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On joint spectra of pairs of analytic Toeplitz operators

by

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Abstract. One computes the joint and essential joint spectra of a pair of multiplication operators with bounded analytic functions on the Hardy spaces of the unit ball in $\mathbb{C}^n$.

1. Introduction. The aim of the present note is to relate the recent results of E. Amar [1] on the Corona Problem for Hardy spaces of the unit ball in $\mathbb{C}^n$ to the computation of certain joint spectra of pairs of bounded analytic functions. More precisely, let B denote the unit ball in $\mathbb{C}^n$ and let $H^p(B)$ be the Hardy space corresponding to $p \in [1,\infty)$. We will compute the joint and essential joint spectra (in the sense of J. L. Taylor) for any pair of bounded analytic functions in B, regarded as multiplication operators on $H^p(B)$. The only novelty here is the use of the non-trivial hard analysis estimates of [1] in the framework of multivariable spectral analysis. Similar prior results, known for the Bergman space [2] or the H^2-space of a strictly pseudoconvex domain [8], have been derived from the much better understood L^2-estimates for the $\overline{\partial}$-operator.

The computation of the joint spectrum (in the sense of Taylor) of a system of multiplication operators with analytic functions is practically an extension of the corresponding Corona Problem. That is, besides the description of the set of all solutions of the Corona Problem one finds all linear relations among given solutions, the relations among these relations and so on. This homological point of view has appeared for the first time related to the Corona Problem in the note [3] of Hörmander and it was put in an abstract setting (of joint spectra of commuting systems of linear operators) by J. L. Taylor [6].

The case of pairs of analytic Toeplitz operators treated in this note is two-fold privileged. First, because the joint and essential joint spectra turn out by simple reasons to be equal in this situation to the corresponding right

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spectra. And second, because a part of the analytic techniques of [1] seem to be specific to pairs of functions. The main result of this note can be stated as follows.

THEOREM 1. *Let $f = (f_1, f_2)$ be a pair of bounded analytic functions in the unit ball B of $\mathbb{C}^n$ and let $p \in [1, \infty)$. Then*

(a) $\sigma(f, H^p(B)) = \sigma_r(f, H^p(B)) = \overline{f(B)}$,

(b) $\sigma_e(f, H^p(B)) = \sigma_{re}(f, H^p(B)) = \bigcap_{0<r<1} \overline{f(B \setminus rB)}$ *and*

$$\operatorname{ind}(f - \lambda) = \operatorname{mult}_\lambda(f|B),$$

for any point $\lambda \in \mathbb{C}^n \setminus \sigma_e(f, H^p(B))$.

Above, $\operatorname{ind}(f - \lambda)$ is the Fredholm index of the pair f at the point λ and $\operatorname{mult}_\lambda(f|B)$ is the multiplicity of the value λ for the analytic function $f : B \to \mathbb{C}^n$. These notions as well as the various joint spectra appearing in the statement of Theorem 1 will be recalled in the next section.

2. Preliminaries. Let $f = (f_1, f_2)$ be a pair of commuting linear bounded operators acting on the Banach space X. By definition, *Taylor's joint spectrum* of f with coefficients in X is the subset $\sigma(f, X)$ of $\mathbb{C}^n$ on which the following Koszul complex $K.(f - z, X)$ (depending on $z \in \mathbb{C}^n$):

$$0 \longrightarrow X \xrightarrow{\delta_2(f-z)} X \oplus X \xrightarrow{\delta_1(f-z)} X \longrightarrow 0, \tag{1}$$

is not exact. We recall the definition of the boundary operators δ:

$$\delta_2(f - z)g = (f_2 - z_2)g \oplus (-(f_1 - z_1)g),$$
$$\delta_1(f - z)(g_1, g_2) = (f_1 - z_1)g_1 + (f_2 - z_2)g_2,$$

where $g, g_1, g_2 \in X$.

The *right joint spectrum* of the pair f on the space X is the set

$$\sigma_r(f, X) = \{z \in \mathbb{C}^n : \operatorname{Im}(\delta_1(f - z)) \neq X\}.$$

It is obvious that $\sigma_r(f, X) \subset \sigma(f, X)$.

Similarly, the *essential joint spectrum* $\sigma_e(f, X)$ is the set of those points $z \in \mathbb{C}^n$ for which the complex $K.(f - z, X)$ does not have finite-dimensional homology. For a point $z \in \mathbb{C}^n \setminus \sigma_e(f, X)$, the *Fredholm index* of f at z is the Euler–Poincaré characteristic of the complex (1):

$$\begin{aligned}\operatorname{ind}(f - z) &= \dim \operatorname{Coker} \delta_1(f - z) \\ &\quad - \dim(\operatorname{Ker} \delta_1(f - z)/\operatorname{Im} \delta_2(f - z)) + \dim \operatorname{Ker} \delta_2(f - z).\end{aligned}$$

The *right essential spectrum* $\sigma_{re}(f, X)$ is the set of those points $z \in \mathbb{C}^n$ for which the algebraic image of $\delta_1(f - z)$ has infinite codimension in X.

These spectra are non-empty and compact and they naturally generalize the known corresponding sets associated with a single linear operator. The reader can consult the monograph [7] for an introduction to the multivariable spectral theory on Banach spaces.

We state first a simple remark which allows one, on some specific spaces of analytic functions, to reduce the computation of the joint spectrum of a pair of multiplication operators with analytic functions to the more accessible right spectrum.

LEMMA 1. *Let $f = (f_1, f_2)$ be a pair of bounded analytic functions in the unit ball B of $\mathbb{C}^n$ and let the constants $p \in [1, \infty)$, $0 < r < 1$ and $\delta > 0$ be fixed. If $|f_1(z)| + |f_2(z)| > \delta$ for $r < |z| < 1$, then the Koszul complex $K.(f, H^p(B))$ is exact in positive degree (that is,* $\operatorname{Ker} \delta_2 = 0$ *and* $\operatorname{Ker} \delta_1 = \operatorname{Im} \delta_2$*).*

Proof. An element $h \in H^p(B)$ belongs to the kernel of $\delta_2(f)$ if and only if $f_1 h = f_2 h = 0$. Since the zeroes of the functions f_1 and f_2 are complex analytic sets of dimension at most one in B, the uniqueness principle forces the function h to be zero.

Let $(h_1, h_2) \in H^p(B) \oplus H^p(B)$ be an element of $\operatorname{Ker} \delta_1(f)$. Then $f_1 h_1 + f_1 h_1 = 0$ in $H^p(B)$. Consider the open subsets of B:

$$U_j = \{z \in B : |f_j(z)| > \delta/2\} \quad (j = 1, 2).$$

By assumption the annulus $A = \{z \in B : r < |z| < 1\}$ is covered by these sets and therefore we can unambiguously define the following analytic function:

$$h'(z) = \begin{cases} h_1(z)/f_2(z), & z \in A \cap U_2, \\ -h_2(z)/f_1(z), & z \in A \cap U_1. \end{cases}$$

By the Hartogs extension theorem, the function h' has an analytic extension, denoted by h, to the unit ball B. In order to estimate the H^p-norm of the function h we remark that for a fixed $\varrho \in (r, 1)$ we have

$$\begin{aligned}\int_S |h(\varrho\zeta)|^p \, ds(\zeta) &\le \int_{S \cap \varrho^{-1} U_1} \frac{|h_2(\varrho\zeta)|^p}{|f_1(\varrho\zeta)|^p} \, ds(\zeta) + \int_{S \cap \varrho^{-1} U_2} \frac{|h_1(\varrho\zeta)|^p}{|f_2(\varrho\zeta)|^p} \, ds(\zeta) \\ &\le \delta^{-1} \Big(\int_S |h_1(\varrho\zeta)|^p \, ds(\zeta) + \int_S |h_2(\varrho\zeta)|^p \, ds(\zeta) \Big) \\ &\le \delta^{-1} (\|h_1\|^p + \|h_2\|^p).\end{aligned}$$

In the above integrals S stands for the unit sphere in $\mathbb{C}^n$ and ds is the normalized rotation invariant measure on S.

Again, by the uniqueness principle for analytic functions, the identity $(h_1, h_2) = \delta_1(f)h$ holds everywhere.

This finishes the proof of Lemma 1.

For the sake of completeness we state the following known result.

LEMMA 2. *Let f be an m-tuple of bounded analytic functions in the unit ball B of $\mathbb{C}^n$ and let $p \in [1, \infty)$. Then*

$$\overline{f(B)} \subset \sigma_{\rm r}(f, H^p(B)) \quad and \quad \bigcap_{0<r<1} \overline{f(B \setminus rB)} \subset \sigma_{\rm re}(f, H^p(B)).$$

To prove the first inclusion one remarks that, if $z \in f(B)$, then the system f has a common zero in B and hence the map $\delta_1(f - z)$ cannot be onto. Thus $f(B) \subset \sigma_{\rm r}(f, H^p(B))$, and since the right spectrum is closed the assertion follows.

The second inclusion is based on a standard one variable H^∞-interpolation trick which we do not repeat here. The paper [2] contains this argument in detail.

Finally, we remark that both Lemmas 1 and 2 are valid on more general domains, for instance at least on bounded strictly pseudoconvex domains with smooth boundary.

3. Proof of Theorem 1. In order to prove the converse inclusions in Lemma 2 we need the L^p-estimates for the tangential Cauchy–Riemann operator given in [1].

Assume that $f = (f_1, f_2)$ is a pair of bounded analytic functions in the unit ball $B \subset \mathbb{C}^n$. If there exists a positive δ with $|f_1(z)| + |f_2(z)| > \delta$ for $z \in B$, then Theorem 3 of [1] asserts that the map $\delta_1(f) : H^p(B) \oplus H^p(B) \to H^p(B)$ is onto. Thus assertion (a) of Theorem 1 is proved.

In order to prove (b) suppose that $0 \notin \bigcap_{0<r<1} \overline{f(B \setminus rB)}$. Then there is a positive δ with $|f_1(z)| + |f_2(z)| > \delta$ whenever $r < |z| < 1$. Let V denote the common zeroes in B of the set pair of functions f. Since V is relatively compact in B, the analytic dimension of V is at most zero (see for instance [4]). Thus the set V is finite.

Let $\mathcal{O}$ be the sheaf of germs of analytic functions in the unit ball B. The support of the analytic sheaf $\mathcal{O}/(f_1, f_2)\mathcal{O}$ coincides with the set V. Hence the space $\mathcal{O}(B)/(f_1, f_2)\mathcal{O}(B)$ is finite-dimensional, say of dimension N. As a consequence of Cartan's Theorem B we infer that a function $h \in \mathcal{O}(B)$ can be written as

$$h = f_1h_1 + f_2h_2, \tag{2}$$

with $h_1, h_2 \in \mathcal{O}(B)$, if and only if the terms of order less than or equal to N in the Taylor expansions of h at the points v of the set V satisfy certain linear relations (expressed more precisely by the conditions (2) mod $(z_1 - v_1)^k(z_2 - v_2)^l$, $k + l \geq N + 1$, $v \in V$). The reader can consult [4] for the general theory needed for this part of the proof.

Our next aim is to strengthen the latter conclusion to involve H^p-functions rather than merely analytic functions in B.

LEMMA 3. *Let $f = (f_1, f_2)$ be a pair of bounded analytic functions in B satisfying the condition $|f_1(z)| + |f_2(z)| > \delta > 0$ for $r < |z| < 1$. Let $p \in [1, \infty)$. Then a function $h \in H^p(B)$ which can be written as $h = f_1h_1 + f_1h_2$ with $h_1, h_2 \in \mathcal{O}(B)$ admits a similar decomposition with $h_1, h_2 \in H^p(B)$.*

Proof. Assume that $h = f_1h_1 + f_2h_2$ with $h_1, h_2 \in \mathcal{O}(B)$. In the annulus $A = \{z \in B : r < |z|\}$ we can construct another similar, but only smooth, decomposition of h by using the functions $k_j = \bar{f}_j h/|f|^2$ $(j = 0, 1)$, where $|f|^2 = |f_1|^2 + |f_2|^2$.

Let $\phi : [0, 1] \to [0, 1]$ be a smooth function with $\phi(t) = 1$ for $t \leq r + (1 - r)/3$ and $\phi(t) = 0$ for $t \geq r + 2(1 - r)/3$. Let $\chi(z) = \phi(|z|)$, $z \in B$.

Then the functions $g_j = \chi h_j + (1 - \chi)k_j$ $(j = 1, 2)$ are smooth and satisfy the identity $f_1g_1 + f_2g_2 = h$. Consequently, $f_1\overline{\partial}g_1 + f_2\overline{\partial}g_2 = 0$ and therefore the differential form

$$\omega(z) = \begin{cases} \frac{\overline{\partial}g_1}{f_2}(z), & z \in \{z \in A : |f_2| \neq 0\}, \\ -\frac{\overline{\partial}g_2}{f_1}(z), & z \in \{z \in A : |f_1| \neq 0\}, \\ 0, & z \in B \setminus A, \end{cases}$$

is well defined, $\overline{\partial}$-closed and with smooth coefficients. Moreover, since the zeroes of f_1 or f_2 are nowhere dense in B, the identities

$$\overline{\partial}g_1 = f_2\omega \quad \text{and} \quad \overline{\partial}g_2 = -f_1\omega$$

hold everywhere in B.

In a neighbourhood of the boundary of B the form ω coincides with

$$\frac{h}{f_2}\overline{\partial}\left(\frac{\bar{f}_1}{|f|^2}\right) = -\frac{h}{f_1}\overline{\partial}\left(\frac{\bar{f}_2}{|f|^2}\right) = h\frac{\overline{f_1\partial f_2} - \overline{f_2\partial f_1}}{|f|^4}.$$

Thus the conditions of Theorem 1 of [1] are satisfied and there exists a function $u \in L^p(\partial B)$ with the property that $\overline{\partial}_b u = \omega$. Therefore, at the level of functions defined on the boundary of B, the elements $l_1 = g_1 - f_2u$ and $l_2 = g_2 + f_1u$ belong to $H^p(B)$ (because of the choice of u) and satisfy the desired identity $f_1l_1 + f_2l_2 = h$.

This completes the proof of Lemma 3.

To finish the proof of assertion (b) in Theorem 1 it suffices to apply Lemma 1 and to remark that, by Lemma 3,

$$\operatorname{Coker} \delta_1(f) \cong \mathcal{O}(B)/(f_1, f_2)\mathcal{O}(B).$$

Thus under our assumptions it follows that 0 is not in the right essential spectrum of f. Moreover, Lemma 1 shows that

$$\operatorname{ind}(f) = \dim(\mathcal{O}(B)/(f_1, f_2)\mathcal{O}(B)) = \operatorname{mult}_0(f|B).$$

(The reader can take the second equality as a definition of the multiplicity of the value 0 of f.)

This completes the proof of Theorem 1.

We remark finally that part (a) of Theorem 1 remains true, with an identical proof, on any bounded domain of $\mathbb{C}^n$ on which the H^p Corona Problem is solvable.

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L^p weighted inequalities for the dyadic square function

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Abstract. We prove that

$$\int (S_d f)^p V\,dx \le C_{p,n} \int |f|^p M_d^{([p/2]+2)} V\,dx,$$

where S_d is the dyadic square function, $M_d^{(k)}$ is the k-fold application of the dyadic Hardy–Littlewood maximal function and $p > 2$.

1. Introduction. Let $V(x) \ge 0$. S. Y. Chang, J. M. Wilson and T. H. Wolff [CWW] showed that if $p = 2$, then

$$\int_{\mathbb{R}^n} S_\psi f(x)^p V(x)\,dx \le C_{p,\psi,n} \int_{\mathbb{R}^n} |f(x)|^p MV(x)\,dx, \tag{1.1}$$

where $S_\psi f$ is the square function of f with respect to the kernel function ψ that satisfies certain strict conditions and where Mf is the Hardy–Littlewood maximal function of f. S. Chanillo and R. L. Wheeden [CW] showed that (1.1) holds for $1 < p \le 2$ and fails for $p > 2$. (Furthermore, they relaxed the conditions on ψ.) J. M. Wilson [W6] extended (1.1) to the case $0 < p \le 1$ by replacing $|f(x)|$ by a certain maximal function of f. Then the remaining problem is to get inequalities that are similar to (1.1) and that hold for the case $p > 2$. In Derrick [D] the following problem is listed. (See also [W6], p. 293.)

J. M. Wilson's Problem. Let S_d be the dyadic square function. Let $M^{(1)}f = Mf$, $M^{(2)}f = M(Mf)$, ... Then, is the following inequality true:

$$\int_{\mathbb{R}^n} S_d f(x)^p V(x)\,dx \le C_{p,n} \int_{\mathbb{R}^n} |f(x)|^p M^{(k(p))} V(x)\,dx,$$

as $p \to \infty$, with $k(p) \sim p/2$? In particular, with $k(p) = -[-p/2]$?

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