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# Tracial states on crossed products associated with Furstenberg transformations on the 2-torus

by

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**Abstract.** Let $\phi_f$ be a Furstenberg transformation on the 2-torus $\mathbb{T}^2$ defined by $\phi_f(x,y) = (e^{2\pi i\theta}x, e^{2\pi i f(x)}xy)$ for any $x, y \in \mathbb{T}$, where $\theta$ is an irrational number and $f$ is a real-valued continuous function on the 1-torus $\mathbb{T}$. Let $A(\phi_f)$ be the crossed product associated with $\phi_f$. We show that $A(\phi_f)$ has a unique tracial state for any irrational number $\theta$ and any real-valued continuous function $f$ on $\mathbb{T}$.

**1. Introduction.** Let $\theta$ be an irrational number in $(0,1)$ and $f$ a real-valued continuous function on the 1-torus $\mathbb{T}$. Let $\phi_f$ be a Furstenberg transformation on the 2-torus $\mathbb{T}^2$ defined by

$$\phi_f(x,y) = (e^{2\pi i\theta}x, e^{2\pi i f(x)}xy)$$

for any $x, y \in \mathbb{T}$. Let $A(\phi_f)$ be the associated crossed product $C(\mathbb{T}^2) \times_{\phi_f} \mathbb{Z}$.

In [5] Rouhani gave the following definition, result and question:

DEFINITION. We say that a real-valued continuous function $f$ on $\mathbb{T}$ *can be split with respect to* $e^{2\pi i\theta} \in \mathbb{T}$ if it can be written as

$$f(x) = g(x) - g(e^{2\pi i\theta}x) + c \quad \text{a.e.}$$

for some real-valued measurable function $g$ on $\mathbb{T}$ and some real constant $c$.

PROPOSITION. *If a real-valued continuous function $f$ on $\mathbb{T}$ can be split with respect to $e^{2\pi i\theta}$, then the associated crossed product $A(\phi_f)$ has a unique tracial state.*

QUESTION. If we drop the assumption that $f$ can be split with respect to $e^{2\pi i\theta}$ in the above proposition, can we still conclude that $A(\phi_f)$ has a unique tracial state?

In this note we give an affirmative answer to this question.

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**2. Result.** Let $C(\mathbb{T})$ be the $C^*$-algebra of all complex-valued continuous functions on $\mathbb{T}$ and $C(\mathbb{T})^{**}$ its enveloping von Neumann algebra, which is identified with its second dual.

Let $u$ be a unitary element in $C(\mathbb{T})$ defined by $u(x) = x$ for any $x \in \mathbb{T}$. Let $P(\mathbb{T})$ be a dense $*$-subalgebra of $C(\mathbb{T})$ generated by $u$ and $P(\mathbb{T})_{\mathrm{sa}}$ a subset of all selfadjoint elements in $P(\mathbb{T})$.

LEMMA 1. *For any $f \in P(\mathbb{T})_{\mathrm{sa}}$ there is an element $g \in P(\mathbb{T})_{\mathrm{sa}}$ such that*

$$f(x) - \int_{\mathbb{T}} f(x)\,dx = g(x) - g(e^{2\pi i\theta}x).$$

Proof. Since $f \in P(\mathbb{T})$, there is a finite set $\{a_n\}_{n=-N}^{N}$ $(N \geq 0)$ of complex numbers such that $f(x) = \sum_{n=-N}^{N} a_n x^n$. Since $f^* = f$,

$$\sum_{n=-N}^{N} a_n x^n = \sum_{n=-N}^{N} \overline{a}_n x^{-n} = \sum_{n=-N}^{N} \overline{a}_{-n} x^n.$$

Hence $a_n = \overline{a}_{-n}$ for $n = -N, \ldots, N$. Let

$$b_n = \frac{a_n}{1 - e^{2\pi in\theta}} \quad \text{if } n \neq 0$$

and $b_0 = 0$. Then if $n \neq 0$,

$$\overline{b}_n = \overline{\frac{a_n}{1 - e^{2\pi in\theta}}} = \frac{a_{-n}}{1 - e^{-2\pi in\theta}} = b_{-n}$$

and $b_0 = \overline{b}_0 = 0$. Let $g(x) = \sum_{n=-N}^{N} b_n x^n$. Then $g \in P(\mathbb{T})_{\mathrm{sa}}$ and for any $x \in \mathbb{T}$,

$$\begin{aligned} g(x) - g(e^{2\pi i\theta}x) &= \sum_{n=-N}^{N} b_n(1 - e^{2\pi in\theta})x^n = \sum_{n\neq 0} a_n x^n \\ &= \sum_{n=-N}^{N} a_n x^n - a_0 = f(x) - \int_{\mathbb{T}} f(x)\,dx. \ \blacksquare \end{aligned}$$

Let $f$ be a real-valued continuous function on $\mathbb{T}$. Then there is a sequence $\{f_n\}$ of selfadjoint elements in $P(\mathbb{T})$ such that $\|f_n - f\| \to 0$ $(n \to \infty)$. By Lemma 1 for $f_n$ there is a $g_n$ in $P(\mathbb{T})_{\mathrm{sa}}$ such that for any $x \in \mathbb{T}$,

$$f_n(x) - \int_{\mathbb{T}} f_n(x)\,dx = g_n(x) - g_n(e^{2\pi i\theta}x).$$

Let $c_n = \int_{\mathbb{T}} f_n(x)\,dx$ and $c = \int_{\mathbb{T}} f(x)\,dx$. Then $c_n \to c$ $(n \to \infty)$ and

$$f_n(x) - c_n = g_n(x) - g_n(e^{2\pi i\theta}x) \quad (x \in \mathbb{T}).$$

Hence we obtain

$$e^{-2\pi ic_n} e^{2\pi if_n(x)} = e^{2\pi ig_n(x)} e^{-2\pi ig_n(e^{2\pi i\theta}x)} \quad (x \in \mathbb{T}).$$



Let $F_n(x) = e^{2\pi if_n(x)}$, $G_n(x) = e^{2\pi ig_n(x)}$, $F(x) = e^{2\pi if(x)}$ and $\lambda_n = e^{2\pi ic_n}$, $\lambda = e^{2\pi ic}$. Then clearly $\|F_n - F\| \to 0$, $\lambda_n \to \lambda$ $(n \to \infty)$ and

$$\overline{\lambda}_n F_n(x) = G_n(x)\overline{G_n(e^{2\pi i\theta}x)}$$

for any $x \in \mathbb{T}$. Thus we obtain

$$F_n(x)G_n(e^{2\pi i\theta}x) = \lambda_n G_n(x)$$

for any $x \in \mathbb{T}$ and $n \in \mathbb{N}$.

LEMMA 2. *With the above notations there is a $G \in L^\infty(\mathbb{T})$ with $|G(x)| = 1$ for any $x \in \mathbb{T}$ such that*

$$F(x)G(e^{2\pi i\theta}x) = \lambda G(x) \quad (x \in \mathbb{T}).$$

Proof. Since $\|G_n\| = 1$ for any $n \in \mathbb{N}$ and the unit ball of $C(\mathbb{T})^{**}$ is weak* compact, there are a $\widetilde{G} \in C(\mathbb{T})^{**}$ and a subsequence $\{G_{n_j}\}$ of $\{G_n\}$ such that $G_{n_j} \to \widetilde{G}$ $(j \to \infty)$ with respect to the weak* topology. For any $x \in \mathbb{T}$ let $\delta_x$ be the pure state of point evaluation on $C(\mathbb{T})$. Since $\delta_x \in C(\mathbb{T})^*$,

$$\delta_x(G_{n_j}) - \delta_x(\widetilde{G}) \to 0 \quad (j \to \infty)$$

for any $x \in \mathbb{T}$. Since $\delta_x(G_{n_j}) = G_{n_j}(x)$,

$$G_{n_j}(x) \to \delta_x(\widetilde{G}) \in \mathbb{C} \quad (j \to \infty)$$

for any $x \in \mathbb{T}$. Let $G$ be the function on $\mathbb{T}$ defined by $G(x) = \delta_x(\widetilde{G}) = \lim_{j\to\infty} G_{n_j}(x)$. Since $G_{n_j}$ is continuous, it is measurable. Hence $G$ is measurable. And for any $x \in \mathbb{T}$,

$$\big||G(x)| - 1\big| = \big||\delta_x(\widetilde{G})| - |G_{n_j}(x)|\big| \leq |\delta_x(\widetilde{G}) - G_{n_j}(x)| \to 0 \quad (j \to \infty).$$

Thus $|G(x)| = 1$ for any $x \in \mathbb{T}$. Hence $G \in L^\infty(\mathbb{T})$. Furthermore, since

$$F_{n_j}(x)G_{n_j}(e^{2\pi i\theta}x) = \lambda_{n_j} G_{n_j}(x)$$

for any $x \in \mathbb{T}$ and $j \in \mathbb{N}$, we obtain $F(x)G(e^{2\pi i\theta}x) = \lambda G(x)$ $(x \in \mathbb{T})$ since $\|F_{n_j} - F\| \to 0$, $\lambda_{n_j} \to \lambda$ $(j \to \infty)$. ∎

Let $\phi_f$ be a Furstenberg transformation induced by a real-valued continuous function $f$ on $\mathbb{T}$.

THEOREM 3. *With the above notations $\phi_f$ is uniquely ergodic.*

Proof. In the same way as in Rouhani [5, Proof of Theorem 2.1], in order to prove that $\phi_f$ is uniquely ergodic it suffices to show that the equation

$$H(e^{2\pi i\theta}x) = e^{2\pi ikf(x)}x^k H(x) \quad (\text{a.e. } x \in \mathbb{T}),$$

for any $k \in \mathbb{Z} \setminus \{0\}$, has no measurable solution $H : \mathbb{T} \to \mathbb{T}$. So let us assume that such an $H$ exists, so that $H \in L^2(\mathbb{T})$. Since $F = e^{2\pi i f}$, we have

$$H(e^{2\pi i\theta}x) = F(x)^k x^k H(x) \quad \text{(a.e. } x \in \mathbb{T})$$

for some $k \in \mathbb{Z} \setminus \{0\}$. Hence

$$H(e^{2\pi i\theta}x)\overline{H(x)} = F(x)^k x^k \quad \text{(a.e. } x \in \mathbb{T}). \tag{1}$$

And by Lemma 2 we also have

$$F(x)^k G(e^{2\pi i\theta}x)^k = \lambda^k G(x)^k \quad (x \in \mathbb{T}). \tag{2}$$

By (1) and (2) we obtain

$$H(e^{2\pi i\theta}x)G(e^{2\pi i\theta}x)^k = \lambda^k x^k H(x)G(x)^k \quad \text{(a.e. } x \in \mathbb{T}). \tag{3}$$

Let $h(x) = H(x)G(x)^k$ for any $x \in \mathbb{T}$. Then $|h(x)| = |H(x)||G(x)|^k = 1$ for any $x \in \mathbb{T}$. Hence $h \in L^\infty(\mathbb{T})$. By (3) we see that

$$h(e^{2\pi i\theta}x) = \lambda^k x^k h(x) \quad \text{(a.e. } x \in \mathbb{T}). \tag{4}$$

Since $L^\infty(\mathbb{T}) \subset L^2(\mathbb{T})$, $h$ can be represented by its Fourier series, $h(x) = \sum_{n=-\infty}^{\infty} a_n x^n$, where $\{a_n\} \in l^2(\mathbb{Z})$. Thus by (4),

$$\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n\theta} x^n = \lambda^k \sum_{n=-\infty}^{\infty} a_n x^{n+k}.$$

Therefore we deduce that $|a_n| = |a_{n-k}|$ for any $n \in \mathbb{Z}$. But since $\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$ and $k \neq 0$, we see that $a_n = 0$ for any $n \in \mathbb{Z}$. Thus $h(x) = 0$ (a.e. $x \in \mathbb{T}$). On the other hand, $|h(x)| = 1$ for any $x \in \mathbb{T}$. This is a contradiction. Thus $\phi_f$ is uniquely ergodic. ∎

Remark. In the above case since $\phi_f$ is minimal, $A(\phi_f)$ has a unique tracial state if and only if $\phi_f$ is uniquely ergodic by Tomiyama [7, Corollary 3.3.10].

COROLLARY 4. *With the above notations $A(\phi_f)$ has a unique tracial state.*

Proof. This is immediate by Remark. ∎

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