(I \in \Id(A)\) is \(\tau_\infty\)-closed iff \(A/I\) has the norm property.

This property is somehow related to minimal norm topologies (see [8]) as will be shown by the following results.

**Proposition 13.** Let \(P\) be a primitive ideal of finite codimension. Then \(\{P\}\) is \(\tau_\infty\)-closed.

**Proof.** We have \(A/P \cong M_m(C)\) for some \(m \in \mathbb{N}\) and this algebra has a unit 1. Let \(p_i \to p\) in \(S_\infty(A/P)\), where each \(p_i\) is a norm. Since \(p(1) = \lim_i p_i(1) \geq 1\) the ideal \(\ker(p)\) must be proper, hence \(\{0\}\).

**Proposition 14.** Let \((A, \| \cdot \|)\) be a Banach *-algebra with a minimal norm topology which stems from a pre-C*-norm on \(A\). Then \(A\) has the norm property.

**Proof.** Let \(p_i \to p\) be a convergent net of norms \(p_i \in S_\infty\). Since \(\| \cdot \|_\infty\) yields the minimal norm topology there are positive constants \(c_i\) satisfying \(\| \cdot \|_{\infty} \leq c_i p_i\) (where \(\| \cdot \|_{\infty}\) is the pre-C*-norm). This implies

\[
\|a\|^2 = \|a^*a\|_{\infty} \leq c_i^{1/n} p_i((a^*a)^{1/n}) \leq c_i^{1/n} p_i(a^*a)^{1/n} \leq c_i^{1/n} p_i(a^*a) \leq c_i^{1/n} p_i(a^*) p_i(a)
\]

for all positive integers \(n\), hence

\[
\|a\|^2 \leq p_i(a^*a)p_i(a) \to p(a^*)p(a).
\]

So \(p(0) = 0\) implies \(a = 0\), which is the desired result.

So if e.g. \(H\) is a Hilbert space and \(A \subset L(H)\) is a *-subalgebra which contains all finite-dimensional operators, then \(A\) (with any Banach algebra norm) has the norm property, since by [8], Th. 3.3, \(A\) satisfies the assumptions of the above proposition. The same conclusion holds for all \(C^*\)-algebras by [8], Th. 3.6.

**Proposition 15.** Let \(A\) be an annihilator algebra. Then \(\Rad(A)\) is a \(\tau_\infty\)-closed point in \(\Id(A)\).

**Proof.** By [7], §32, Prop. 15, \(A/\Rad(A)\) is an annihilator algebra, hence we may reduce to the semisimple case. If \(p_i \to p\) is a convergent net of norms \(p_i \in S_\infty\) then by [7], §32, Lemma 23, we have for the spectral radius

\[
r(a) = \lim_n p_i(a^n)^{1/n} \leq p_i(a) \to p(a).
\]

Therefore \(\ker(p)\) consists entirely of quasiregular elements, hence \(\ker(p) \subset \Rad(A) = \{0\}\), which proves the assertion.

Next one may ask when \((\Id(A), \tau_\infty)\) is a Hausdorff space. This of course is the case if \(A\) is a \(C^*\)-algebra since then we have \(\tau_\infty = \tau_1\). I would like to mention without proof that the Banach algebras \(L^p(G)\), \(1 \leq p \leq \infty\), with componentwise multiplication and the convolution algebras \(L^p(G)\), \(G\) a compact group, \(1 < p < \infty\), have \(\tau_\infty\) Hausdorff. It can be shown (using [11], Th. 1.5) that each \(\tau_\infty\)-closed ideal of \(L^p(G)\) is the intersection of the maximal ideals containing it. Therefore \(\Id(L^p(G))\) corresponds bijectively to the subgroups of the dual group \(\Gamma\), and hence to \(\{0, 1\}\) \(\Gamma\), where each subset of \(\Gamma\) is identified with its characteristic function \(\Gamma \to \{0, 1\}\). Then the \(\tau_\infty\)-topology on the space \(\Id(L^p(G))\) corresponds to the product topology on \(\{0, 1\}^\Gamma\) which clearly is a Hausdorff space. The details are left to the reader.

**Proposition 16.** Let \(A\) be a Banach algebra. Then the following are equivalent:

(i) \((\Id(A), \tau_\infty)\) is a Hausdorff space.

(ii) If \(I_i \to I\) in \((\Id(A), \tau_\infty)\), then for all \(x \in A\)

\[
\lim_i \sup \|x + I_i\| \leq \|x + I\|, \quad \lim_i \inf \|x + I_i\| = 0 \Rightarrow x \in I.
\]

**Proof.** (i)⇒(ii). Let \(q_i\) be the quotient seminorm of \(I_i\). Assume that \(\lim sup_i q_i(x) \geq \|x + I\|\). Then we can find a subset \((q_j)\) such that \(\lim_j q_j(x) > \|x + I\|\) and \(q_j \to p \in \Delta_1\), since \(\Delta_1\) is compact. Since \(I_i = \ker(q_j) \to \ker(p)\) with respect to \(\tau_1\), hence with respect to \(\tau_\infty\), we have \(I = \ker(p)\) because \(\tau_\infty\) is Hausdorff. But \(p \in \Delta_1\) and this implies

\[
\|x + I\| = \|x + \ker(p)\| \geq p(x) = \lim_j q_j(x) > \|x + I\|.
\]

This contradiction shows \(\lim sup_i \|x + I_i\| \leq \|x + I\|\).

Now consider the situation \(\lim_i q_i(x) = 0\). Again we may find a subset \((q_j)\) such that \(\lim_j q_j(x) = 0\) and \(q_j \to p \in \Delta_1\). Since \(\tau_\infty\) is Hausdorff we have \(I = \ker(p)\), and this implies \(p(x) = \lim_i q_i(x) = 0\), i.e. \(x \in I\).

(ii)⇒(i). Let \(I_i \to I\) and \(I_i \to J\) in \((\Id(A), \tau_\infty)\). We have to prove \(I = J\). For this, let \(x \in I\). Then by the assumption in (ii),

\[
\lim_i \inf \|x + I_i\| \leq \lim_i \sup \|x + I_i\| \leq \|x + I\| = 0,
\]

and this implies \(x \in J\). Hence \(I \subset J\) and then \(I = J\).

6. **Another topology.** We are in need of still another topology in order to say more about the case of a separable Banach algebra. For a compact set \(K \subset A\) define

\[
U(K) := \{I \in \Id(A) : I \cap K = \emptyset\}.
\]
Obviously we have \( U(K_1 \cap K_2) = U(K_1) \cup U(K_2) \), hence the sets \( U(K) \), \( K \subseteq A \) compact, form a base for a topology \( \tau_0 \) on \( \text{Id}(A) \).

**Lemma 17.** For all Banach algebras we have \( \tau_1 \subseteq \tau_0 \subseteq \tau \).

**Proof.** Since \( U(\{x\}) = \{y \in \text{Id}(A) : x \notin I\} \) we have \( \tau_1 \subseteq \tau_0 \). Let \( p \rightarrow p \) in \( S_0 \), and let \( \text{ker}(p) \subseteq U(K) \) for some compact set \( K \subseteq A \). Then \( \text{inf}_{x \in K} p(x) > 0 \). Since \( p \rightarrow p \) pointwise on compact sets, there is an index \( i_0 \) such that \( \text{inf}_{x \in K} p_i(x) > 0 \) for all \( i \geq i_0 \). This means that \( \kappa_{\xi} : S_0 \rightarrow \text{Id}(A), \tau_0 \) is continuous. By definition of the quotient topology \( \tau_0 \) we deduce \( \tau_0 \subseteq \tau_0 \). This holds for all \( k \), so we have \( \tau_0 \subseteq \tau_0 \).

**Lemma 18.** Let \( (I_i) \) be a net in \( \text{Id}(A) \) and \( I \in \text{Id}(A) \) such that
\[
\forall x \in A \setminus I : \liminf_i \|x + I_i\| > 0.
\]
Then \( I \rightarrow I \) with respect to \( \tau_0 \).

**Proof.** Let \( K \subseteq A \) be compact and \( I \cap K = \emptyset \). For each \( x \in K \) we then have \( \tau_x := \liminf_i \|x + I_i\| > 0 \). Hence there is an index \( i(x) \) such that \( \|x + I_i\| > \frac{3}{4} \tau_x \) for all \( i \geq i(x) \). Let \( B(x, r) \subseteq A \) be the open ball around \( x \) with radius \( r \). Then by compactness of \( K \) there are finitely many points \( x_1, \ldots, x_n \in K \) such that \( K \subseteq B(x_1, \frac{3}{4} \tau_{x_1}) \cup \ldots \cup B(x_n, \frac{3}{4} \tau_{x_n}) \). Let \( i_0 \) be an index larger than \( i(x_1), \ldots, i(x_n) \). Then for any \( y \in K \) there is a \( j \) such that \( y \in B(x_j, \frac{3}{4} \tau_{x_j}) \) and then for \( i \geq i_0 \) we have
\[
\|y + I_i\| \geq \|x_j + I_i\| - \|x_j - y\| \geq \frac{3}{4} \tau_{x_j} - \frac{3}{4} \tau_{x_j} = \frac{3}{4} \tau_{x_j} > 0,
\]
and hence we have shown \( I \cap K = \emptyset \) for \( i \geq i_0 \).

**Corollary 19.** If \( A \) is a C*-algebra then \( \tau_0 = \tau_0 \).

**Proof.** If \( I \rightarrow I \) with respect to \( \tau_0 \) then we know that \( \|x + I_i\| \leq \liminf_i \|x + I_i\| \) for all \( x \in A \), and the result follows.

In general we have \( \tau_0 \neq \tau_\). For instance, in Example 1 it can be shown that \( I_B \) is not in the \( \tau_0 \)-closure of \( \{I(x) : x \in 2^D \setminus D\} \), hence we must have \( \tau_0 \neq \tau_\) in this example.

7. Separable Banach algebras. In this section let \( A \) be a separable Banach algebra. Then we know that all spaces \( S_0 \) are metrizable compact spaces by Lemma 2.

**Theorem 20.** Let \( W \subset \text{Id}(A) \) be any subspace. Then \( W \) is a separable Lindelöf space in the \( \tau_0 \)-topology, hence the same holds true for all weaker topologies, for example \( \tau_1 \), \( \tau_0 \) or \( \tau_\).

**Proof.** \( \kappa^{-1}_\xi(W) \subset S_0 \) is a second countable and metricizable space, hence it is a separable Lindelöf space, and so is the continuous image \( W = \kappa_\xi(\kappa^{-1}_\xi(W)) \).

This theorem extends the results of [14], Cor. 4.5, to a large extent. We are interested in other properties of \( \text{Id}(A), \tau_\). I do not know whether this space must be first (or even better second) countable, but the following holds:

**Proposition 21.** Sequentially closed sets in \( \text{Id}(A), \tau_\) are already closed, and sequentially continuous maps on \( \text{Id}(A), \tau_\) are already continuous.

**Proof.** Let \( W \subset \text{Id}(A) \) be sequentially closed. Let \( p \in \text{ker}(\kappa_\xi(W)) \subset S_0 \). Since \( S_0 \) is metrizable, there is a sequence \( (p_n) \) in \( \kappa_\xi^{-1}(W) \) such that \( p_n \rightarrow p \), and this implies \( W \supseteq \ker(p_n) \rightarrow \ker(p) \). Since \( W \) is sequentially closed we have \( \ker(p) \subseteq W \), hence \( p \in \kappa_\xi(W) \). So we have proved that \( \kappa_\xi^{-1}(W) \) is closed in \( S_0 \) and this proves that \( W \) is \( \tau_\)-closed for all \( k \in \mathbb{N} \), hence \( \tau_\)-closed. The second assertion follows from the first.

**Proposition 22.** For an sequence \( (I_n) \) and an ideal \( I \) in \( \text{Id}(A) \) the following are equivalent:

(i) \( I_n \rightarrow I \) with respect to \( \tau_\).

(ii) For all \( x \in A \setminus I \) we have \( \liminf_n \|x + I_n\| > 0 \).

**Proof.** (i)\(\Rightarrow\)(ii) is clear from Lemma 18. Conversely, assume that (ii) holds and let \( x \in A \setminus I \). We have to show \( \liminf_n \|x + I_n\| > 0 \). Assume the contrary. Then there is a sequence \( (I_n) \) such that \( \liminf_n \|x + I_n\| = 0 \), hence there are \( x_m \in I_n \) such that \( \|x - x_m\| \rightarrow 0 \). Since \( x \notin I \) and \( I \) is closed, we have \( x_m \notin I \) for large \( m \). Then \( K := \{x\} \cup \{x_m : m \geq m_0\} \subset A \) is compact and disjoint from \( I \). So we have \( I_n \cap K = \emptyset \) for large \( m \), but this contradicts \( x_m \in K \cap I_n \) for \( m \geq m_0 \).

**Theorem 23.** For a separable Banach algebra \( A \) the following assertions are equivalent:

(i) \( \text{Id}(A), \tau_\) is a Hausdorff space.

(ii) \( \text{Id}(A), \tau_\) is a metrizable space.

(iii) If the sequence \( (I_n) \) is \( \tau_\)-convergent to \( I \), then \( \lim sup_n \|x + I_n\| \leq \|x - I\| \) for all \( x \in A \).

(iv) If \( p_n \rightarrow p \) and \( \tau_\rightarrow \tau \) in some \( S_k \) such that \( \ker(p_n) = \ker(p) \) for all \( n \in \mathbb{N} \) then \( \ker(p) = \ker(\tau) \).

**Proof.** (i)\(\Rightarrow\)(ii). Since comparable, compact Hausdorff topologies are equal we have \( \tau = \tau_0 \) for all \( k \). Then \( \text{Id}(A), \tau_0 = \kappa_{\xi}(S_0) \) is a Suslin space and (ii) follows because compact Suslin spaces are metrizable.

The converse (ii)\(\Rightarrow\)(i) is trivial.
Again by compactness of $S_1$ we may assume $r_n \to r$ in $S_1$. Let $(V_i)$ be a countable open neighbourhood base of $r$ in $S_1$. There are $n_1 < n_2 < \ldots$ satisfying $r_{n_i} \in V_i$. Hence there are $n_2 \in \mathbb{N}$ such that $q_{n_2}^m \in V_i$. This implies $q_{n_2}^m \to r$ as $m \to \infty$, and therefore $\ker(r) \in \overline{W}$.

Since $I_n^m \to I_n$ and $I_n^m \to \ker(r)$ we have $I_n = \ker(r)$ by (i). Since $I_n \to I$ and $\ker(r) \to \ker(r)$ we have $I = \ker(r)$, again by (i). Therefore $I \in \overline{W}$, and this proves (**) again.

Now let $I_i \to I$ be a $\tau$-convergent net in $\operatorname{Id}(A)$. We must show that $I$ is the unique limit. Restricting to a subnet we may assume $I_i \to p$ in $S_1$ and it is enough to prove $I = \ker(p)$.

Let $(W_n)_n$ be a countable closed neighbourhood base of $p$. There are indices $i_k$ such that $q_{i_k}^m \in W_n$ for all $i \geq i_k$. Then $I$ is in the closure of $\{I_i : i \geq i_k\}$ and so by (**) there are sequences $(I_n^m)_{n_i} \in \{I_i : i \geq i_k\}$ converging to $I$, and additionally we may assume that the quotient seminorms $q_{n_i}^m$ of $I_n^m$ converge to some seminorm $p_n$ in $S_1$. Then we have $I_n^m \to I_n^m \to \ker(p_n)$, and by (i) we conclude $I = \ker(p_n)$. But $p_n = \lim m \in \overline{W} = W_n$ for all $n$, implying $I_n \to p$, hence $I = \ker(p_n) = \ker(p)$. By (i) we now may conclude $I = \ker(p)$.

**LEMMA 24.** Let $(\operatorname{Id}(A), \tau_\infty)$ be a Hausdorff space. Then the intersection $\cap : \operatorname{Id}(A) \times \operatorname{Id}(A) \to \operatorname{Id}(A)$ is $\tau_\infty$-continuous.

**Proof.** Let $I_n \to I$ and $J_n \to J$ be $\tau_\infty$-convergent sequences. If $q_n$ resp. $q_n'$ are the quotient seminorms of $I_n$ resp. $J_n$ then each subsequence $(I_{n_i})_{n_i}$ contains another subsequence $(I_{n_i} \cap J_{n_i})_{n_i}$ such that the sequences $(q_{n_i})_i$ and $(q_{n_i}')_i$ converge to $I$ in $S_1$ and $J$ respectively. Since $\tau_\infty$ is assumed to be Hausdorff we have $I = \ker(p)$ and $J = \ker(p')$. We have $\max\{q_n, q_n'\} \to \max(p, p')$ in $S_1$, and this implies $I_n \cap J_n \to \ker(p) \cap \ker(p') = I \cap J$.

Let us consider the space $\operatorname{Min-Primal}(A)$ of minimal primal ideals. This is always a Hausdorff space with respect to $\tau_\infty$. To see this, let $(I_i)_i$ be a net in $\operatorname{Min-Primal}(A)$ which converges to $I$ and $J$ in $\operatorname{Min-Primal}(A)$. Then we also have $I_i \to I \cap J$, in particular $I \cap J$ is primal. Since $I$ and $J$ are minimal primal we conclude $I = I \cap J = J$.

In the case of a $C^*$-algebra we know by [1, Cor. 4.3(a)], that $\tau_\infty$ and $\tau_\infty$ coincide on $\operatorname{Min-Primal}(A)$. A possible generalization to Banach algebras would be that $\tau_\infty$ and $\tau_\infty$ coincide on $\operatorname{Min-Primal}(A)$, but this is not the case as can be seen by Example 1; there $I_0$ is in the $\tau_\infty$-closure of $\operatorname{Min-Primal}(A) \setminus \{I_0\}$ while $I_0$ is an $\tau_\infty$-isolated point in $\operatorname{Min-Primal}(A)$ by Theorem 11. For $C^*$-algebras we have $\tau_\infty = \tau_\infty$ by Proposition 19, and one could replace $\tau_\infty$ by $\tau_0$ in the above considerations. Following this idea one gets the following

**THEOREM 25.** For a separable Banach algebra the following assertions hold:
For $E \subset \text{Id}(A)$ define $\overline{E} := \{I \in \text{Id}(A) : I \supset J \text{ for some } J \in E\}$. Then $(\overline{E^k})^\omega = (\overline{E^k})^{\omega^k}$ for all $k$ and this is the $\tau_\omega$-sequential closure of $E$ (i.e. the smallest $\tau_\omega$-sequentially closed set containing $E$).

(ii) The $\tau_\omega$-sequentially closed sets are the closed sets of a topology $\tau_\omega$.

(iii) $\tau_\omega$, $\tau_\omega$ and all the $\tau_k$ coincide when restricted to $\text{Min-Primal}(A)$ and these topologies make $\text{Min-Primal}(A)$ a Suslin space.

(iv) If $\tau_\omega(A)$ is Hausdorff, then $(\text{Min-Primal}(A), \tau_\omega)$ is a Polish space, i.e. a $G_\delta$-subset of $(\text{Id}(A), \tau_\omega)$.

Proof. (i) Let $E_0$ be the $\tau_\omega$-sequential closure of $E$. Because $\tau_\omega \subset \tau_k$ and by Proposition 21, $\overline{E^k} \subset \overline{E_\omega} \subset E_0$, hence $(\overline{E^k})^\omega \subset (\overline{E_\omega})^\omega \subset E_0$. Therefore it is enough to show that $(\overline{E^k})^\omega$ is $\tau_\omega$-sequentially closed. To this end, let $(I_n)$ be a sequence in $(\overline{E^k})^\omega$ which is $\tau_\omega$-convergent to some $I \in \text{Id}(A)$. Then we have $J_n \subset E^k$ such that $J_n \subset I_n$ and considering subsequences we may assume $q_{I_n} \to p$ in $S_k$. From Proposition 22 we conclude $p(x) = \lim_n q_{I_n}(x) = \lim q_{I_k}(x) > 0$ if $x \in A \setminus I$, and this means $\ker(p) \subset I$. Since $\ker(q_{I_n}) \to \ker(p)$ with respect to $\tau_\omega$ we have $\ker(p) \subset \overline{E^k}$.

Hence $I \in (\overline{E^k})^\omega$, and this proves (i).

(ii) Since obviously $(E \cup F)^\omega = \overline{E \cup F}$ for subsets $E, F \subset \text{Id}(A)$ it is easy to see by (i) that the operation of taking the $\tau_\omega$-sequential closure satisfies the Kuratowski closure axioms.

(iii) $\tau_\omega$-sequentially closed sets are clearly $\tau_\omega$-closed by Proposition 21 and so $\tau_\omega$-closed for all $k$. Conversely, let $E \subset \text{Min-Primal}(A)$ be relatively $\tau_\omega$-closed. Then $\overline{E \cap \text{Min-Primal}(A)} = E$ and this clearly implies $(\overline{E^k})^\omega \cap \text{Min-Primal}(A) = E$ because $\overline{E^k} \subset \text{Primal}(A)$. By (i), $E$ is relatively $\tau_\omega$-closed. Thus all topologies in question coincide on $\text{Min-Primal}(A)$.

Let $H := \kappa_1^{-1}(\text{Primal}(A))$, which is a closed subset of $S_k$, and let $\max(H)$ be the set of maximal elements in $H$. Clearly $H = \{q : I \in \text{Min-Primal}(A)\}$ and so $\text{Min-Primal}(A)$, which is a Hausdorff space (this is even true for the relative $\tau_\omega$-topology), is a continuous image of $\max(H)$. Therefore it is sufficient to show that $\max(H)$ is a Polish space, i.e. a $G_\delta$-set in $S_k$. Define

$$H_1 := \{(q, p) \in H^2 : p \le q\}, \quad H_2 := \{(q, p) \in H^2 : p \ge q\}.$$

We have $p \in H \setminus \max(H)$ iff there is a $q \in H$ such that $p \le q$ and $p \neq q$ iff $p \in \text{pr}_1(H^2 \cap H_1 \setminus H_2)$, where $\text{pr}_1$ denotes the projection onto the first coordinate. Since open sets in $S_k$ are $F_\sigma$-sets, $H^2 \cap H_1 \setminus H_2$ is $\sigma$-compact, and so is the continuous image $\text{pr}_1(H^2 \cap H_1 \setminus H_2)$. Therefore $\max(H) = H \setminus \text{pr}_1(H^2 \cap H_1 \setminus H_2)$ is a $G_\delta$-set. This proves (iii).

(iv) Now let $(\text{Id}(A), \tau_\omega)$ be Hausdorff, hence metrizable by Theorem 23. Let $X := \text{Primal}(A)$, which is $\tau_\omega$-closed. Then

$$G_1 := \{(I, J) \in X^2 : I \subset J\}, \quad G_2 := \{(I, J) \in X^2 : I \supset J\}$$

are $\tau_\omega$-closed by Lemma 24. The same arguments as in (iii) now yield that

$$\text{Min-Primal}(A) = X \setminus \text{pr}_1(X^2 \cap G_2 \setminus G_1)$$

is a $G_\delta$-set in $X$. This finishes the proof of the theorem.

Part (iii) of the above theorem obviously generalizes Corollary 4.3(a) of [1], and part (iv) is a generalization of Corollary 5.2 of [5]. For all this it would be an advantage to know whether $\tau_\omega$ and/or $\tau_{k}$ satisfy the first (or even better the second) countability axiom if the Banach algebra is separable, but this must remain an open question here.

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