(The reader can take the second equality as a definition of the multiplicity of the value 0 of f.)

This completes the proof of Theorem 1.

We remark finally that part (a) of Theorem 1 remains true, with an identical proof, on any bounded domain of $\mathbb{C}^n$ on which the H^p Corona Problem is solvable.

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L^p weighted inequalities for the dyadic square function

by

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Abstract. We prove that

$$\int (S_d f)^p V\,dx \le C_{p,n} \int |f|^p M_d^{([p/2]+2)} V\,dx,$$

where S_d is the dyadic square function, $M_d^{(k)}$ is the k-fold application of the dyadic Hardy–Littlewood maximal function and $p > 2$.

1. Introduction. Let $V(x) \ge 0$. S. Y. Chang, J. M. Wilson and T. H. Wolff [CWW] showed that if $p = 2$, then

$$\int_{\mathbb{R}^n} S_\psi f(x)^p V(x)\,dx \le C_{p,\psi,n} \int_{\mathbb{R}^n} |f(x)|^p MV(x)\,dx, \tag{1.1}$$

where $S_\psi f$ is the square function of f with respect to the kernel function ψ that satisfies certain strict conditions and where Mf is the Hardy–Littlewood maximal function of f. S. Chanillo and R. L. Wheeden [CW] showed that (1.1) holds for $1 < p \le 2$ and fails for $p > 2$. (Furthermore, they relaxed the conditions on ψ.) J. M. Wilson [W6] extended (1.1) to the case $0 < p \le 1$ by replacing $|f(x)|$ by a certain maximal function of f. Then the remaining problem is to get inequalities that are similar to (1.1) and that hold for the case $p > 2$. In Derrick [D] the following problem is listed. (See also [W6], p. 293.)

J. M. Wilson's Problem. Let S_d be the dyadic square function. Let $M^{(1)}f = Mf$, $M^{(2)}f = M(Mf)$, ... Then, is the following inequality true:

$$\int_{\mathbb{R}^n} S_d f(x)^p V(x)\,dx \le C_{p,n} \int_{\mathbb{R}^n} |f(x)|^p M^{(k(p))} V(x)\,dx,$$

as $p \to \infty$, with $k(p) \sim p/2$? In particular, with $k(p) = -[-p/2]$?

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In this paper we investigate this problem. Our result is still incomplete.

2. Results

NOTATION. $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of all real numbers, integers and natural numbers, respectively. We fix the dimension n ($\in \mathbb{N}$). For $k \in \mathbb{Z}$, let D_k be the set of all cubes in $\mathbb{R}^n$ of the form

$$[2^{-k}j_1, 2^{-k}(j_1+1)) \times \ldots \times [2^{-k}j_n, 2^{-k}(j_n+1)),$$

where $j_1, \ldots, j_n \in \mathbb{Z}$. Let

$$D = \bigcup_{k\in\mathbb{Z}} D_k,$$

that is, D is the set of all dyadic cubes in $\mathbb{R}^n$. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let

$$E_k f(x) = 2^{kn} \int_{I(x,k)} f(y)\,dy,$$

where $I(x,k) \in D_k$ and $I(x,k) \ni x$, that is, E_k is the conditional expectation with respect to the sub-σ-field generated by D_k. Let

$$S_d f(x) = \Big(\sum_{k\in\mathbb{Z}} (E_k f(x) - E_{k-1} f(x))^2\Big)^{1/2},$$

$$M_d f(x) = \sup_{k\in\mathbb{Z}} E_k|f|(x).$$

Let

$$M_d^{(1)} f = M_d f, \quad M_d^{(k+1)} f = M_d(M_d^{(k)} f) \quad (k = 1, 2, \ldots).$$

Remark 2.1. All functions considered in this paper are real-valued.

Our result is the following.

THEOREM. *Let $2 < p < \infty$, $f \in \bigcup_{1\le q<\infty} L^q(\mathbb{R}^n)$, $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $V(x) \ge 0$. Then*

$$\int_{\mathbb{R}^n} S_d f(x)^p V(x)\,dx \le C \int_{\mathbb{R}^n} M_d f(x)^p M_d^{([p/2]+1)} V(x)\,dx, \tag{2.1}$$

$$\int_{\mathbb{R}^n} S_d f(x)^p V(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p M_d^{([p/2]+2)} V(x)\,dx, \tag{2.2}$$

where $[p/2]$ is the greatest integer not exceeding $p/2$ and where C is a constant depending only on p and n.

Remark 2.2. (2.1) and (2.2) can be extended to the cases $0 < p \le 2$ and $1 < p \le 2$, respectively. But in these cases better results are known.

The arguments of [CWW], [CW] and [W6] show that

$$\int S_d f(x)^p V(x)\,dx \le C \int |f(x)|^p M_d V(x)\,dx \quad \text{if } 1 < p \le 2,$$

$$\int S_d f(x)^p V(x)\,dx \le C \int \sup_k |E_k f(x)|^p M_d V(x)\,dx \quad \text{if } 0 < p \le 1.$$

Remark 2.3. For our case $2 < p < \infty$, the argument of [CW], Theorem 2, shows

$$\int S_d f(x)^p V(x)\,dx \le C \int |f(x)|^p \Big(\frac{M_d V(x)}{V(x)}\Big)^{p/2} V(x)\,dx.$$

[W6], Theorem 6, gave a little bit more complicated result.

Remark 2.4. The classical theory using the A_1-condition shows

$$\int S_d f(x)^p V(x)\,dx \le C \int |f(x)|^p M_d(V^{1+\varepsilon})(x)^{1/(1+\varepsilon)}\,dx$$

if $1 < p < \infty$ and $\varepsilon > 0$. Our $M_d^{([p/2]+2)} V$ is smaller than $CM_d(V^{1+\varepsilon})^{1/(1+\varepsilon)}$. (In Remarks 2.2–2.4, the C's depend only on p, n (and ε).)

For the proof of our Theorem we need more notation.

NOTATION (continued). For a measurable set $\Omega \subset \mathbb{R}^n$ let Ω^c, χ_Ω and $|\Omega|$ denote the complement, the characteristic function and the Lebesgue measure of Ω, respectively. For $V(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ let

$$V\{\Omega\} = \int_\Omega V(x)\,dx, \quad \mathrm{av}(V, \Omega) = \int_\Omega V\,dx/|\Omega|.$$

For a nonnegative function $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $Q \in D$ and $\eta > 0$ let

$$Y(V,Q,\eta) = \begin{cases} \frac{1}{V\{Q\}} \int_Q V(x)\big(1 + \log^+ \frac{V(x)}{\mathrm{av}(V,Q)}\big)^\eta\,dx & \text{if } V\{Q\} > 0,\\ 1 & \text{if } V\{Q\} = 0.\end{cases}$$

Very often we abbreviate $L^p(\mathbb{R}^n)$, $\|\cdot\|_{L^p}$, $\int_{\mathbb{R}^n} f(x)\,dx$ and $\{x \in \mathbb{R}^n : f(x) > \lambda\}$ to L^p, $\|\cdot\|_p$, $\int f\,dx$ and $\{f > \lambda\}$, respectively.

Remark 2.5. We borrowed $Y(V,Q,\eta)$ and the main idea of our proof from J. M. Wilson [W1]–[W7], where he investigated the inequalities of the type

$$\int \sup_k |E_k f|^p \cdot V\,dx \le C \int (Sf)^p MV\,dx$$

as well as our type (1.1).

3. Preliminaries I

LEMMA 3.1 *Let $f \in \bigcup_{1\le q<\infty} L^q$. Then there exist $\{a_Q\}_{Q\in D} \subset L^\infty$ and $\{\lambda_Q\}_{Q\in D} \subset \mathbb{R}$ so that*

$$a_Q(x) = 0 \quad \text{on } Q^c, \tag{3.1}$$

$$\int a_Q\,dx = 0, \tag{3.2}$$

$$(3.3)\qquad \|a_Q\|_{L^\infty} \le 1,$$

$$(3.4)\qquad \lambda_Q \in \{2^k : k \in \mathbb{Z}\} \cup \{0\},$$

$$(3.5)\qquad \textit{if } \lambda_P = \lambda_Q \neq 0, \textit{ then } P \cap Q = \emptyset \textit{ or } P = Q,$$

$$(3.6)\qquad \sum_{Q\in D} \lambda_Q \chi_Q(x) \le C M_d f(x),$$

$$(3.7)\qquad f(x) = \sum_{Q\in D} \lambda_Q a_Q(x) \quad \textit{a.e.},$$

where D is the collection of all dyadic cubes in $\mathbb{R}^n$ and where C depends only on n.

Proof. Let $k \in \mathbb{Z}$. Let $\{Q_{k,j}\}_{j=1,2,\ldots}$ be the maximal elements with respect to inclusion among the cubes Q satisfying $Q \in D$ and $\mathrm{av}(|f|, Q) > 2^{kn}$. Then

$$(3.8)\qquad \{Q_{k,j}\}_j \text{ are mutually disjoint},$$

$$(3.9)\qquad \mathrm{av}(|f|, Q_{k,j}) \le 2^{(k+1)n}.$$

Since $f \in L^q$ for some $q \in [1, \infty)$, we have

$$(3.10)\qquad \bigcup_j Q_{k,j} = \{M_d f > 2^{kn}\},$$

in particular,

$$(3.11)\qquad \Big\|\Big(1 - \sum_j \chi_{Q_{k,j}}\Big) f\Big\|_\infty \le 2^{kn}.$$

Moreover,

$$(3.12)\qquad \Big|\bigcup_j Q_{k,j}\Big| \to 0 \quad (k \to \infty),$$

(3.13) for each $Q_{k+1,i}$ there exists $Q_{k,j}$ so that

$$Q_{k,j} \supset Q_{k+1,i} \text{ and } Q_{k,j} \neq Q_{k+1,i}.$$

By (3.8) and (3.13) the $Q_{k,j}$ ($k \in \mathbb{Z}$, $j = 1, 2, \ldots$) are all distinct.

Next, we take the "good part" of the Calderón–Zygmund decomposition of f with respect to 2^k, namely let

$$g_k = \Big(1 - \sum_j \chi_{Q_{k,j}}\Big) f + \sum_j \mathrm{av}(f, Q_{k,j}) \chi_{Q_{k,j}}$$

for $k \in \mathbb{Z}$. (If $\|f\|_\infty \le 2^{kn}$, then $\{Q_{k,j}\}_j$ is empty and $g_k = f$.) Then

$$(3.14)\qquad \|g_k\|_\infty \le 2^{(k+1)n} \quad \text{by (3.9) and (3.11)},$$

$$(3.15)\qquad \int_{Q_{k,j}} g_{k+1}\,dx = \int_{Q_{k,j}} f\,dx = \int_{Q_{k,j}} g_k\,dx \quad \text{by (3.13) and (3.8)},$$

$$(3.16)\qquad g_{k+1} - g_k = 0 \quad \text{on } \Big(\bigcup_j Q_{k,j}\Big)^{\mathrm{c}} \quad \text{by (3.13)},$$

$$(3.17)\qquad f = \lim_{k\to+\infty} g_k \quad \text{by (3.12)}$$

$$= \lim_{k\to+\infty} (g_k - g_{-k}) \quad \text{by (3.14)}$$

$$= \sum_{k=-\infty}^{+\infty} (g_{k+1} - g_k) \quad \text{a.e.}$$

For each $Q_{k,j}$ set $b_{k,j} = (g_{k+1} - g_k)\chi_{Q_{k,j}}$. Then

$$(3.18)\qquad b_{k,j} = 0 \quad \text{on } Q_{k,j}^{\mathrm{c}},$$

$$(3.19)\qquad \int b_{k,j}\,dx = 0 \quad \text{by (3.15)},$$

$$(3.20)\qquad \|b_{k,j}\|_\infty \le 2^{3n} \cdot 2^{kn} \quad \text{by (3.14)},$$

$$(3.21)\qquad \sum_j b_{k,j} = g_{k+1} - g_k \quad \text{by (3.16) and (3.8)}.$$

Finally, we define $\{a_Q\}$ and $\{\lambda_Q\}$. Let $Q \in D$.

Case 1. If there exists $Q_{k,j}$ so that

$$(3.22)\qquad Q = Q_{k,j},$$

then set

$$a_Q = 2^{-(k+3)n} b_{k,j} \quad \text{and} \quad \lambda_Q = 2^{(k+3)n}.$$

(Recall that for each $Q \in D$ at most one $Q_{k,j}$ satisfies (3.22).)

Case 2. If there does not exist $Q_{k,j}$ that satisfies (3.22), then set

$$a_Q \equiv 0 \quad \text{and} \quad \lambda_Q = 0.$$

Then the desired properties (3.1)–(3.7) follow from (3.8), (3.10) and (3.17)–(3.21). ∎

Remark 3.1. This is an application of the argument of [C]. This kind of argument might be implicit in [Gs].

LEMMA 3.2 *Let $Q \in D$. Let $a_Q \in L^\infty$ satisfy* (3.1)–(3.3). *Let $\lambda > 0$. Then*

$$(3.23)\qquad S_d a_Q(x) = 0 \quad \textit{on } Q^{\mathrm{c}},$$

$$(3.24)\qquad |\{x \in Q : S_d a_Q(x) > \lambda\}| \le C \exp(-\lambda^2/C)|Q|,$$

where C depends only on n.

Proof. (3.23) is clear from (3.1)–(3.2). Take any $P \in D$. Set

$$c_0 = \sum_{k:\, 2^{-k} \ge l(P)} (E_k a_Q(x_0) - E_{k-1} a_Q(x_0))^2,$$

where $x_0 \in P$ and where $l(P)$ denotes the edge length of P. Then

$$\int_P |S_d a_Q(x)^2 - c_0|\,dx = \int_P \sum_{k:\,2^{-k}<l(P)} (E_k a_Q(x) - E_{k-1}a_Q(x))^2\,dx$$
$$= \int_P (a_Q(x) - \mathrm{av}(a_Q,P))^2 dx \le |P| \quad \text{by (3.3).}$$

So, the dyadic-BMO norm of $(S_d a_Q)^2$ is at most 1. Then (3.24) follows from the John–Nirenberg inequality and from

$$\mathrm{av}((S_d a_Q)^2, Q) \le 1,$$

which follows from (3.1)–(3.3). (For the dyadic BMO and the John–Nirenberg inequality see [Gn], pp. 274 and 230.) ∎

The following two lemmas are easy. We omit their proofs.

LEMMA 3.3. *Let G ($\neq \emptyset$) be a subset of D. Let $G' \subset G$. Suppose that to each $Q \in G$ there corresponds $a_Q \in L^1$. Let $\{a_Q\}_{Q\in G}$ satisfy* (3.1), (3.2) *and*

$$\sum_{Q\in G} |a_Q(x)| \in L^1_{\mathrm{loc}}.$$

Then

$$S_d\Big(\sum_{Q\in G} a_Q\Big)(x) = S_d\Big(\sum_{Q\in G\setminus G'} a_Q\Big)(x) \quad \text{on } \Big(\bigcup_{Q\in G'} Q\Big)^c.$$

LEMMA 3.4. *Let $(\emptyset \neq) G \subset D$. Suppose that to each $Q \in G$ there corresponds $\lambda_Q \in \mathbb{R}$. Let $\{\lambda_Q\}_{Q\in G}$ satisfy* (3.4) *and* (3.5). *Let $0 < p < \infty$. Then*

$$C^{-1}\sum_{Q\in G} \lambda_Q^p \chi_Q(x) \le \Big(\sum_{Q\in G}\lambda_Q\chi_Q(x)\Big)^p \le C\sum_{Q\in G}\lambda_Q^p\chi_Q(x),$$

where C depends only on p.

4. Preliminaries II. Recall the definition of $Y(V,Q,\eta)$.

LEMMA 4.1. *Let $\eta > 0$. Let $Q \in D$. Let $E \subset Q$ be a measurable set. Let $V \in L^1_{\mathrm{loc}}$, $V(x) \ge 0$ and $V\{Q\} > 0$. Then*

$$V\{E\}/V\{Q\} \le CY(V,Q,\eta)(\log(|Q|/|E|))^{-\eta}, \tag{4.1}$$

where C depends only on η.

Proof. We may assume $V\{E\} > 0$. Set

$$E' = \{x \in E : V(x) > \mathrm{av}(V,E)/2\}.$$

Then

$$V\{E'\} = V\{E\} - V\{E\setminus E'\} \ge V\{E\} - V\{E\}/2 = V\{E\}/2. \tag{4.2}$$

So,

$$Y(V,Q,\eta) \ge \frac{1}{V\{Q\}}\int_{E'} V(x)\Big(1 + \log^+\frac{\mathrm{av}(V,E)}{2\mathrm{av}(V,Q)}\Big)^\eta dx$$
$$\ge \frac{1}{2}\frac{V\{E\}}{V\{Q\}}\Big(\log^+\frac{V\{E\}|Q|}{2V\{Q\}|E|}\Big)^\eta \quad \text{by (4.2).}$$

So,

$$\frac{V\{E\}/V\{Q\}}{2|E|/|Q|}\Big(\log^+\frac{V\{E\}/V\{Q\}}{2|E|/|Q|}\Big)^\eta \le \frac{Y(V,Q,\eta)}{|E|/|Q|}. \tag{4.3}$$

Put $\mathrm{h}(t) = t(\log t)^{-\eta}$. If

$$\frac{V\{E\}/V\{Q\}}{|E|/|Q|} > C_\eta,$$

then

h(the left-hand side of (4.3)) $\le$ h(the right-hand side of (4.3)),

which implies (4.1); else (4.1) is clear. ∎

LEMMA 4.2. *Let $(\emptyset \neq) G \subset D$. Suppose that to each $Q \in G$ there correspond $a_Q \in L^\infty$ and $\lambda_Q \in \mathbb{R}$. Let $\{a_Q\}_{Q\in G}$ and $\{\lambda_Q\}_{Q\in G}$ satisfy* (3.1)–(3.5) *and*

$$\sum_{Q\in G}\lambda_Q\chi_Q \in L^1_{\mathrm{loc}}.$$

Set

$$u(x) = \sum_{Q\in G}\lambda_Q a_Q(x).$$

Let $\eta > 0$, $V \in L^1_{\mathrm{loc}}$, $V(x) \ge 0$ and set

$$A = \sup_{Q\in G} Y(V,Q,\eta).$$

Then the following hold.

(i) *If $k \in \mathbb{Z}$, $m \in \mathbb{N}$ and $\varepsilon \in (0,1)$, then*

$$V\{x \in \mathbb{R}^n : S_d u(x) > 2^k\} \tag{4.4}$$
$$\le V\Big\{x \in \mathbb{R}^n : \sum_{Q\in G}\lambda_Q\chi_Q(x) > 2^{k-m}\Big\}$$
$$+ \sum_{h=-\infty}^{k-m} \min\{CA2^{-2\eta\varepsilon m}2^{-2\eta(1-\varepsilon)(k-h)}, 1\}$$
$$\times V\Big\{x \in \mathbb{R}^n : \sum_{Q\in G}\lambda_Q\chi_Q(x) > 2^h\Big\},$$

where C depends only on η, ε and n.

(ii) *If*

$$p \in (0, 2\eta), \tag{4.5}$$

then

$$\int S_d u(x)^p V(x)\,dx \le CA^{p/(2\eta)} \int \Big(\sum_{Q\in G} \lambda_Q \chi_Q(x)\Big)^p V(x)\,dx, \tag{4.6}$$

where C depends only on p, η and n.

Proof of (i). Let

$$\widetilde{u} = \sum_{Q\in G:\,\lambda_Q \le 2^{k-m}} \lambda_Q a_Q \quad \text{and} \quad \Omega = \bigcup_{Q\in G:\,\lambda_Q > 2^{k-m}} Q.$$

Then Lemma 3.3 implies $S_d u(x) = S_d \widetilde{u}(x)$ on Ω^c. So,

$$\{S_d u > 2^k\} \subset \Omega \cup \{S_d \widetilde{u} > 2^k\} \subset \Big\{\sum_{Q\in G} \lambda_Q \chi_Q > 2^{k-m}\Big\} \cup \{S_d \widetilde{u} > 2^k\}. \tag{4.7}$$

On the other hand,

$$\begin{aligned} \{S_d \widetilde{u} > 2^k\} &\subset \Big\{\sum_{h=-\infty}^{k-m} 2^h \sum_{Q\in G:\,\lambda_Q = 2^h} S_d a_Q > 2^k\Big\} \\ &\subset \bigcup_{h=-\infty}^{k-m} \Big\{\sum_{Q\in G:\,\lambda_Q = 2^h} S_d a_Q > c_\varepsilon 2^{k-h-\varepsilon(k-m-h)}\Big\} \\ &= \bigcup_{h=-\infty}^{k-m} \bigcup_{Q\in G:\,\lambda_Q = 2^h} \{S_d a_Q > c_\varepsilon 2^{k-h-\varepsilon(k-m-h)}\} \\ &= \bigcup\bigcup E_Q, \quad \text{say.} \end{aligned} \tag{4.8}$$

The first equality of (4.8) follows from the fact that the sets $\{S_d a_Q > 0\}$, where $Q \in G$ and $\lambda_Q = 2^h$, are mutually disjoint by (3.5) and (3.23). Note that $E_Q \subset Q$ by (3.23). Then

$$\begin{aligned} &\sum_{Q\in G:\,\lambda_Q = 2^h} V\{E_Q\} \\ &\le \sum \min\{CA(\log(|Q|/|E_Q|))^{-\eta}, 1\} V\{Q\} \quad \text{by Lemma 4.1} \\ &\le \min\{CA(\log^+(C^{-1}\exp((c2^{k-h-\varepsilon(k-m-h)})^2/C)))^{-\eta}, 1\} \\ &\quad \times \sum_{Q\in G:\,\lambda_Q = 2^h} V\{Q\} \quad \text{by (3.24) with } \lambda = c2^{k-h-\varepsilon(k-m-h)} \\ &\le \min\{CA\max\{c' 2^{2\varepsilon m + 2(1-\varepsilon)(k-h)} - C, 0\}^{-\eta}, 1\} V\Big\{\sum_{Q\in G} \lambda_Q \chi_Q \ge 2^h\Big\} \\ &\hspace{20em} \text{by (3.5)} \end{aligned} \tag{4.9}$$

$$\begin{aligned} &= \min\{CA\max\{\ldots, 1\}^{-\eta}, 1\} V\{\ldots\} \quad \text{by } CA \ge 1 \\ &\le \min\{CA2^{-2\eta\varepsilon m} 2^{-2\eta(1-\varepsilon)(k-h)}, 1\} V\Big\{\sum_{Q\in G} \lambda_Q \chi_Q \ge 2^h\Big\}. \end{aligned}$$

So, substituting (4.8)–(4.9) into (4.7) gives (4.4). ■

Proof of (ii). Take $\varepsilon \in (0,1)$ and $m \in \mathbb{N}$ so that

$$p < 2\eta(1-\varepsilon), \tag{4.10}$$

$$2^{2\eta m} \approx A. \tag{4.11}$$

Then

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} 2^{kp} V\{S_d u > 2^k\} \\ &\le \sum_{k\in\mathbb{Z}} 2^{kp} V\Big\{\sum_{Q\in G} \lambda_Q \chi_Q > 2^{k-m}\Big\} \\ &\quad + CA2^{-2\eta\varepsilon m} \sum_{k\in\mathbb{Z}} 2^{kp} \sum_{h=-\infty}^{k-m} 2^{-2\eta(1-\varepsilon)(k-h)} V\Big\{\sum_{Q\in G} \lambda_Q \chi_Q > 2^h\Big\} \quad \text{by (4.4)} \\ &= 2^{mp} \sum_{k\in\mathbb{Z}} 2^{kp} V\Big\{\sum \lambda_Q \chi_Q > 2^k\Big\} \\ &\quad + CA2^{-2\eta\varepsilon m} \sum_{h\in\mathbb{Z}} V\Big\{\sum \lambda_Q \chi_Q > 2^h\Big\} \sum_{k=h+m}^{\infty} 2^{kp} 2^{-2\eta(1-\varepsilon)(k-h)} \\ &= \ldots + CA2^{m(p-2\eta)} \sum_{h\in\mathbb{Z}} 2^{hp} V\Big\{\sum \lambda_Q \chi_Q > 2^h\Big\} \quad \text{by (4.10)} \\ &= 2^{mp}(1 + CA2^{-2\eta m}) \sum_{h\in\mathbb{Z}} 2^{hp} V\Big\{\sum \lambda_Q \chi_Q > 2^h\Big\} \\ &\le CA^{p/(2\eta)} \sum_{h\in Z} 2^{hp} V\Big\{\sum \lambda_Q \chi_Q > 2^h\Big\} \quad \text{by (4.11)} \\ &\le CA^{p/(2\eta)} \int \Big(\sum_{Q\in G} \lambda_Q \chi_Q\Big)^p V\,dx. \ ■ \end{aligned}$$

LEMMA 4.3. *Let $\{a_Q(x)\}_{Q\in D}$ and $\{\lambda_Q\}_{Q\in D}$ satisfy* (3.1)–(3.5) *and*

$$\sum_{Q\in D} \lambda_Q \chi_Q \in L^1_{\mathrm{loc}}.$$

Set

$$u(x) = \sum_{Q\in D} \lambda_Q a_Q(x).$$

Let $V \in L^1_{\mathrm{loc}}$, $V(x) \ge 0$ *and* $0 < p < 2\eta$. *Then*

$$\int S_d u(x)^p V(x)\,dx \le C \sum_{Q\in D} \lambda_Q^p V\{Q\} Y(V,Q,\eta),$$

where C *depends only on* p, η *and* n.

Proof. For $j \in \mathbb{N}$ set

$$G_j = \{Q \in D : 2^{j-1} \le Y(V,Q,\eta) < 2^j\}, \qquad u_j = \sum_{Q\in G_j} \lambda_Q a_Q.$$

(If $G_j = \emptyset$, we define $u_j \equiv 0$.) Then

$$u = \sum_{j\in\mathbb{N}} u_j \tag{4.12}$$

and

$$\begin{aligned}(4.13)\quad \int (S_d u_j)^p V\,dx &\le C(2^j)^{p/(2\eta)} \int \Big(\sum_{Q\in G_j} \lambda_Q \chi_Q\Big)^p V\,dx \quad \text{by (4.6)}\\ &\le C2^{jp/(2\eta)} \sum_{Q\in G_j} \lambda_Q^p V\{Q\} \quad \text{by Lemma 3.4}\\ &\le C2^{j(p/(2\eta)-1)} \sum_{Q\in G_j} \lambda_Q^p V\{Q\} Y(V,Q,\eta)\end{aligned}$$

since $Y(V,Q,\eta) \approx 2^j$ for $Q \in G_j$. Take

$$\varepsilon \in (0, 1 - p/(2\eta)]. \tag{4.14}$$

Then

$$\begin{aligned}\int (S_d u)^p V\,dx &\le \int \Big(\sum_{j\in N} S_d u_j\Big)^p V\,dx \quad \text{by (4.12)}\\ &\le C \sum 2^{\varepsilon j} \int (S_d u_j)^p V\,dx \quad \text{by Hölder's inequality (if } p>1)\\ &\le C \sum_{j\in\mathbb{N}} \sum_{Q\in G_j} \lambda_Q^p V\{Q\} Y(V,Q,\eta) \quad \text{by (4.13)–(4.14).} \ \blacksquare\end{aligned}$$

5. Preliminaries III.

The lemmas in this section are known.

LEMMA 5.1. *Let* $k \in \mathbb{N}$ *and* $\gamma > 1$. *Let* $V \in L^1$ *and* $V(x) \ge 0$. *Let* $\lambda > 0$. *Then*

$$\begin{aligned}(5.1)_k\quad C^{-1}\lambda|\{x\in\mathbb{R}^n : M_d^{(k)}V(x) > \gamma\lambda\}| &\le \int_{\{x\in\mathbb{R}^n : V(x)>\lambda\}} V(x)\Big(\log\frac{V(x)}{\lambda}\Big)^{k-1} dx\\ &\le C\lambda|\{x\in\mathbb{R}^n : M_d^{(k)}V(x) > \lambda\}|,\end{aligned}$$

$$\begin{aligned}(5.2)_k\quad C^{-1} \int_{\{x\in\mathbb{R}^n : M_d^{(k)}V(x)>\gamma\lambda\}} M_d^{(k)}V(x)\,dx &\le \int_{\{x\in\mathbb{R}^n : V(x)>\lambda\}} V(x)\Big(\log\frac{V(x)}{\lambda}\Big)^k dx\\ &\le C \int_{\{x\in\mathbb{R}^n : M_d^{(k)}V(x)>\lambda\}} M_d^{(k)}V(x)\,dx,\end{aligned}$$

where C *depends only on* k, γ *and* n.

The case $k = 1$ of Lemma 5.1 is well known. The general case will be explained in Section 7.

LEMMA 5.2. *Let* $k \in \mathbb{N}$ *and*

$$Q_0 = [0,1) \times \ldots \times [0,1) \quad (\subset \mathbb{R}^n). \tag{5.3}$$

Let $V \in L^1$, $V(x) \ge 0$ *and*

$$V(x) = 0 \quad \text{on } Q_0^c. \tag{5.4}$$

Then

$$\begin{aligned}(5.5)\quad cM_d^{(k)}V(x) \le\ & (\chi_{Q_0}M_d)^{(k)}V(x)\\ &+ \sum_{j=0}^{k-1} (\log(2+|x|))^{k-1-j}(1+|x|)^{-n}\|(\chi_{Q_0}M_d)^{(j)}V\|_{L^1},\end{aligned}$$

where $c > 0$ *depends only on* k *and* n *and where*

$$(\chi_{Q_0}M_d)^{(0)}V(x) = V(x),$$
$$(\chi_{Q_0}M_d)^{(j)}V(x) = \chi_{Q_0}(x)M_d((\chi_{Q_0}M_d)^{(j-1)}V)(x) \quad (j \in \mathbb{N}).$$

The case $k = 1$ of Lemma 5.2 is clear. The rest of the proof is by induction on k.

LEMMA 5.3. *Let* $k \in \mathbb{N}$, $Q \in D$, $V \in L^1_{\mathrm{loc}}$, $V(x) \ge 0$ *and* $V\{Q\} > 0$. *Then*

$$\int_Q V(x)\Big(\log^+\frac{V(x)}{\mathrm{av}(V,Q)}\Big)^k dx \le C \int_Q M_d^{(k)}V(x)\,dx, \tag{5.6}$$

where C *depends only on* k *and* n.

Proof. We may assume that

$$Q = Q_0 \quad \text{in (5.3)}, \tag{5.7}$$

that (5.4) holds and

$$\mathrm{av}(V,Q_0) = V\{Q_0\} = 1. \tag{5.8}$$

Set

$$A = \int_{Q_0} V(x)(\log^+ V(x))^k \, dx.$$

For the proof of (5.6) (under (5.7), (5.4) and (5.8)) we may assume that

(5.9) A is very large.

If $1 \le j \le k-1$, then

$$(5.10)\quad \|(\chi_{Q_0} M_d)^{(j)} V\|_1 \le \|\chi_{Q_0} M_d^{(j)} V\|_1 \le \int_{\{M_d^{(j)} V \ge 1\}} M_d^{(j)} V \, dx \quad \text{by (5.8)}$$

$$\le C \int_{Q_0} V(\log^+ V)^j \, dx + C$$

by the first inequality of (5.2), (5.4) and (5.8)

$$\le CA^{j/k} \quad \text{by Hölder's inequality and (5.8)–(5.9).}$$

Moreover,

$$(5.10)_{j=0} \quad \|(\chi_{Q_0} M_d)^{(0)} V\|_1 \le 1$$

is clear. Substituting (5.10) into (5.5) gives

$$(5.11)\quad cM_d^{(k)} V(x) \le \chi_{Q_0}(x) M_d^{(k)} V(x) + (\log(2+|x|))^{k-1}(1+|x|)^{-n} A^{(k-1)/k}.$$

So,

$$A \le C \int_{\{M_d^{(k)} V > 1\}} M_d^{(k)} V \, dx \quad \text{by the second inequality of (5.2)}$$

$$\le C \int_{\{|x|<A\}} M_d^{(k)} V \, dx \quad \text{by } \{M_d^{(k)} V > 1\} \subset \{|x| < A\}$$

which follows from (5.11) and (5.9)

$$\le C \int_{\{|x|<A\}} (\chi_{Q_0}(x) M_d^{(k)} V(x) + (\log(2+|x|))^{k-1}(1+|x|)^{-n} A^{(k-1)/k}) \, dx \quad \text{by (5.11)}$$

$$\le C\|\chi_{Q_0} M_d^{(k)} V\|_1 + CA^{(k-1)/k}(\log A)^k.$$

This and (5.9) yield

$$cA \le \|\chi_{Q_0} M_d^{(k)} V\|_1,$$

which implies (5.6). ■

6. Proof of the Theorem. Applying Lemma 3.1 to our f gives $\{a_Q(x)\}_{Q\in D}$ and $\{\lambda_Q\}_{Q\in D}$ that satisfy (3.1)–(3.7). Set

$$\eta = [p/2] + 1, \qquad H = \{Q \in D : V\{Q\} > 0\}.$$

Then

$$\int S_d f(x)^p V(x) \, dx \le C \sum_{Q\in D} \lambda_Q^p V\{Q\} Y(V, Q, \eta) \quad \text{by Lemma 4.3}$$

$$= C \sum_{Q\in H} \lambda_Q^p V\{Q\} Y(V, Q, \eta)$$

$$= C \sum \lambda_Q^p \int_Q V(x)\Big(1 + \log^+ \frac{V(x)}{\operatorname{av}(V,Q)}\Big)^\eta dx$$

$$\le C \sum \lambda_Q^p \int \chi_Q(x) M_d^{(\eta)} V(x) \, dx \quad \text{by Lemma 5.3}$$

$$\le C \int \Big(\sum_{Q\in H} \lambda_Q \chi_Q(x)\Big)^p M_d^{(\eta)} V(x) \, dx \quad \text{by Lemma 3.4}$$

$$\le C \int M_d f(x)^p M_d^{(\eta)} V(x) \, dx \quad \text{by (3.6),}$$

which implies (2.1). (2.2) follows from (2.1) and from the following inequality of C. Fefferman and Stein (see [S2], p. 53):

$$\int M_d f(x)^p W(x) \, dx \le C_p \int |f(x)|^p M_d W(x) \, dx \quad (1 < p < \infty). \ ■$$

7. Appendix. We outline of the proof of Lemma 5.1. By induction it is enough to show $(5.1)_{k=1}$ and two implications "$(5.1)_k \Rightarrow (5.2)_k$" and "$(5.2)_k \Rightarrow (5.1)_{k+1}$". Firstly,

$$(5.1)_{k=1}\quad C^{-1}\lambda|\{M_d V > \gamma\lambda\}| \le \int_{\{V>\lambda\}} V \, dx \le C\lambda|\{M_d V > \lambda\}|$$

can be proved by the argument of [S1], p. 7 (5), and [S1], p. 23 (b).

$(5.1)_k \Rightarrow (5.2)_k$. Since $\gamma > 1$ is arbitrary, it is enough to show $(5.2)_k$ with γ replaced by γ^2. Then

$$\int_{\{M_d^{(k)} V > \gamma^2\lambda\}} M_d^{(k)} V \, dx = \int_{\gamma^2\lambda}^{\infty} |\{M_d^{(k)} V > \mu\}| \, d\mu + \gamma^2\lambda|\{M_d^{(k)} V > \gamma^2\lambda\}|$$

$$\le C \int_{\gamma^2\lambda}^{\infty} d\mu \int_{\{V>\mu/\gamma\}} \frac{V}{\mu}\Big(\log \frac{V}{\mu/\gamma}\Big)^{k-1} dx + C \int_{\{V>\gamma\lambda\}} V\Big(\log \frac{V}{\gamma\lambda}\Big)^{k-1} dx$$

by the first inequality of $(5.1)_k$

$$= C\frac{1}{k}\int_{\{V>\gamma\lambda\}} V\Big(\log\frac{V}{\gamma\lambda}\Big)^k\,dx + \ldots \quad \text{by Fubini's theorem}$$

$$\le C\int_{\{V>\lambda\}} V\Big(\log\frac{V}{\lambda}\Big)^k\,dx.$$

This implies the first inequality of $(5.2)_k$. The second inequality follows from the second inequality of $(5.1)_k$ and from a similar argument. ■

$(5.2)_k \Rightarrow (5.1)_{k+1}$. Note that $(5.2)_k$ can be written as

$$C^{-1}\int_{\{V>\lambda\}} V\Big(\log\frac{V}{\lambda}\Big)^k\,dx \le \int_{\{M_d^{(k)}V>\lambda\}} M_d^{(k)}V\,dx$$
$$\le C\int_{\{V>\lambda/\gamma\}} V\Big(\log\frac{V}{\lambda/\gamma}\Big)^k\,dx.$$

Note that $(5.1)_{k=1}$ with V replaced by $M_d^{(k)}V$ implies

$$C^{-1}\lambda|\{M_d^{(k+1)}V>\gamma\lambda\}| \le \int_{\{M_d^{(k)}V>\lambda\}} M_d^{(k)}V\,dx$$
$$\le C\lambda|\{M_d^{(k+1)}V>\lambda\}|.$$

Then combining these two estimates implies $(5.1)_{k+1}$ (with γ replaced by γ^2). ■

Note. C. Pérez [P] showed similar weighted inequalities for the singular integral operator instead of our dyadic square function.

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