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Topologies on the space of ideals of a Banach algebra

by

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Abstract. Some topologies on the space $\mathrm{Id}(A)$ of two-sided and closed ideals of a Banach algebra are introduced and investigated. One of the topologies, namely τ_∞, coincides with the so-called strong topology if A is a C^*-algebra. We prove that for a separable Banach algebra τ_∞ coincides with a weaker topology when restricted to the space Min-Primal(A) of minimal closed primal ideals and that Min-Primal(A) is a Polish space if τ_∞ is Hausdorff; this generalizes results from [1] and [5]. All subspaces of $\mathrm{Id}(A)$ with the relative hull kernel topology turn out to be separable Lindelöf spaces if A is separable, which improves results from [14].

1. Introduction. In [14] D. W. B. Somerset has started the investigation of the space Min-Primal(A) of minimal primal closed ideals of a general Banach algebra. He proved, among other things, that if A is a separable Banach algebra, then the space Prime(A) of closed prime ideals with the hull kernel topology (or weak topology τ_{w}) is separable, and if additionally A is topologically semiprimal, then Min-Primal(A) is also separable. We will prove the much stronger result that all subspaces of the space $\mathrm{Id}(A)$ of closed two-sided ideals of A are separable Lindelöf spaces if A is separable. On page 50 of [14] the author left open the question whether Min-Primal(A) is second countable if A is separable. We will give an example of a unital separable subalgebra A of a commutative C^*-algebra such that Min-Primal(A) is not even first countable.

The methods used here are rather different from those of [14]. Here we try to generalize the idea of the strong topology τ_{s} in $\mathrm{Id}(A)$ (see [1] for this or next paragraph). If A is a C^*-algebra, then this topology makes $\mathrm{Id}(A)$ a compact Hausdorff space. This useful topology has been investigated and applied in the theory of C^*-algebras (see e.g. [1]–[6]).

If $(A, \|\cdot\|)$ is a Banach algebra, then τ_{s} is by definition the weak topology of all maps

$$\mathrm{Id}(A) \to \mathbb{R}_0^+, \quad I \mapsto \|x + I\|, \quad x \in A.$$

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For a general Banach algebra τ_s need not be compact, and obviously τ_s depends on the special norm on A. Indeed, it is easy to find equivalent algebra norms on the C^*-algebra of convergent sequences such that the τ_s-topology defined by this new norm is not compact. We will define another topology τ_∞ on $\mathrm{Id}(A)$ with the following properties:

(i) $(\mathrm{Id}(A), \tau_\infty)$ is compact (not Hausdorff in general).

(ii) $\tau_w \subset \tau_\infty \subset \tau_s$.

(iii) τ_∞ only depends on the norm topology of A, not on the special norm.

(iv) $\tau_\infty = \tau_s$ if A is a C^*-algebra.

(v) If A is a commutative Banach algebra with a bounded approximate identity then on the Gelfand space τ_∞ coincides with the Gelfand topology.

This topology can be used to answer the above mentioned questions. Further properties and examples are given in the following sections.

2. An example

EXAMPLE 1. Let $\mathbb{D}$ be the closed unit disc in the plane, $2\mathbb{D}$ the disc with radius 2. Let $A(\mathbb{D})$ be the disc algebra, i.e. the Banach algebra of continuous functions $\mathbb{D} \to \mathbb{C}$ that are holomorphic in the interior of $\mathbb{D}$. Let $\mathcal{C}(2\mathbb{D})$ be the commutative C^*-algebra of continuous functions on $2\mathbb{D}$. Define

$$A := \{f \in \mathcal{C}(2\mathbb{D}) : f|\mathbb{D} \in A(\mathbb{D})\}.$$

For $M \subset 2\mathbb{D}$ let I_M be the ideal of functions in A vanishing on M. Then it is not difficult to prove that

$$\text{Min-Primal}(A) = \{I_{\{z\}} : z \in 2\mathbb{D} \setminus \mathbb{D}\} \cup \{I_{\mathbb{D}}\}.$$

ASSERTION. *$I_{\mathbb{D}}$ is in the τ_w-closure of $M_0 := \{I_{\{z\}} : z \in 2\mathbb{D} \setminus \mathbb{D}\}$ but no sequence in M_0 τ_w-converges to $I_{\mathbb{D}}$.*

Proof. Let U be an open τ_w-neighbourhood of $I_{\mathbb{D}}$. Then U contains a neighbourhood of the form

$$V := \{I \in \text{Min-Primal}(A) : f_1 \notin I, \ldots, f_n \notin I\}, \quad f_1, \ldots, f_n \in A,\ n \in \mathbb{N}.$$

Since $I_{\mathbb{D}} \in V$ we have $f_j|\mathbb{D} \neq 0$. Since $A(\mathbb{D})$ is an integral domain we have $f_1 \ldots f_n \neq 0$, and by the maximum modulus principle there is a point $t \in \partial\mathbb{D}$ such that $f_1(t) \ldots f_n(t) \neq 0$. By the continuity of the f_j we can conclude that there is a point $s \in 2\mathbb{D} \setminus \mathbb{D}$ such that $f_1(s) \ldots f_n(s) \neq 0$, and this means $I_{\{s\}} \in V \subset U$. Hence $I_{\mathbb{D}}$ is in the τ_w-closure of M_0.

Now assume that there were a sequence $(t_n)_n$ in $2\mathbb{D} \setminus \mathbb{D}$ such that $I_{\{t_n\}}$ converges to $I_{\mathbb{D}}$. Since $2\mathbb{D}$ is compact we may assume that $t_n \to t \in 2\mathbb{D}$. Then $F := \mathbb{D} \cup \{t_n : n \in \mathbb{N}\} \cup \{t\}$ is a closed subset of $2\mathbb{D}$. Define

$$f : F \to \mathbb{C}, \quad f(z) := \begin{cases} z - t & \text{if } z \in \mathbb{D}, \\ 0 & \text{if } z \in F \setminus \mathbb{D}. \end{cases}$$

This clearly is continuous and by the Tietze extension theorem it can be extended to an element $g \in \mathcal{C}(2\mathbb{D})$, and obviously $g \in A$. Since $g \notin I_{\mathbb{D}}$ we must have $g \notin I_{t_n}$ for large n by the assumed τ_w-convergence, but this is not the case. This contradiction finishes the proof of the assertion. ∎

Hence $I_{\mathbb{D}} \in$ Min-Primal(A) cannot have a countable neighbourhood base, and so Min-Primal(A) is not a first countable space in the relative τ_w-topology. This answers a question on page 50 of [14] in the negative.

3. Construction of the topology and simple properties. Let $(A, \|\cdot\|)$ be a Banach algebra. For $k \in \mathbb{N}$ let $\mathcal{S}_k(A, \|\cdot\|)$ be the set of all algebra seminorms bounded by k, i.e. the set of seminorms $p : A \to \mathbb{R}_0^+$ such that $p(ab) \le p(a)p(b)$ and $p(a) \le k\|a\|$ for all $a, b \in A$. Write only $\mathcal{S}_k$ if $(A, \|\cdot\|)$ is clear. We have in the obvious manner

$$\mathcal{S}_k(A, \|\cdot\|) \subset \prod_{a \in A} [0, k\|a\|].$$

Since the conditions for a real-valued function to be an element of $\mathcal{S}_k$ are pointwise conditions, $\mathcal{S}_k$ is a closed, hence compact subspace of the product space, i.e. $\mathcal{S}_k$ is compact with respect to the topology of pointwise convergence.

LEMMA 2. *If A is separable, then $\mathcal{S}_k$ is metrizable.*

Proof. If $(a_n)_n$ is a dense sequence in A then the injection

$$\mathcal{S}_k \hookrightarrow \prod_{n \in \mathbb{N}} [0, k\|a_n\|]$$

defines the same topology on $\mathcal{S}_k$. Indeed, if $(p_i)_i$ is a net in $\mathcal{S}_k$, $p \in \mathcal{S}_k$, and if $p_i(a_n) \to p(a_n)$ for all n, then for $a \in A$,

$$\begin{aligned} |p_i(a) - p(a)| &\le p_i(a - a_n) + |p_i(a_n) - p(a_n)| + p(a_n - a) \\ &\le 2k\|a - a_n\| + |p_i(a_n) - p(a_n)|, \end{aligned}$$

and this is small for large i if n is chosen appropriately. ∎

Now define

$$\kappa_k : \mathcal{S}_k(A, \|\cdot\|) \to \mathrm{Id}(A), \quad p \mapsto \ker(p).$$

This map is surjective since if $I \in \mathrm{Id}(A)$ then the corresponding quotient seminorm

$$q_I(a) := \|a + I\|, \quad a \in A,$$

obviously is in $\mathcal{S}_1 \subset \mathcal{S}_k$. Let $\tau_k(A, \|\cdot\|)$ be the quotient topology of this map on $\mathrm{Id}(A)$, and finally $\tau_\infty(A, \|\cdot\|) := \bigcap_k \tau_k(A, \|\cdot\|)$. Simply write τ_k or τ_∞ if no confusion can arise.

τ_∞ may alternatively be described as follows: Equip $\bigcup_k \mathcal{S}_k$ with the inductive topology; then τ_∞ is the quotient topology of the map $p \mapsto \ker(p)$.

LEMMA 3. *For any Banach algebra $(A, \|\cdot\|)$ the topologies τ_k, $k \in \mathbb{N}_\infty$, are compact (in general not Hausdorff) and*

$$\tau_{\mathrm{w}} \subset \tau_\infty \subset \ldots \subset \tau_{k+1} \subset \tau_k \subset \ldots \subset \tau_1 \subset \tau_{\mathrm{s}}.$$

Proof. Since the $\mathcal{S}_k$ are compact, it is clear that the topologies τ_k and hence τ_∞ are also compact.

Let $I_i \to I$ in $(\mathrm{Id}(A), \tau_{\mathrm{s}})$, i.e. $q_{I_i} \to q_I$ in $\mathcal{S}_1$. This implies $I_i \to I$ with respect to τ_1, thus proving the inclusion $\tau_1 \subset \tau_{\mathrm{s}}$.

Since $\kappa_{k+1} : \mathcal{S}_{k+1} \to (\mathrm{Id}(A), \tau_{k+1})$ is continuous, so is the restriction

$$\kappa_k = \kappa_{k+1}|\mathcal{S}_k : \mathcal{S}_k \to (\mathrm{Id}(A), \tau_{k+1}).$$

By definition τ_k is the finest topology on $\mathrm{Id}(A)$ making this map continuous, and this implies $\tau_{k+1} \subset \tau_k$.

The only thing left to show is $\tau_{\mathrm{w}} \subset \tau_k$ for all $k \in \mathbb{N}$. If $p_i \to p$ in $\mathcal{S}_k$ and $x \notin \ker(p)$, then $p_i(x) \to p(x) \neq 0$ and therefore $x \notin \ker(p_i)$ for large i. This proves the τ_{w}-convergence $\kappa_k(p_i) \to \kappa_k(p)$, and again by the definition of the quotient topology τ_k we conclude $\tau_{\mathrm{w}} \subset \tau_k$. ■

The topologies τ_k seem to depend on the special norm chosen on A although I do not know any example for this. But we have the following

PROPOSITION 4. *Let $(A, \|\cdot\|)$ is a Banach algebra. The topology $\tau_\infty(A, \|\cdot\|)$ on $\mathrm{Id}(A)$ is compact, $\tau_{\mathrm{w}} \subset \tau_\infty \subset \tau_{\mathrm{s}}$, and if $\|\cdot\|_0$ is another equivalent algebra norm then $\tau_\infty(A, \|\cdot\|) = \tau_\infty(A, \|\cdot\|_0)$.*

Proof. We only have to show the independence on the special norm. There are constants $\alpha, \beta > 0$ such that $\alpha\|\cdot\|_0 \le \|\cdot\| \le \beta\|\cdot\|_0$. For $k \in \mathbb{N}$ let $l \in \mathbb{N}$ be such that $l \ge \beta k$. Then $\mathcal{S}_k(A, \|\cdot\|) \subset \mathcal{S}_l(A, \|\cdot\|_0)$. Since the restriction

$$\kappa_l : \mathcal{S}_k(A, \|\cdot\|) \to (\mathrm{Id}(A), \tau_l(A, \|\cdot\|_0))$$

is continuous we have $\tau_k(A, \|\cdot\|) \supset \tau_l(A, \|\cdot\|_0) \supset \tau_\infty(A, \|\cdot\|_0)$ by the definition of the quotient topologies. As k was arbitrary we see $\tau_\infty(A, \|\cdot\|) \supset \tau_\infty(A, \|\cdot\|_0)$; the reverse inclusion is similar. ■

PROPOSITION 5. *Let $\varphi : A \to B$ be a continuous homomorphism between Banach algebras and define*

$$\widetilde{\varphi} : \mathrm{Id}(B) \to \mathrm{Id}(A), \qquad I \mapsto \varphi^{-1}(I).$$

Then $\widetilde{\varphi}$ is τ_∞-continuous. If φ is surjective then $\widetilde{\varphi}$ is a homeomorphism onto its image.

Proof. Since φ is continuous, $\widetilde{\varphi}$ maps closed ideals to closed ideals. For $k \in \mathbb{N}$ let $l \in \mathbb{N}$ be such that $l \ge k\|\varphi\|$. Then we have a map

$$\overline{\varphi} : \mathcal{S}_k(B) \to \mathcal{S}_l(A), \qquad p \mapsto p \circ \varphi,$$

and the diagram

$$\begin{array}{ccc} \mathcal{S}_k(B) & \xrightarrow{\overline{\varphi}} & \mathcal{S}_l(A) \\ {\scriptstyle \kappa_k}\downarrow & & \downarrow{\scriptstyle \kappa_l} \\ \mathrm{Id}(B) & \xrightarrow{\widetilde{\varphi}} & \mathrm{Id}(A) \end{array}$$

is obviously commutative. Since $\widetilde{\varphi} \circ \kappa_k = \kappa_l \circ \overline{\varphi}$ is continuous, we deduce the τ_k-τ_l-continuity of $\widetilde{\varphi}$, and hence the τ_k-τ_∞-continuity. As k was arbitrary the first assertion follows.

Now let φ be surjective. By Proposition 4 we may assume that $B = A/J$ for $J = \ker(\varphi)$ and that φ is the quotient map. Then

$$\mathrm{im}(\widetilde{\varphi}) = \{I \in \mathrm{Id}(A) : I \supset J\} \subset \mathrm{Id}(A)$$

is τ_{w}-closed, hence τ_k-closed for all $k \in \mathbb{N}$. Then the restricted topology $\tau_k|\mathrm{im}(\widetilde{\varphi})$ coincides with the quotient topology of the map

$$\kappa_k : \kappa_k^{-1}(\mathrm{im}(\widetilde{\varphi})) \to \mathrm{im}(\widetilde{\varphi}).$$

Therefore it is enough to show that

$$\widetilde{\varphi}^{-1} \circ \kappa_k : \kappa_k^{-1}(\mathrm{im}(\widetilde{\varphi})) \to \mathrm{Id}(A/J, \tau_\infty)$$

is continuous. To this end let $p_i \to p$ in $\kappa_k^{-1}(\mathrm{im}(\widetilde{\varphi}))$. Then $\ker(p_i), \ker(p) \supset J$,

$$\overline{p}_i(a + J) := p_i(a), \qquad \overline{p}(a + J) := p(a)$$

are well-defined elements of $\mathcal{S}_k(A/J)$ and we have $\overline{p}_i \to \overline{p}$; moreover,

$$\begin{aligned} \widetilde{\varphi}^{-1} \circ \kappa_k(p_i) &= \widetilde{\varphi}^{-1}(\ker(p_i)) = \ker(p_i)/J = \ker(\overline{p}_i) \\ &\to \ker(\overline{p}) = \ldots = \widetilde{\varphi}^{-1} \circ \kappa_k(p), \end{aligned}$$

and this finishes the proof of the second assertion. ■

PROPOSITION 6. *Let I be a two-sided closed ideal in a Banach algebra A. Then*

(i) *The intersection map $i : \mathrm{Id}(A) \to \mathrm{Id}(I)$, $J \mapsto J \cap I$, is τ_∞-continuous.*
(ii) *If I has an approximate identity then $\mathrm{Id}(I) \subset \mathrm{Id}(A)$.*
(iii) *If I has a bounded approximate identity then $(\mathrm{Id}(I), \tau_\infty(I))$ carries the subspace topology from $(\mathrm{Id}(A), \tau_\infty(A))$.*

Proof. Let $r : \mathcal{S}_k(A) \to \mathcal{S}_k(I)$ be the restriction map. Then the diagram

$$\begin{array}{ccc} \mathcal{S}_k(A) & \xrightarrow{r} & \mathcal{S}_k(I) \\ {\scriptstyle \kappa_k}\downarrow & & \downarrow{\scriptstyle \kappa_k} \\ \mathrm{Id}(A) & \xrightarrow{i} & \mathrm{Id}(I) \end{array}$$

is obviously commutative, and the τ_k-continuity is easily deduced from this. Since k was arbitrary this proves (i).

(ii) is easy.

(iii) Let $(e_j)_j$ be an approximate identity with bound $c \geq 1$. For $k \in \mathbb{N}$ let $l \in \mathbb{N}$ be such that $l \geq ck$. First let us prove

$$(*) \qquad \forall p \in \mathcal{S}_k(I) : \exists \widetilde{p} \in \mathcal{S}_l(A) : \quad p \leq \widetilde{p}|_I \leq ckp.$$

Define $\widehat{p}(a) := \sup\{p(ax) : x \in I, \|x\| \leq 1\}$. Then $\widehat{p}$ obviously is a seminorm on A, and for $a, b \in A$, $x \in I$, $\|x\| \leq 1$ we have

$$p(abx) = \lim_j p(ae_j bx) \leq \lim_j p(ae_j)p(bx) \leq c\widehat{p}(a)\widehat{p}(b).$$

Hence $\widehat{p}(ab) \leq c\widehat{p}(a)\widehat{p}(b)$ and this implies that $\widetilde{p} := c\widehat{p}$ is an algebra seminorm on A. It is easily seen that this $\widetilde{p}$ satisfies $(*)$.

Let $\iota : \mathrm{Id}(I) \subset \mathrm{Id}(A)$ be the inclusion, and let $p_i \to p$ be any convergent net in $\mathcal{S}_k(I)$. Given any subnet $(p_j)_j$ choose $\widetilde{p}_j$ as in $(*)$, and find a convergent subnet $(\widetilde{p}_m)_m$ by compactness of $\mathcal{S}_l(A)$, $\widetilde{p}_m \to q$ say. For $x \in I$ we have $p_m(x) \leq \widetilde{p}_m(x) \leq ckp_m(x)$, and this yields $p(x) \leq q(x) \leq ckp(x)$. This implies $\ker(\widetilde{p}_m) \cap I = \ker(p_m)$ and $\ker(q) \cap I = \ker(p)$. Then we see

$$\begin{aligned} \iota(\kappa_k(p_m)) &= \ker(\widetilde{p}_m) \cap I \xrightarrow{m} \ker(q) \cap I \quad \text{by (i)} \\ &= \iota(\ker(p)) = \iota(\kappa_k(p)). \end{aligned}$$

Therefore $\iota \circ \kappa_k : \mathcal{S}_k(I) \to (\mathrm{Id}(A), \tau_\infty)$ is continuous for all k, and this implies the τ_∞-continuity of ι. Together with (i) this proves the claim since the intersection map i is obviously the continuous inverse of ι. ∎

Without proof I would like to mention the following

PROPOSITION 7. *Let $(A_n)_n$ be a sequence of Banach algebras having approximate identities and let A be the c_0-sum of the A_n. For $I \in \mathrm{Id}(A)$ and $n \in \mathbb{N}$ let $I(n) \subset A_n$ be the nth projection of I. Then*

$$(\mathrm{Id}(A), \tau_\infty) \to \prod_{n\in\mathbb{N}} (\mathrm{Id}(A_n), \tau_\infty), \qquad I \mapsto (I(n))_{n\in\mathbb{N}},$$

is a continuous bijection.

4. Comparison with other topologies

PROPOSITION 8. *Let A be a C^*-algebra. Then $\tau_s = \tau_k$ for all $k \in \mathbb{N}$, in particular $\tau_\infty = \tau_s$.*

Proof. We have to prove $\tau_s \subset \tau_k$ for all positive integers k. To this end, let $p_i \to p$ in $\mathcal{S}_k$. The claim is

$$\|a + \ker(p_i)\| \xrightarrow{i} \|a + \ker(p)\| \quad \forall a \in A.$$

Since $\|a + I\|^2 = \|a^*a + I\|$ for all $I \in \mathrm{Id}(A)$ we may assume that a is selfadjoint. By an old theorem of Kaplansky (see [13], Th. I.2.4) we have

$$\|a + \ker(q)\| \leq q(a) \quad \text{for all seminorms } q \in \mathcal{S}_k,$$

because $a + \ker(q)$ generates a commutative C^*-algebra in $A/\ker(q)$. By the definition of $\mathcal{S}_k$ we have

$$\|a + \ker(q)\| \leq q(a) \leq k\|a + \ker(q)\| \quad \text{for all } q \in \mathcal{S}_k,\ a \in A \text{ selfadjoint.}$$

Therefore for all $n \in \mathbb{N}$ we see

$$\|a + \ker(q)\|^n = \|a^n + \ker(q)\| \leq q(a^n) \leq k\|a^n + \ker(q)\| \leq k\|a + \ker(q)\|^n,$$

and from this by taking the nth root

$$(1) \qquad \|a + \ker(q)\| \leq q(a^n)^{1/n} \leq k^{1/n}\|a + \ker(q)\|$$

for all selfadjoint $a \in A$ and all seminorms $q \in \mathcal{S}_k$.

Back to our convergent net $p_i \to p$. For $\varepsilon > 0$ find $n \in \mathbb{N}$ such that $|k^{1/n} - 1|\|a\| < \varepsilon$. There is an index i_0 such that $|p_i(a^n)^{1/n} - p(a^n)^{1/n}| < \varepsilon$ for all $i \geq i_0$. Then by (1),

$$\begin{aligned} &\big|\, \|a + \ker(p_i)\| - \|a + \ker(p)\| \,\big| \\ &\quad \leq \big|\, \|a + \ker(p_i)\| - p_i(a^n)^{1/n}\big| + |p_i(a^n)^{1/n} - p(a^n)^{1/n}| \\ &\qquad + |p(a^n)^{1/n} - \|a + \ker(p)\|\,| \\ &\quad \leq (k^{1/n} - 1)\|a + \ker(p_i)\| + |p_i(a^n)^{1/n} - p(a^n)^{1/n}| \\ &\qquad + (k^{1/n} - 1)\|a + \ker(p)\| < 3\varepsilon \end{aligned}$$

for all $i \geq i_0$, and this proves the proposition. ∎

Next we will compare τ_∞ with the Gelfand topology. Let A be a Banach algebra. Let $\mathcal{M}$ be the space of maximal modular ideals with codimension 1 together with the trivial ideal A. By the Gelfand theory this corresponds to the space of homomorphisms $A \to \mathbb{C}$, and so carries the relative w^*-topology from the dual A' which is known as the Gelfand topology. We will show that $\mathcal{M} \subset \mathrm{Id}(A)$ is τ_∞-closed and that the relative τ_∞-topology coincides with the Gelfand topology, provided A has a bounded approximate identity $(e_i)_i$.

LEMMA 9. *Let A be a Banach algebra and $\varphi : A \to \mathbb{C}$ a non-zero homomorphism, $I := \ker(\varphi)$. Then*

$$\kappa_k^{-1}(I) = \{t|\varphi(\cdot)| : t \in [1, k/\|\varphi\|]\}.$$

Proof. The inclusion "$\supset$" is obvious. If $p \in \kappa_k^{-1}(I)$, then p induces a norm $\overline{p}$ on A/I via $\overline{p}(a + I) := p(a)$. But $A/I \cong \mathbb{C}$ via $\overline{\varphi} : A/I \to \mathbb{C}$, $a + I \mapsto \varphi(a)$, hence $\overline{p} \circ \overline{\varphi}^{-1}$ is an algebra norm on $\mathbb{C}$, and this implies $\overline{p} \circ \overline{\varphi}^{-1} = t|\cdot|$ for some $t \geq 1$. But then $p = t|\varphi(\cdot)|$, and from $p \in \mathcal{S}_k$ we conclude $t \leq k/\|\varphi\|$. ∎

LEMMA 10. *Let A be a Banach algebra, and $p \in \mathcal{S}_k(A)$. Then the following assertions are equivalent:*

(i) *There is a homomorphism $\varphi : A \to \mathbb{C}$ and a real number $t > 0$ such that $p = t|\varphi(\cdot)|$.*

(ii) $\exists s > 0 : \forall a, b \in A : p(ab) \geq sp(a)p(b)$.

Proof. The implication from (i) to (ii) is trivial, just let $s = 1/t$. Conversely, assume that (ii) holds. Then p induces a norm $\overline{p}$ on $A/\ker(p)$ with the property $\overline{p}(xy) \geq s\overline{p}(x)\overline{p}(y)$ for all $x, y \in A/\ker(p)$, and this property also holds in the completion B of $(A/\ker(p), \overline{p})$. From $\overline{p}(x^2) \geq s\overline{p}(x)^2$ we get by induction

$$\overline{p}(x^{2^n}) \geq s^{2^n-1}\overline{p}(x)^{2^n},$$

and then by the Beurling formula for the spectral radius $r_B(x) \geq s\overline{p}(x)$. By the theorem from [10] we know that B is commutative. See also [9], p. 345, for this argument. Let $\widetilde{B} = B$ if B has a unit and $\widetilde{B} = B_1$, the algebra which emerges from the process of adjoining a unit e, otherwise. By Lemma 2 of [10] we may introduce a norm q on $\widetilde{B}$ in such a way that $\overline{p}$ and $q|_B$ are equivalent (and $r_{\widetilde{B}}(x) \geq s^3 q(x)$ for all $x \in \widetilde{B}$).

In the case where B does not have a unit we have

$$q(x + \lambda e) := \sup\{\overline{p}(xz + \lambda z) : z \in B,\ \overline{p}(z) \leq 1\} \quad (x \in A,\ \lambda \in \mathbb{C}).$$

Then for $z_1, z_2 \in B$ with $\|z_1\|, \|z_2\| \leq 1$ we get

$$\begin{aligned} q((x + \lambda e)(y + \mu e)) &\geq \overline{p}((x + \lambda e)(y + \mu e)z_1 z_2) \\ &= \overline{p}((x + \lambda e)z_1(y + \mu e)z_2) \geq s\overline{p}((x + \lambda e)z_1)\overline{p}((y + \mu e)z_2) \end{aligned}$$

and so

$$q((x + \lambda e)(y + \mu e)) \geq sq(x + \lambda e)q(y + \mu e) \quad \text{for all } x + \lambda e, y + \mu e \in \widetilde{B}.$$

Hence we may assume $q(xy) \geq sq(x)q(y)$ for all $x, y \in \widetilde{B}$ in either case.

We now follow the argument of Theorem 10.19 of [12] to conclude that $\widetilde{B}$ is isomorphic to the complex numbers. We have $q(1) = q(xx^{-1}) \geq sq(x)q(x^{-1})$, and hence

$$(1) \qquad q(x^{-1}) \leq \frac{q(1)}{sq(x)} \quad \text{for all invertible elements } x \in \widetilde{B}.$$

If $(x_n)_n$ is a sequence of invertibles converging to $x \in \widetilde{B} \setminus \{0\}$, then

$$\begin{aligned} q(x_n^{-1} - x_m^{-1}) &= q(x_m^{-1}(x_n - x_m)x_n^{-1}) \leq q(x_m^{-1})q(x_n - x_m)q(x_n^{-1}) \\ &\overset{(1)}{\leq} \frac{q(1)^2}{s^2 q(x_n)q(x_m)} q(x_n - x_m) \xrightarrow{n,m} 0. \end{aligned}$$

Hence the sequence $(x_n^{-1})_n$ converges to an element $y \in \widetilde{B}$ which is easily seen to be the inverse of x.

So the invertible elements of $\widetilde{B}$ are open and closed in $\widetilde{B} \setminus \{0\}$. Since the latter set clearly is connected we conclude that $\widetilde{B}$ is a division algebra, hence $\widetilde{B} \cong \mathbb{C}$ by Mazur's theorem.

Of course this implies $A/\ker(p) \cong \mathbb{C}$. Therefore $\ker(p) = \ker(\varphi)$ for a homomorphism $\varphi : A \to \mathbb{C}$, and since a norm on $\mathbb{C}$ necessarily is a multiple of the absolute value, p must be of the form $t|\varphi(\cdot)|$ for some $t > 0$. ∎

THEOREM 11. *Let A be a Banach algebra with a bounded approximate identity $(e_i)_i$. Then the Gelfand space $\mathcal{M}$ is τ_∞-closed, and the Gelfand topology coincides with all τ_k, hence with τ_∞.*

Proof. Let $\mathcal{H}$ be the set of homomorphisms $A \to \mathbb{C}$. Then the norms of $\varphi \in \mathcal{H} \setminus \{0\}$ stay away from zero. For this, let β be a bound for $(e_i)_i$. Since $\varphi(e_i) \to 1$ for $0 \neq \varphi \in \mathcal{H}$, we easily see that $\|\varphi\| \geq 1/\beta$.

By a combination of the lemmas above we have

$$\kappa_k^{-1}(\mathcal{M}) = \left\{p \in \mathcal{S}_k : \forall a, b \in A : p(ab) \geq \frac{1}{k\beta}p(a)p(b)\right\}.$$

In particular, $\kappa_k^{-1}(\mathcal{M})$ is closed and this means that $\mathcal{M}$ is τ_k-closed for all k, hence τ_∞-closed.

But then the τ_k-topology on $\mathcal{M}$ coincides with the quotient topology of the map

$$(1) \qquad \kappa_k|\kappa_k^{-1}(\mathcal{M}) : \kappa_k^{-1}(\mathcal{M}) \to \mathcal{M}.$$

Let $p_i \to p$ in $\kappa_k^{-1}(\mathcal{M})$. We have $p_i = t_i|\varphi_i(\cdot)|$ and $p = t|\varphi(\cdot)|$ for $\varphi, \varphi_i \in \mathcal{H}$ and $t, t_i \in [1, \beta]$ by Lemma 9. Given any subnet $(p_j)_j$ we may find a finer subnet $(p_l)_l$ such that $t_l \to s$ in $[1, \beta]$ and $\varphi_l \to \psi$ in $(\mathcal{H}, w^*)$. This implies

$$t|\varphi(\cdot)| = p = \lim_l p_l = \lim_l t_l|\varphi_l(\cdot)| = s|\psi(\cdot)|.$$

So φ and ψ are proportional homomorphisms, hence equal. This implies

$$\kappa_k(p_l) = \ker(\varphi_l) \to \ker(\varphi) = \kappa_k(p)$$

in the Gelfand topology. Therefore the map (1) is continuous if $\mathcal{M}$ carries the Gelfand topology, and this in turn means that τ_k is finer than the Gelfand topology τ_G for all k, and then $\tau_\infty \supset \tau_G$.

Conversely, if $\varphi_i \to \varphi$ in $(\mathcal{H}, w^*)$ then $|\varphi_i(\cdot)| \to |\varphi(\cdot)|$ in $\mathcal{S}_1$, and this implies $\ker(\varphi_i) \to \ker(\varphi)$ with respect to τ_1. But this yields $\tau_G \supset \tau_1$ on $\mathcal{M}$. This finally proves the theorem. ∎

5. Topological properties. When is $(\mathrm{Id}(A), \tau_\infty)$ a T_1-space, i.e. when are points closed? This of course is the case iff all topologies τ_k are T_1, and this is the case iff

$$\forall I \in \mathrm{Id}(A) : \quad \{p \in \mathcal{S}_k : \ker(p) = I\} \text{ is closed in } \mathcal{S}_k.$$

Say that a Banach algebra $(A, \|\cdot\|)$ has the *norm property* iff any pointwise limit of a uniformly $\|\cdot\|$-bounded net of norms on A is again a norm. Since the seminorms $p \in \mathcal{S}_k(A)$ with $\ker(p) = I$ correspond bijectively to the norms in $\mathcal{S}_k(A/I)$ (where A/I carries the quotient norm), a simple reformulation of the above consideration yields:

PROPOSITION 12. *A point $I \in \mathrm{Id}(A)$ is τ_∞-closed iff A/I has the norm property.*

This property is somehow related to minimal norm topologies (see [8]) as will be shown by the following results.

PROPOSITION 13. *Let P be a primitive ideal of finite codimension. Then $\{P\}$ is τ_∞-closed.*

Proof. We have $A/P \cong M_m(\mathbb{C})$ for some $m \in \mathbb{N}$ and this algebra has a unit 1. Let $p_i \to p$ in $\mathcal{S}_k(A/P)$, where each p_i is a norm. Since $p(1) = \lim_i p_i(1) \geq 1$ the ideal $\ker(p)$ must be proper, hence $\{0\}$. ∎

PROPOSITION 14. *Let $(A, \|\cdot\|)$ be a Banach *-algebra with a minimal norm topology which stems from a pre-C^*-norm on A. Then A has the norm property.*

Proof. Let $p_i \to p$ be a convergent net of norms p_i in $\mathcal{S}_k$. Since $\|\cdot\|_*$ yields the minimal norm topology there are positive constants c_i satisfying $\|\cdot\|_* \leq c_i p_i$ (where $\|\cdot\|_*$ is the pre-C^*-norm). This implies

$$\begin{aligned}\|a\|_*^2 = \|a^*a\|_* = \|(a^*a)^n\|_*^{1/n} &\leq c_i^{1/n} p_i((a^*a)^n)^{1/n} \\ &\leq c_i^{1/n} p_i(a^*a) \leq c_i^{1/n} p_i(a^*) p_i(a)\end{aligned}$$

for all positive integers n, hence

$$\|a\|_*^2 \leq p_i(a^*) p_i(a) \to p(a^*) p(a).$$

So $p(a) = 0$ implies $a = 0$, which is the desired result. ∎

So if e.g. H is a Hilbert space and $A \subset \mathcal{L}(H)$ is a *-subalgebra which contains all finite-dimensional operators, then A (with any Banach algebra norm) has the norm property, since by [8], Th. 3.3, A satisfies the assumptions of the above proposition. The same conclusion holds for all C^*-algebras by [8], Th. 3.6.

PROPOSITION 15. *Let A be an annihilator algebra. Then $\mathrm{Rad}(A)$ is a τ_∞-closed point in $\mathrm{Id}(A)$.*

Proof. By [7], §32, Prop. 15, $A/\mathrm{Rad}(A)$ is an annihilator algebra, hence we may reduce to the semisimple case. If $p_i \to p$ is a convergent net of norms $p_i \in \mathcal{S}_k$ then by [7], §32, Lemma 23, we have for the spectral radius

$$r(a) = \lim_n p_i(a^n)^{1/n} \leq p_i(a) \to p(a).$$

Therefore $\ker(p)$ consists entirely of quasiregular elements, hence $\ker(p) \subset \mathrm{Rad}(A) = \{0\}$, which proves the assertion. ∎

Next one may ask when $(\mathrm{Id}(A), \tau_\infty)$ is a Hausdorff space. This of course is the case if A is a C^*-algebra since then we have $\tau_\infty = \tau_s$. I would like to mention without proof that the Banach algebras l^p, $1 \leq p \leq \infty$, with componentwise multiplication and the convolution algebras $L^p(G)$, G a compact group, $1 \leq p < \infty$, have τ_∞ Hausdorff. It can be shown (using [11], 7.1.5) that each two-sided closed ideal of $L^p(G)$ is the intersection of the maximal ideals containing it. Therefore $\mathrm{Id}(L^p(G))$ corresponds bijectively to the subsets of the dual group Γ, and hence to $\{0,1\}^\Gamma$, where each subset of Γ is identified with its characteristic function $\Gamma \to \{0,1\}$. Then the τ_∞-topology on the space $\mathrm{Id}(L^p(G))$ corresponds to the product topology on $\{0,1\}^\Gamma$ which clearly is a Hausdorff space. The details are left to the reader.

PROPOSITION 16. *Let A be a Banach algebra. Then the following are equivalent*:

(i) *$(\mathrm{Id}(A), \tau_\infty)$ is a Hausdorff space.*
(ii) *If $I_i \to I$ in $(\mathrm{Id}(A), \tau_\infty)$, then for all $x \in A$*

$$\limsup_i \|x + I_i\| \leq \|x + I\|, \quad \liminf_i \|x + I_i\| = 0 \Rightarrow x \in I.$$

Proof. (i)⇒(ii). Let q_i be the quotient seminorm of I_i. Assume that $\limsup_i q_i(x) > \|x + I\|$. Then we can find a subnet $(q_j)_j$ such that $\lim_j q_j(x) > \|x + I\|$ and $q_j \to p \in \mathcal{S}_1$, since $\mathcal{S}_1$ is compact. Since $I_j = \ker(q_j) \to \ker(p)$ with respect to τ_1, hence with respect to τ_∞, we have $I = \ker(p)$ because τ_∞ is Hausdorff. But $p \in \mathcal{S}_1$ and this implies

$$\|x + I\| = \|x + \ker(p)\| \geq p(x) = \lim q_j(x) > \|x + I\|.$$

This contradiction shows $\limsup_i \|x + I_i\| \leq \|x + I\|$.

Now consider the situation $\liminf_i q_i(x) = 0$. Again we may find a subnet $(q_j)_j$ such that $\lim_j q_j(x) = 0$ and $q_j \to p$ in $\mathcal{S}_1$. Since τ_∞ is Hausdorff we have $I = \ker(p)$, and this implies $p(x) = \lim_j q_j(x) = 0$, i.e. $x \in I$.

(ii)⇒(i). Let $I_i \to I$ and $I_i \to J$ in $(\mathrm{Id}(A), \tau_\infty)$. We have to prove $I = J$. For this, let $x \in I$. Then by the assumption in (ii),

$$\liminf_i \|x + I_i\| \leq \limsup_i \|x + I_i\| \leq \|x + I\| = 0,$$

and this implies $x \in J$. Hence $I \subset J$ and then $I = J$. ∎

6. Another topology. We are in need of still another topology in order to say more about the case of a separable Banach algebra. For a compact set $K \subset A$ define

$$U(K) := \{I \in \mathrm{Id}(A) : I \cap K = \emptyset\}.$$

Obviously we have $U(K_1) \cap U(K_2) = U(K_1 \cup K_2)$, hence the sets $U(K)$, $K \subset A$ compact, form a base for a topology τ_c on $\mathrm{Id}(A)$.

LEMMA 17. *For all Banach algebras we have* $\tau_w \subset \tau_c \subset \tau_\infty$.

Proof. Since $U(\{x\}) = \{I \in \mathrm{Id}(A) : x \notin I\}$ we have $\tau_w \subset \tau_c$. Let $p_i \to p$ in $\mathcal{S}_k$, and let $\ker(p) \in U(K)$ for some compact set $K \subset A$. Then $\inf_{x\in K} p(x) > 0$. Since $p_i \to p$ pointwise we have $p_i \to p$ uniformly on compact sets, hence there is an index i_0 such that $\inf_{x\in K} p_i(x) > 0$ for all $i \geq i_0$. But this means that $\ker(p_i) \in U(K)$ for all $i \geq i_0$. So we have shown that $\kappa_k : \mathcal{S}_k \to (\mathrm{Id}(A), \tau_c)$ is continuous. By definition of the quotient topology τ_k we deduce $\tau_c \subset \tau_k$. This holds for all k, so we have $\tau_c \subset \tau_\infty$. ∎

LEMMA 18. *Let* $(I_i)_i$ *be a net in* $\mathrm{Id}(A)$ *and* $I \in \mathrm{Id}(A)$ *such that*

$$\forall x \in A \setminus I : \quad \liminf_i \|x + I_i\| > 0.$$

Then $I_i \to I$ *with respect to* τ_c.

Proof. Let $K \subset A$ be compact and $I \cap K = \emptyset$. For each $x \in K$ we then have $r_x := \liminf_i \|x + I_i\| > 0$. Hence there is an index $i(x)$ such that $\|x + I_i\| > \frac{2}{3} r_x$ for all $i \geq i(x)$. Let $B(x, r) \subset A$ denote the open ball around $x \in A$ with radius r. Then by compactness of K there are finitely many points $x_1, \ldots, x_n \in K$ such that $K \subset B(x_1, \frac{1}{3} r_{x_1}) \cup \ldots \cup B(x_n, \frac{1}{3} r_{x_n})$. Let i_0 be an index larger than $i(x_1), \ldots, i(x_n)$. Then for any $y \in K$ there is a j such that $y \in B(x_j, \frac{1}{3} r_{x_j})$ and then for $i \geq i_0$ we have

$$\|y + I_i\| \geq \|x_j + I_i\| - \|x_j - y\| \geq \tfrac{2}{3} r_{x_j} - \tfrac{1}{3} r_{x_j} = \tfrac{1}{3} r_{x_j} > 0,$$

hence we have shown $I_i \cap K = \emptyset$ for $i \geq i_0$. ∎

COROLLARY 19. *If A is a C^*-algebra then* $\tau_w = \tau_c$.

Proof. If $I_i \to I$ with respect to τ_w then we know that $\|x + I\| \leq \liminf \|x + I_i\|$ for all $x \in A$, and the result follows. ∎

In general we have $\tau_w \neq \tau_c$. For instance, in Example 1 it can be shown that $I_{\mathbb{D}}$ is not in the τ_c-closure of $\{I_{\{z\}} : z \in 2\mathbb{D} \setminus \mathbb{D}\}$, hence we must have $\tau_w \neq \tau_c$ in this example.

7. Separable Banach algebras. In this section let A be a separable Banach algebra. Then we know that all spaces $\mathcal{S}_k$ are metrizable compact spaces by Lemma 2.

THEOREM 20. *Let $W \subset \mathrm{Id}(A)$ be any subspace. Then W is a separable Lindelöf space in the τ_1-topology, hence the same holds true for all weaker topologies, for example τ_w, τ_c or τ_∞.*

Proof. $\kappa_1^{-1}(W) \subset \mathcal{S}_1$ is a second countable and metrizable space, hence it is a separable Lindelöf space, and so is the continuous image $W = \kappa_1(\kappa_1^{-1}(W))$. ∎

This theorem extends the results of [14], Cor. 4.5, to a large extent. We are interested in other properties of $(\mathrm{Id}(A), \tau_\infty)$. I do not know whether this space must be first (or even better second) countable, but the following holds:

PROPOSITION 21. *Sequentially closed sets in $(\mathrm{Id}(A), \tau_\infty)$ are already closed, and sequentially continuous maps on $(\mathrm{Id}(A), \tau_\infty)$ are already continuous.*

Proof. Let $W \subset \mathrm{Id}(A)$ be sequentially closed. Let p be in the closure of $\kappa_k^{-1}(W) \subset \mathcal{S}_k$. Since $\mathcal{S}_k$ is metrizable, there is a sequence $(p_n)_n$ in $\kappa_k^{-1}(W)$ such that $p_n \to p$, and this implies $W \ni \ker(p_n) \to \ker(p)$. Since W is sequentially closed we have $\ker(p) \in W$, hence $p \in \kappa_k^{-1}(W)$. So we have proved that $\kappa_k^{-1}(W)$ is closed in $\mathcal{S}_k$ and this proves that W is τ_k-closed for all $k \in \mathbb{N}$, hence τ_∞-closed. The second assertion follows from the first. ∎

PROPOSITION 22. *For a sequence $(I_n)_n$ and an ideal I in $\mathrm{Id}(A)$ the following are equivalent:*

(i) *$I_n \to I$ with respect to τ_c.*
(ii) *For all $x \in A \setminus I$ we have $\liminf_n \|x + I_n\| > 0$.*

Proof. (ii)⇒(i) is clear from Lemma 18. Conversely, assume that (i) holds and let $x \in A \setminus I$. We have to show $\liminf_n \|x + I_n\| > 0$. Assume the contrary. Then there is a subsequence $(I_{n_m})_m$ such that $\lim_m \|x + I_{n_m}\| = 0$, hence there are $x_m \in I_{n_m}$ such that $\|x - x_m\| \to 0$. Since $x \notin I$ and I is closed, we have $x_m \notin I$ for large m, $m \geq m_0$ say. Then $K := \{x\} \cup \{x_m : m \geq m_0\} \subset A$ is compact and disjoint from I. So we have $I_n \cap K = \emptyset$ for large n, but this contradicts $x_m \in K \cap I_{n_m}$ for $m \geq m_0$. ∎

THEOREM 23. *For a separable Banach algebra A the following assertions are equivalent:*

(i) *$(\mathrm{Id}(A), \tau_\infty)$ is a Hausdorff space.*
(ii) *$(\mathrm{Id}(A), \tau_\infty)$ is a metrizable space.*
(iii) *If the sequence $(I_n)_n$ is τ_∞-convergent to I, then $\limsup_n \|x + I_n\| \leq \|x + I\|$ for all $x \in A$.*
(iv) *If $p_n \to p$ and $r_n \to r$ in some $\mathcal{S}_k$ such that $\ker(p_n) = \ker(r_n)$ for all $n \in \mathbb{N}$ then $\ker(p) = \ker(r)$.*

Proof. (i)⇒(ii). Since comparable, compact Hausdorff topologies are equal we have $\tau_\infty = \tau_k$ for all k. Then $(\mathrm{Id}(A), \tau_k) = \kappa_k(\mathcal{S}_k)$ is a Suslin space and (ii) follows because compact Suslin spaces are metrizable.

The converse (ii)⇒(i) is trivial.

(i)⇒(iii) follows from Proposition 16.

(i)⇒(iv). The assumptions of (iv) imply

$$\ker(p_n) = \ker(r_n) \to \ker(p), \ker(r)$$

hence $\ker(p) = \ker(r)$ by the Hausdorff property.

(iv)⇒(iii). Let $\widetilde{\tau}_{\mathrm{uc}}$ be the topology on $\mathbb{R}$ which is generated by the intervals $(-\infty, \lambda)$, $\lambda \in \mathbb{R}$. Define

$$\varphi_x : \mathrm{Id}(A) \to (\mathbb{R}, \widetilde{\tau}_{\mathrm{uc}}), \quad I \mapsto \|x + I\|, \quad x \in A.$$

Let $p_n \to p$ in $\mathcal{S}_k$ and let q_n be the quotient seminorm of $\ker(p_n)$. For any subsequence $(p_{n_i})_i$ we may find another subsequence $(p_{n_{i_j}})_j$ such that $q_{n_{i_j}} \to r$ in $\mathcal{S}_1$. Since $\ker(p_n) = \ker(q_n)$ for all n we deduce $\ker(p) = \ker(r)$ from (iv). Since $r \in \mathcal{S}_1$ we have $r(x) \le \|x + \ker(r)\|$, hence

$$\|x + \ker(p)\| = \|x + \ker(r)\| \ge r(x) = \lim_j q_{n_{i_j}}(x) = \lim_j \|x + \ker(p_{n_{i_j}})\|.$$

Hence we proved that $\mathcal{S}_k \to (\mathbb{R}, \widetilde{\tau}_{\mathrm{uc}})$, $p \mapsto \|x + \ker(p)\|$, is continuous, hence $\varphi_x : (\mathrm{Id}(A), \tau_k) \to (\mathbb{R}, \widetilde{\tau}_{\mathrm{uc}})$ is continuous for all k. Therefore $\varphi_x : (\mathrm{Id}(A), \tau_\infty) \to (\mathbb{R}, \widetilde{\tau}_{\mathrm{uc}})$ is continuous, and this is nothing but (iii).

(iii)⇒(i). First let us prove the following assertion:

(*) Limits of sequences are unique, i.e. $(\mathrm{Id}(A), \tau_\infty)$ is a US-space in the sense of [15].

Let $I_n \to I$ be a τ_∞-convergent sequence. Then there is a subsequence $(I_{n_m})_m$ such that $q_{I_{n_m}} \to p$ in $\mathcal{S}_1$. Let us prove $I = \ker(p)$. By the assumption (iii) we have

$$p(x) = \lim_m \|x + I_{n_m}\| \le \|x + I\|,$$

and so $I \subset \ker(p)$. On the other hand, we have $I_{n_m} \to I$ with respect to τ_c, hence by Proposition 22,

$$x \in A \setminus I \quad \text{implies} \quad \liminf_m \|x + I_{n_m}\| > 0, \text{ hence } p(x) > 0.$$

Therefore $\ker(p) \subset I$. So we have $I = \ker(p)$, and this proves (*).

Since we do not know whether $(\mathrm{Id}(A), \tau_\infty)$ is first countable, the following assertion deserves a proof.

(**) Let $W \subset \mathrm{Id}(A)$. Then $\widetilde{W} := \{I\colon$ there is a sequence $I_n \in W$ such that $I_n \to I\}$ is the τ_∞-closure of W.

It is clear that $\widetilde{W} \subset \overline{W}$, hence by Proposition 21 we only have to show that $\widetilde{W}$ is sequentially closed. Let $(I_n)_n$ be a sequence in $\widetilde{W}$ such that $I_n \to I$ and let us show $I \in \widetilde{W}$. There are sequences $(I^n_m)_m$ in W such that $I^n_m \xrightarrow{m} I_n$ as $m \to \infty$. Let q_n and q^n_m be the quotient seminorms of I_n resp. I^n_m. Considering subsequences we may assume $q_n \to p$ and $q^n_m \xrightarrow{m} r_n$ in $\mathcal{S}_1$.

Again by compactness of $\mathcal{S}_1$ we may assume $r_n \to r$ in $\mathcal{S}_1$. Let $(V_t)_t$ be a countable open neighbourhood base of r in $\mathcal{S}_1$. There are $n_1 < n_2 < \ldots$ satisfying $r_{n_t} \in V_t$. Hence there are $m_t \in \mathbb{N}$ such that $q^{n_t}_{m_t} \in V_t$. This implies $q^{n_t}_{m_t} \to r$ as $t \to \infty$, and therefore $\ker(r) \in \widetilde{W}$.

Since $I^n_m \xrightarrow{m} I_n$ and $I^n_m = \ker(q^n_m) \xrightarrow{m} \ker(r_n)$ we have $I_n = \ker(r_n)$ by (*). Since $I_n \to I$ and $\ker(r_n) \to \ker(r)$ we have $I = \ker(r)$, again by (*). Therefore $I \in \widetilde{W}$, and this proves (**).

Now let $I_i \to I$ be a τ_∞-convergent net in $\mathrm{Id}(A)$. We must show that I is the unique limit. Restricting to a subnet we may assume $q_{I_i} \to p$ in $\mathcal{S}_1$ and it is enough to prove $I = \ker(p)$.

Let $(W_n)_n$ be a countable closed neighbourhood base of p. There are indices i_n such that $q_{I_i} \in W_n$ for all $i \ge i_n$. Then I is in the closure of $\{I_i : i \ge i_n\}$ and so by (**) there are sequences $(J^n_m)_m$ in $\{I_i : i \ge i_n\}$ converging to I, and additionally we may assume that the quotient seminorms q^n_m of J^n_m converge to some seminorm p_n in $\mathcal{S}_1$. Then we have $J^n_m \xrightarrow{m} I$ and $J^n_m \xrightarrow{m} \ker(p_n)$, and by (*) we conclude $I = \ker(p_n)$. But $p_n = \lim_m q^n_m \in \overline{W}_n = W_n$ for all n, implying $p_n \to p$, hence $I = \ker(p_n) \to \ker(p)$. By (*) we now may conclude $I = \ker(p)$. ■

LEMMA 24. *Let $(\mathrm{Id}(A), \tau_\infty)$ be a Hausdorff space. Then the intersection $\cap : \mathrm{Id}(A) \times \mathrm{Id}(A) \to \mathrm{Id}(A)$ is τ_∞-continuous.*

Proof. Let $I_n \to I$ and $J_n \to J$ be τ_∞-convergent sequences. If q_n resp. q'_n are the quotient seminorms of I_n resp. J_n then each subsequence $(I_m \cap J_m)_m$ contains another subsequence $(I_t \cap J_t)_t$ such that the sequences $(q_t)_t$ and $(q'_t)_t$ converge in $\mathcal{S}_1$, to p resp. p' say. Since τ_∞ is assumed to be Hausdorff we have $I = \ker(p)$ and $J = \ker(p')$. We have $\max\{q_t, q'_t\} \to \max\{p, p'\}$ in $\mathcal{S}_1$, and this implies $I_t \cap J_t \to \ker(p) \cap \ker(p') = I \cap J$. ■

Let us consider the space Min-Primal(A) of minimal primal ideals. This is always a Hausdorff space with respect to τ_{w}. To see this, let $(I_i)_i$ be a net in Min-Primal(A) which converges to I and J in Min-Primal(A). Then we also have $I_i \to I \cap J$, in particular $I \cap J$ must be primal. Since I and J are minimal primal we conclude $I = I \cap J = J$.

In the case of a C^*-algebra we know by [1], Cor. 4.3(a), that τ_{w} and τ_{s} coincide on Min-Primal(A). A possible generalization to Banach algebras would be that τ_∞ and τ_{w} coincide on Min-Primal(A), but this is not the case as can be seen by Example 1; there $I_{\mathbb{D}}$ is in the τ_{w}-closure of Min-Primal$(A) \setminus \{I_{\mathbb{D}}\}$ while $I_{\mathbb{D}}$ is an τ_∞-isolated point in Min-Primal(A) by Theorem 11. For C^*-algebras we have $\tau_{\mathrm{w}} = \tau_{\mathrm{c}}$ by Proposition 19, and one could replace τ_{w} by τ_{c} in the above considerations. Following this idea one gets the following

THEOREM 25. *For a separable Banach algebra the following assertions hold*:

(i) *For* $E \subset \mathrm{Id}(A)$ *define* $\widetilde{E} := \{I \in \mathrm{Id}(A) : I \supset J$ *for some* $J \in E\}$. *Then* $(\overline{E}^{\tau_k})^\sim = (\overline{E}^{\tau_\infty})^\sim$ *for all* k *and this is the* τ_c*-sequential closure of* E (*i.e. the smallest* τ_c*-sequentially closed set containing* E).

(ii) *The* τ_c*-sequentially closed sets are the closed sets of a topology* τ_{cs}.

(iii) τ_∞, τ_{cs} *and all the* τ_k *coincide when restricted to* Min-Primal(A) *and these topologies make* Min-Primal(A) *a Suslin space.*

(iv) *If* $\tau_\infty(A)$ *is Hausdorff, then* $(\text{Min-Primal}(A), \tau_\infty)$ *is a Polish space, i.e. a* G_δ*-subset of* $(\mathrm{Id}(A), \tau_\infty)$.

Proof. (i) Let E_0 be the τ_c-sequential closure of E. Because $\tau_\infty \subset \tau_k$ and by Proposition 21, $\overline{E}^{\tau_k} \subset \overline{E}^{\tau_\infty} \subset E_0$, hence $(\overline{E}^{\tau_k})^\sim \subset (\overline{E}^{\tau_\infty})^\sim \subset E_0$. Therefore it is enough to show that $(\overline{E}^{\tau_k})^\sim$ is τ_c-sequentially closed. To this end, let $(I_n)_n$ be a sequence in $(\overline{E}^{\tau_k})^\sim$ which is τ_c-convergent to some $I \in \mathrm{Id}(A)$. Then we have $J_n \in \overline{E}^{\tau_k}$ such that $J_n \subset I_n$ and considering subsequences we may assume $q_{J_n} \to p$ in $\mathcal{S}_1$. From Proposition 22 we conclude $p(x) = \lim_n q_{J_n}(x) \geq \liminf_n q_{I_n}(x) > 0$ if $x \in A \setminus I$, and this means $\ker(p) \subset I$. Since $\ker(q_{J_n}) \to \ker(p)$ with respect to τ_k we have $\ker(p) \in \overline{E}^{\tau_k}$. Hence $I \in (\overline{E}^{\tau_k})^\sim$, and this proves (i).

(ii) Since obviously $(E \cup F)^\sim = \widetilde{E} \cup \widetilde{F}$ for subsets $E, F \subset \mathrm{Id}(A)$ it is easy to see by (i) that the operation of taking the τ_c-sequential closure satisfies the Kuratowski closure axioms.

(iii) τ_c-sequentially closed sets are clearly τ_∞-closed by Proposition 21 and so τ_k-closed for all k. Conversely, let $E \subset \text{Min-Primal}(A)$ be relatively τ_k-closed. Then $\overline{E}^{\tau_k} \cap \text{Min-Primal}(A) = E$ and this clearly implies $(\overline{E}^{\tau_k})^\sim \cap \text{Min-Primal}(A) = E$ because $\overline{E}^{\tau_k} \subset \mathrm{Primal}(A)$. By (i), E is relatively τ_{cs}-closed. So all topologies in question coincide on Min-Primal(A).

Let $H := \kappa_1^{-1}(\mathrm{Primal}(A))$, which is a closed subset of $\mathcal{S}_1$, and let $\max(H)$ be the set of maximal elements in H. Clearly $H = \{q_I : I \in \text{Min-Primal}(A)\}$ and so Min-Primal(A), which is a Hausdorff space (this is even true for the relative τ_w-topology), is a continuous image of $\max(H)$. Therefore it is sufficient to show that $\max(H)$ is a Polish space, i.e. a G_δ-set in $\mathcal{S}_1$. Define

$$H_1 := \{(p,q) \in H^2 : p \leq q\}, \quad H_2 := \{(p,q) \in H^2 : p \geq q\}.$$

We have $p \in H \setminus \max(H)$ iff there is a $q \in H$ such that $p \leq q$ and $p \ngeq q$ iff $p \in \mathrm{pr}_1(H^2 \cap H_1 \setminus H_2)$, where pr_1 denotes the projection onto the first coordinate. Since open sets in $\mathcal{S}_1$ are F_σ-sets, $H^2 \cap H_1 \setminus H_2$ is F_σ, hence σ-compact, and so is the continuous image $\mathrm{pr}_1(H^2 \cap H_1 \setminus H_2)$. Therefore $\max(H) = H \setminus \mathrm{pr}_1(H^2 \cap H_1 \setminus H_2)$ is a G_δ-set. This proves (iii).

(iv) Now let $(\mathrm{Id}(A), \tau_\infty)$ be Hausdorff, hence metrizable by Theorem 23. Let $X := \mathrm{Primal}(A)$, which is τ_∞-closed. Then

$$G_1 := \{(I,J) \in X^2 : I \subset J\}, \quad G_2 := \{(I,J) \in X^2 : I \supset J\}$$

are τ_∞-closed by Lemma 24. The same arguments as in (iii) now yield that Min-Primal$(A) = X \setminus \mathrm{pr}_1(X^2 \cap G_2 \setminus G_1)$ is a G_δ-set in X. This finishes the proof of the theorem. ∎

Part (iii) of the above theorem obviously generalizes Corollary 4.3(a) of [1], and part (iv) is a generalization of Corollary 5.2 of [5]. For all this it would be an advantage to know whether τ_∞ and/or τ_c satisfy the first (or even better the second) countability axiom if the Banach algebra is separable, but this must remain an open question here.

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