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Convergence in the generalized sense relative to Banach algebras of operators and in LMC-algebras

by

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Abstract. The notion of convergence in the generalized sense of a sequence of closed operators is generalized to the situation where the closed operators involved are affiliated with a Banach algebra of operators. Also, the concept of convergence in the generalized sense is extended to the context of a LMC-algebra, where it applies to the spectral theory of the algebra.

1. Introduction. Let X be a Banach space, and let $B(X)$ denote the algebra of all bounded linear operators on X. Assume $\mathscr{B} \subseteq B(X)$ is a Banach algebra of operators (the complete norm on $\mathscr{B}$ need not be the operator norm). A linear operator T with domain $D(T)$ in X, $T : D(T) \to X$, is affiliated with $\mathscr{B}$ if for some $\lambda \in \mathbb{C}$, $(\lambda - T)^{-1} \in \mathscr{B}$. The spectral and Fredholm theory of operators affiliated with $\mathscr{B}$ is developed in [B1] and [B2]. Also, examples and applications of the theory are given in [B1], [B2], and [PR] (where affiliated semigroups of operators are considered).

In this paper we study convergence in the generalized sense (GS-convergence) of a sequence of operators affiliated with $\mathscr{B}$. This is a natural type of convergence which can be usefully applied to the spectral theory of affiliated operators. When $\mathscr{B} = B(X)$, this notion of convergence has been widely used; see [K]. We develop the basic properties of GS-convergence of affiliated operators in §2.

In addition to applications to the spectral theory of affiliated operators, this paper is also motivated by the fact that the concept of GS-convergence applies directly to analysis in LMC-algebras. In fact, from the point of view of analysis, GS-convergence is the most useful type of convergence in the context of LMC-algebras. GS-convergence in an LMC-algebra is studied in §3. The results there are exactly analogous to those for GS-convergence of sequences of affiliated operators as developed in §2.

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2. GS-convergence relative to a Banach algebra of operators. Throughout this section, $\mathscr{B} \subseteq B(X)$ is a fixed Banach algebra of operators which contains the identity operator I. The $\mathscr{B}$-norm of an operator $R \in \mathscr{B}$ is denoted by $\|R\|_{\mathscr{B}}$, and it is assumed that $\|\cdot\|_{\mathscr{B}}$ dominates the usual operator norm.

Let $\mathscr{A}_{\mathscr{B}}$ be the collection of all closed operators with domain in X which are affiliated with $\mathscr{B}$. For an operator T we let

$$\text{res}_{\mathscr{B}}(T) \equiv \{\lambda \in \mathbb{C} : (\lambda - T)^{-1} \in \mathscr{B}\} \quad \text{and} \quad \sigma_{\mathscr{B}}(T) \equiv \mathbb{C} \setminus \text{res}_{\mathscr{B}}(T).$$

Of course, $\text{res}_{\mathscr{B}}(T)$ may be empty, and by definition, $T \in \mathscr{A}_{\mathscr{B}}$ exactly when $\text{res}_{\mathscr{B}}(T)$ is nonempty.

DEFINITION 1. Assume $\{T_n\}$ and T are linear operators with domain in X. The sequence $\{T_n\}$ *converges to T in the generalized sense* (*relative to* $\mathscr{B}$) if there exist $\lambda_0 \in \text{res}_{\mathscr{B}}(T)$ and N such that $\lambda_0 \in \text{res}_{\mathscr{B}}(T_n)$ for $n \ge N$, and

$$\|(\lambda_0 - T_n)^{-1} - (\lambda_0 - T)^{-1}\|_{\mathscr{B}} \to 0 \quad \text{as } N \le n \to \infty.$$

We use the notation $T_n \to T$ (GS) when T_n converges to T in the generalized sense.

The notion of GS-convergence (relative to $\mathscr{B}$) is a generalization of a well-known type of convergence used in the spectral theory of closed operators; see [K, Chapter 2, especially Theorem 2.23, p. 206]. Since $\mathscr{B}$ is fixed, we drop the statement "(relative to $\mathscr{B}$)" in what follows.

PROPOSITION 2. *Let $T \in \mathscr{A}_{\mathscr{B}}$.*

(1) *Assume $T^{-1} \in \mathscr{B}$, $S = T + R$ where $R \in \mathscr{B}$, and $\|T - S\|_{\mathscr{B}} < \|T^{-1}\|_{\mathscr{B}}^{-1}$. Then $S^{-1} \in \mathscr{B}$, and*

$$\|T^{-1} - S^{-1}\|_{\mathscr{B}} \le \frac{\|T^{-1}\|_{\mathscr{B}}^2 \|T - S\|_{\mathscr{B}}}{1 - (\|T - S\|_{\mathscr{B}} \|T^{-1}\|_{\mathscr{B}})}.$$

(2) $\text{res}_{\mathscr{B}}(T)$ *is open and* $\sigma_{\mathscr{B}}(T)$ *is closed.*

Proof. Part (2) is an immediate consequence of (1). Now we give a proof of (1) (although the argument is a standard one). Assume S and T are as stated in (1). Note that $T^{-1}S \in \mathscr{B}$, and

$$\|I - T^{-1}S\|_{\mathscr{B}} \le \|T^{-1}\|_{\mathscr{B}} \|T - S\|_{\mathscr{B}} < 1.$$

By [BD, Cor. 10, p. 12] it follows that $T^{-1}S$ has an inverse W in $\mathscr{B}$. Then

$$(WT^{-1})Sx = x \quad \text{for all } x \in D(S) = D(T);$$

and for all $x \in X$, $T^{-1}S(WT^{-1})x = T^{-1}x$, so

$$S(WT^{-1})x = x \quad \text{for all } x \in X.$$

Therefore $S^{-1} = WT^{-1} \in \mathscr{B}$. Now

$$(3) \qquad \begin{aligned} \|S^{-1}\|_{\mathscr{B}} - \|T^{-1}\|_{\mathscr{B}} \le \|S^{-1} - T^{-1}\|_{\mathscr{B}} &= \|T^{-1}(S - T)S^{-1}\|_{\mathscr{B}} \\ &\le \|T^{-1}\|_{\mathscr{B}} \|S - T\|_{\mathscr{B}} \|S^{-1}\|_{\mathscr{B}}. \end{aligned}$$

Therefore

$$(4) \qquad \|S^{-1}\|_{\mathscr{B}}(1 - \|S - T\|_{\mathscr{B}} \|T^{-1}\|_{\mathscr{B}}) \le \|T^{-1}\|_{\mathscr{B}}.$$

Substituting (4) into (3), we have the inequality in (1).

PROPOSITION 3. *Assume $T_n \to T$ (GS). If $\lambda \in \text{res}_{\mathscr{B}}(T)$, then there exists N such that $\lambda \in \text{res}_{\mathscr{B}}(T_n)$ for $n \ge N$, and*

$$\|(\lambda - T_n)^{-1} - (\lambda - T)^{-1}\|_{\mathscr{B}} \to 0 \quad \textit{as } N \le n \to \infty.$$

Proof. We use the following elementary equality repeatedly in the proof: For $\mu \in \text{res}_{\mathscr{B}}(T)$,

$$(\eta - T)(\mu - T)^{-1} = [(\eta - \mu)I + (\mu - T)](\mu - T)^{-1} = I + (\eta - \mu)(\mu - T)^{-1}.$$

Assume $\lambda_0, \lambda \in \text{res}_{\mathscr{B}}(T)$, and

$$(1) \qquad \|(\lambda_0 - T_n)^{-1} - (\lambda_0 - T)^{-1}\|_{\mathscr{B}} \to 0.$$

Set $S = (\lambda - T)(\lambda_0 - T)^{-1} = I + (\lambda - \lambda_0)(\lambda_0 - T)^{-1}$. Then $S^{-1} \in \mathscr{B}$, and $S^{-1} = (\lambda_0 - T)(\lambda - T)^{-1} = I + (\lambda_0 - \lambda)(\lambda - T)^{-1}$. Now $(\lambda - T_n)(\lambda_0 - T_n)^{-1} = I + (\lambda - \lambda_0)(\lambda_0 - T_n)^{-1}$, and so, $\|(\lambda - T_n)(\lambda_0 - T_n)^{-1} - S\|_{\mathscr{B}} \to 0$. By Proposition 2, there exists N such that $(\lambda - T_n)(\lambda_0 - T_n)^{-1}$ has an inverse in $\mathscr{B}$ for $n \ge N$. This implies $\lambda - T_n$ has an inverse in $\mathscr{B}$ for $n \ge N$. Also, $\|(\lambda_0 - T_n)(\lambda - T_n)^{-1} - S^{-1}\|_{\mathscr{B}} \to 0$, and $\|(\lambda - T_n)^{-1} - (\lambda_0 - T_n)^{-1}S^{-1}\|_{\mathscr{B}} \to 0$ as $N \ge n \to \infty$. Combining this with (1), we have

$$\|(\lambda - T_n)^{-1} - (\lambda - T)^{-1}\|_{\mathscr{B}} \to 0 \quad \text{as } N \le n \to \infty.$$

The next proposition contains some basic properties of GS-convergence.

PROPOSITION 4. (1) *If $T_n \to T$ (GS), then $T_n + \lambda I \to T + \lambda I$ (GS) for all $\lambda \in \mathbb{C}$.*

(2) *If $T_n \to T$ (GS), $S_n \to S$ (GS) and $T^{-1}, S^{-1} \in \mathscr{B}$, then $T_n S_n \to TS$ (GS).*

(3) *If $T_n \to T$ (GS) and $\{\lambda_n\} \subseteq \mathbb{C}$ with $\lambda_n \to \lambda$, $\lambda \ne 0$, then $\lambda_n T_n \to \lambda T$ (GS).*

(4) *If $\{R_n\} \subseteq \mathscr{B}$, $R \in \mathscr{B}$, $\|R_n - R\|_{\mathscr{B}} \to 0$, and $T_n + R \to T + R$ (GS), then $T_n + R_n \to T + R$ (GS).*

Proof. The verification of (1) is completely elementary. To verify (2), note that for all n sufficiently large, T_n^{-1} and S_n^{-1} are in $\mathscr{B}$, and that $\|T_n^{-1} - T^{-1}\|_{\mathscr{B}} \to 0$, $\|S_n^{-1} - S^{-1}\|_{\mathscr{B}} \to 0$. It follows immediately that $\|(T_n S_n)^{-1} - (TS)^{-1}\|_{\mathscr{B}} \to 0$. This implies $T_n S_n \to TS$ (GS).

Suppose $V_n^{-1} \in \mathscr{B}$ for all n, $V^{-1} \in \mathscr{B}$ with $\|V_n^{-1} - V^{-1}\|_{\mathscr{B}} \to 0$. Furthermore, assume $\{W_n\} \subseteq \mathscr{B}$ with $\|W_n\|_{\mathscr{B}} \to 0$. Since $\|V_n^{-1}\|_{\mathscr{B}} \to \|V^{-1}\|_{\mathscr{B}}$, there exists N such that $\|W_n\|_{\mathscr{B}}\|V_n^{-1}\|_{\mathscr{B}} \le 1/2$ for $n \ge N$. Using Proposition 2, we have for $n \ge N$, $(W_n + V_n)^{-1} \in \mathscr{B}$ and

$$\|(W_n + V_n)^{-1} - V_n^{-1}\|_{\mathscr{B}} \le \|V_n^{-1}\|_{\mathscr{B}}^2 \|W_n\|_{\mathscr{B}}(1 - \|W_n\|_{\mathscr{B}}\|V_n^{-1}\|_{\mathscr{B}})^{-1} \to 0$$

as $N \le n \to \infty$. Therefore $\|(W_n + V_n)^{-1} - V^{-1}\|_{\mathscr{B}} \to 0$. Now assume as in (3) that $T_n \to T$ (GS) and $\lambda_n \to \lambda \ne 0$. Fix $\lambda_0 \in \mathrm{res}_{\mathscr{B}}(T)$. Then $\|(\lambda_n\lambda_0 - \lambda_n T_n)^{-1} - (\lambda\lambda_0 - \lambda T)^{-1}\|_{\mathscr{B}} \to 0$. Let $W_n = (-\lambda_n\lambda_0 + \lambda\lambda_0)I \to 0$. Applying the previous argument we have

$$\|(W_n + \lambda_n\lambda_0 - \lambda_n T_n)^{-1} - (\lambda_0\lambda - \lambda T)^{-1}\|_{\mathscr{B}} \to 0,$$

so

$$\|(\lambda\lambda_0 - \lambda_n T_n)^{-1} - (\lambda_0\lambda - \lambda T)^{-1}\|_{\mathscr{B}} \to 0.$$

This implies $\lambda_n T_n \to \lambda T$ (GS).

To prove (4), set $W_n = R - R_n$, so by hypothesis $\|W_n\|_{\mathscr{B}} \to 0$. Choose $\lambda_0 \in \mathrm{res}_{\mathscr{B}}(T + R)$, so by assumption

$$\|(\lambda_0 - (T_n + R))^{-1} - (\lambda_0 - (T + R))^{-1}\|_{\mathscr{B}} \to 0.$$

Applying the argument in the previous paragraph with

$$V_n = \lambda_0 - (T_n + R), \qquad V = \lambda_0 - (T + R),$$

one concludes that

$$\|(\lambda_0 - (T_n + R_n))^{-1} - (\lambda_0 - (T + R))^{-1}\|_{\mathscr{B}} \to 0.$$

It is not always true that when $T_n \to T$ (GS) and $R \in \mathscr{B}$, then $T_n + R \to T + R$ (GS). However, if $\mathrm{res}_{\mathscr{B}}(T + R)$ is nonempty (i.e. $T + R \in \mathscr{A}_{\mathscr{B}}$) and $\mathrm{res}_{\mathscr{B}}(T)$ is unbounded, then this statement is true.

THEOREM 5. *Assume $T_n \to T$ (GS) and $R \in \mathscr{B}$. Then there exists $\varepsilon > 0$ such that whenever $|\delta| < \varepsilon$, then $T_n + \delta R \to T + \delta R$ (GS). If $\mathrm{res}_{\mathscr{B}}(T + R)$ is nonempty and $\mathrm{res}_{\mathscr{B}}(T)$ is unbounded, then $T_n + R \to T + R$ (GS).*

Proof. Fix $-\lambda \in \mathrm{res}_{\mathscr{B}}(T)$ and $-\mu \in \mathrm{res}_{\mathscr{B}}(R)$. By Proposition 4(2),

$$(\mu + R)^{-1}(\lambda + T_n) \to (\mu + R)^{-1}(\lambda + T) \text{ (GS)}.$$

Now $(\mu + R)^{-1}(\lambda + T)$ has an inverse in $\mathscr{B}$, so by Proposition 2, there exists $\varepsilon > 0$ such that

$$|\delta| < \varepsilon \Rightarrow -\delta \in \mathrm{res}_{\mathscr{B}}((\mu + R)^{-1}(\lambda + T)).$$

Fix any such δ. By Proposition 4,

$$(\mu + R)^{-1}(\lambda + T_n) + \delta \to (\mu + R)^{-1}(\lambda + T) + \delta \text{ (GS)},$$

and again by Proposition 4(2),

$$(\lambda + T_n) + \delta(\mu + R) \to (\lambda + T) + \delta(\mu + R) \text{ (GS)}.$$

Thus, $T_n + \delta R \to T + \delta R$ (GS).

The proof above shows that $T_n + \delta R \to T + \delta R$ (GS) whenever $-\lambda \in \mathrm{res}_{\mathscr{B}}(T)$, $-\mu \in \mathrm{res}_{\mathscr{B}}(R)$, and $-\delta \in \mathrm{res}_{\mathscr{B}}((\mu+R)^{-1}(\lambda+T))$. Suppose now that $\mathrm{res}_{\mathscr{B}}(T)$ is unbounded and $\mathrm{res}_{\mathscr{B}}(T+R)$ is nonempty. Fix $-\gamma \in \mathrm{res}_{\mathscr{B}}(T+R)$. Choose $-\lambda \in \mathrm{res}_{\mathscr{B}}(T)$ such that $|\lambda - \gamma| > \|R\|_{\mathscr{B}}$. Set $\mu = \gamma - \lambda$; so $-\mu \in \mathrm{res}_{\mathscr{B}}(R)$. Then

$$I + (\mu+R)^{-1}(\lambda+T) = (\mu+R)^{-1}[(\mu+\lambda) + (T+R)] = (\mu+R)^{-1}[\gamma + (T+R)].$$

This last operator has an inverse in $\mathscr{B}$; so $-1 \in \mathrm{res}_{\mathscr{B}}((\mu + R)^{-1}(\lambda + T))$. Therefore, as remarked above, $T_n + R \to T + R$ (GS).

The next result has useful application to the operational calculus for operators in $\mathscr{A}_{\mathscr{B}}$ (see Theorem 7) and to results concerning variation of $\sigma_{\mathscr{B}}(\cdot)$ (as in Theorem 9).

THEOREM 6. *Assume $T_n \to T$ (GS). Let Γ be a compact subset of $\mathbb{C}$ with $\Gamma \subseteq \mathrm{res}_{\mathscr{B}}(T)$.*

(1) *There exists N such that for all $n \ge N$, $\Gamma \subseteq \mathrm{res}_{\mathscr{B}}(T_n)$.*

(2) *Let N be as in (1). Then $\|(\lambda - T_n)^{-1} - (\lambda - T)^{-1}\|_{\mathscr{B}} \to 0$ as $N \le n \to \infty$ uniformly in $\lambda \in \Gamma$.*

Proof. Suppose that (1) does not hold. Then there exists a subsequence $\{T_{n_k}\}$ and a sequence $\{\lambda_k\} \subseteq \Gamma$ with $\lambda_k \in \sigma_{\mathscr{B}}(T_{n_k})$, $k \ge 1$. Since Γ is compact, there is a subsequence of $\{\lambda_k\}$ that converges to some $\lambda_0 \in \Gamma$. To simplify notation, we assume that $\lambda_n \in \sigma_{\mathscr{B}}(T_n) \cap \Gamma$ and $\lambda_n \to \lambda_0$ as $n \to \infty$. By Proposition 3, we may assume $\lambda_0 \in \mathrm{res}_{\mathscr{B}}(T_n)$ for all n. Now

$$\begin{aligned}\|I - (\lambda_n - T_n)(\lambda_0 - T_n)^{-1}\|_{\mathscr{B}} &= \|I - [(\lambda_n - \lambda_0)(\lambda_0 - T_n)^{-1} + I]\|_{\mathscr{B}} \\ &= |\lambda_n - \lambda_0|\, \|(\lambda_0 - T_n)^{-1}\|_{\mathscr{B}} \to 0.\end{aligned}$$

By Proposition 2, this implies $(\lambda_n - T_n)(\lambda_0 - T_n)^{-1}$ is invertible in $\mathscr{B}$ for all n sufficiently large. Therefore $(\lambda_n - T_n)^{-1} \in \mathscr{B}$ for large n, contradicting the fact that $\lambda_n \in \sigma_{\mathscr{B}}(T_n)$. This proves (1).

To prove (2), we first show that there is $M > 0$ such that $\|(\lambda - T_n)^{-1}\|_{\mathscr{B}} \le M$ whenever $\lambda \in \Gamma$ and $n \ge N$ (N chosen as in (1)). For suppose no such M exists. Then there exists a subsequence $\{T_{n_k}\}$ and a sequence of scalars $\{\lambda_k\}$ such that $\|(\lambda_k - T_{n_k})^{-1}\|_{\mathscr{B}} > k$, $k \ge 1$. Also, some subsequence of $\{\lambda_k\}$ converges to $\mu \in \Gamma$. Again, for convenience of notation, we assume $\|(\lambda_n - T_n)^{-1}\|_{\mathscr{B}} \ge n$ and $\mu \in \mathrm{res}_{\mathscr{B}}(T_n)$, $n \ge 1$, and $\lambda_n \to \mu$. Now

$$\begin{aligned}\|(\lambda_n - T_n)^{-1}\|_{\mathscr{B}} - \|(\mu - T_n)^{-1}\|_{\mathscr{B}} &\le \|(\lambda_n - T_n)^{-1} - (\mu - T_n)^{-1}\|_{\mathscr{B}} \\ &= \|(\mu - \lambda_n)(\lambda_n - T_n)^{-1}(\mu - T_n)^{-1}\|_{\mathscr{B}} \\ &\le |\mu - \lambda_n|\, \|(\lambda_n - T_n)^{-1}\|_{\mathscr{B}}\|(\mu - T_n)^{-1}\|_{\mathscr{B}}.\end{aligned}$$

Thus for $n \geq 1$, $\|(\lambda_n - T_n)^{-1}\|_{\mathscr{B}}\{1 - |\mu - \lambda_n| \, \|(\mu - T_n)^{-1}\|_{\mathscr{B}}\} \leq \|(\mu - T_n)^{-1}\|_{\mathscr{B}}$. But $\{\|(\mu - T_n)^{-1}\|_{\mathscr{B}}\}$ is bounded in n (Proposition 3). This is a contradiction.

Now assume N is as in (1) and

$$\|(\lambda - T_n)^{-1}\|_{\mathscr{B}} \leq M \quad \text{whenever } \lambda \in \Gamma \text{ and } n \geq N.$$

Fix $\lambda_0 \in \Gamma$, so that $\|(\lambda_0 - T_n)^{-1} - (\lambda_0 - T)^{-1}\|_{\mathscr{B}} \to 0$ as $N \leq n \to \infty$. Then for $n \geq N$ and $\lambda \in \Gamma$,

$$\begin{aligned}(\lambda - T_n)^{-1} - (\lambda - T)^{-1} \\ &= (\lambda - T_n)^{-1}(\lambda_0 - T_n)[(\lambda_0 - T_n)^{-1} - (\lambda_0 - T)^{-1}](\lambda_0 - T)(\lambda - T)^{-1} \\ &= (I + (\lambda_0 - \lambda)(\lambda - T_n)^{-1})[(\lambda_0 - T_n)^{-1} - (\lambda_0 - T)^{-1}] \\ &\quad \circ (I + (\lambda_0 - \lambda)(\lambda - T)^{-1}).\end{aligned}$$

Since $\|(\lambda - T_n)^{-1}\|_{\mathscr{B}}$ and $\|(\lambda - T)^{-1}\|_{\mathscr{B}}$ have a uniform bound for $n \geq N$ and all $\lambda \in \Gamma$, we have $\|(\lambda - T_n)^{-1} - (\lambda - T)^{-1}\|_{\mathscr{B}} \to 0$ as $N \leq n \to \infty$ uniformly for $\lambda \in \Gamma$.

When $T \in \mathscr{A}_{\mathscr{B}}$, then there is an operational calculus for T completely analogous to that for unbounded closed operators with nonempty resolvent set. The properties of this operational calculus can be proved by applying the proofs in [DS, §9].

For $T \in \mathscr{A}_{\mathscr{B}}$, let $\mathcal{F}(T)$ be the set of all holomorphic functions F with the properties:

(i) F is holomorphic on some open set U with U^c (the complement of U) compact and $\sigma_{\mathscr{B}}(T) \subseteq U$, and

(ii) $F(\infty) = \lim_{|\lambda| \to \infty} F(\lambda)$ exists. Choose γ to be a sum of suitable closed curves with image in $U \setminus \sigma_{\mathscr{B}}(T)$ such that

$$\operatorname{Ind}_\gamma(z) = \begin{cases} 0 & \text{for all } z \in \sigma_{\mathscr{B}}(T), \\ -1 & \text{for all } z \notin U. \end{cases}$$

Define

$$F(T) = F(\infty)I + (2\pi i)^{-1} \int_\gamma F(\lambda)(\lambda - T)^{-1}\, d\lambda.$$

In this case $F(T) \in \mathscr{B}$.

THEOREM 7. *Assume $T \in \mathscr{A}_{\mathscr{B}}$ and assume $F \in \mathcal{F}(T)$. If $T_n \to T$ (GS), then there exists N such that $F \in \mathcal{F}(T_n)$ for $n \geq N$ and*

$$\|F(T_n) - F(T)\|_{\mathscr{B}} \to 0 \quad \text{as } N \leq n \to \infty.$$

Proof. By definition there exists an open set U such that U^c is compact, F is holomorphic in U, and $F(\infty) = \lim_{|\lambda| \to \infty} F(\lambda)$ exists. Fix γ to be a sum of closed curves in $U \setminus \sigma_{\mathscr{B}}(T)$ such that

$$\operatorname{Ind}_\gamma(z) = \begin{cases} 0, & z \in \sigma_{\mathscr{B}}(T), \\ -1, & z \in U^c. \end{cases}$$

By definition

$$F(T) = F(\infty)I + (2\pi i)^{-1} \int_\gamma F(\lambda)(\lambda - T)^{-1}\, d\lambda.$$

Let γ^* be the image of γ in $\mathbb{C}$. Set $V = \{z \notin \gamma^* : \operatorname{Ind}_\gamma(z) = 0\}$. Then V is an open set with V^c compact (V contains the unbounded component of $(\gamma^*)^c$). By Theorem 6, there exists N such that $\sigma_{\mathscr{B}}(T_n) \subseteq U \cap V$ for $n \geq N$. Also, $\|(\lambda - T_n)^{-1} - (\lambda - T)^{-1}\|_{\mathscr{B}} \to 0$ uniformly on $(U \cap V)^c$, and so on γ^*. Thus, for $n \geq N$, $F(T_n)$ is defined as above with T_n in place of T. It follows easily that

$$\|F(T_n) - F(T)\|_{\mathscr{B}} \to 0 \quad \text{as } N \leq n \to \infty.$$

COROLLARY 8. *Assume Γ is a nonempty compact and relatively open subset of $\sigma_{\mathscr{B}}(T)$. Let V be any open subset of $\mathbb{C}$ with $\Gamma \subseteq V$. If $T_n \to T$ (GS), then $\sigma_{\mathscr{B}}(T_n) \cap V$ is nonempty for all n sufficiently large.*

Proof. Choose γ to be an appropriate sum of closed curves in $V \cap \operatorname{res}_{\mathscr{B}}(T)$ surrounding Γ such that

$$P \equiv (2\pi i)^{-1} \int_\gamma (\lambda - T)^{-1}\, d\lambda \neq 0$$

is the corresponding spectral projection. Denote by γ^* the image of γ in V. By Theorem 6, there exists N such that $(\lambda - T_n)^{-1} \in \mathscr{B}$ whenever $n \geq N$ and $\lambda \in \gamma^*$, and

$$\|(\lambda - T_n)^{-1} - (\lambda - T)^{-1}\|_{\mathscr{B}} \to 0 \quad \text{uniformly on } \gamma^* \quad \text{as } N \leq n \to \infty.$$

Applying Theorem 7, we have

$$P \leftarrow P_n \equiv (2\pi i)^{-1} \int_\gamma (\lambda - T_n)^{-1}\, d\lambda$$

in $\mathscr{B}$-norm as $N \leq n \to \infty$. If $\sigma_{\mathscr{B}}(T_n) \cap V$ were empty for an infinite number of n, then for these n, $P_n = 0$, so $P = 0$, a contradiction.

For $R > 0$ we use the notation $D_R = \{z \in \mathbb{C} : |z| \leq R\}$. The upper semicontinuity of the spectrum in a Banach algebra is a basic result in spectral theory; see [R, Theorem (1.6.16)] or [BD, Proposition 17, p. 26]. For Γ a subset of $\mathbb{C}$ and $\lambda \in \mathbb{C}$, let

$$d(\lambda, \Gamma) = \inf\{|\lambda - \gamma| : \gamma \in \Gamma\}.$$

Now we prove a type of upper semicontinuity result that holds for $\sigma_{\mathscr{B}}(\cdot)$ relative to GS-convergence.

THEOREM 9 (local upper semicontinuity of $\sigma_{\mathscr{B}}$). *Assume $T_n \to T$ (GS). If $R > 0$ and U is any open set with $\sigma_{\mathscr{B}}(T) \subseteq U$, then there exists N such that*

$$n \geq N \Rightarrow \sigma_{\mathscr{B}}(T_n) \cap D_R \subseteq U \cap D_R.$$

Proof. Assume R and U are as stated. Let $\Gamma_R = U^c \cap D_R$, and note that Γ_R is compact, and $\Gamma_R \subseteq U^c \subseteq \text{res}_{\mathscr{B}}(T)$. Therefore by Theorem 6, there exists N such that $\Gamma_R \subseteq \text{res}_{\mathscr{B}}(T_n)$ whenever $n \geq N$. Thus, for $n \geq N$, $\sigma_{\mathscr{B}}(T_n) \subseteq \Gamma_R^c = U \cup D_R^c$. It follows that

$$n \geq N \Rightarrow \sigma_{\mathscr{B}}(T_n) \cap D_R \subseteq U \cap D_R.$$

Now we prove a stronger variational result when $\sigma_{\mathscr{B}}(T)$ is totally disconnected (therefore this result holds when $\sigma_{\mathscr{B}}(T)$ is countable).

THEOREM 10. *Assume $T_n \to T$ (GS) and $\sigma_{\mathscr{B}}(T)$ is totally disconnected. Let $\varepsilon > 0$ and $R > 0$ be arbitrary. Fix δ, $0 < \delta < \varepsilon$, and set $S = R + \delta$. Then there exists N such that*

$$\mu \in \sigma_{\mathscr{B}}(T) \cap D_R \quad \textit{and} \quad n \geq N \Rightarrow d(\mu, \sigma_{\mathscr{B}}(T_n) \cap D_S) < \varepsilon.$$

Proof. Suppose on the contrary that no such N exists. Then there is an increasing sequence of positive integers, $\{n_k\}$, and a sequence $\{\mu_{n_k}\} \subseteq \sigma_{\mathscr{B}}(T) \cap D_R$ such that

$$(1) \qquad d(\mu_{n_k}, \sigma_{\mathscr{B}}(T_{n_k}) \cap D_S) \geq \varepsilon \quad \text{for all } k.$$

We may assume (by taking a subsequence if necessary) that $\mu_{n_k} \to \mu_0 \in \sigma_{\mathscr{B}}(T) \cap D_R$. Since $\sigma_{\mathscr{B}}(T)$ is totally disconnected, there exists a compact open and closed subset Γ of $\sigma_{\mathscr{B}}(T)$ such that

$$\mu_0 \in \Gamma \subseteq B(\mu_0, \delta) \equiv \{\lambda : |\lambda - \mu_0| < \delta\}.$$

By Corollary 8, there exists N_1 such that for all $n \geq N_1$, there are $\lambda_n \in \sigma_{\mathscr{B}}(T_n) \cap B(\mu_0, \delta)$. Choose n_k so large that

$$n_k \geq N_1 \quad \text{and} \quad \mu_{n_k} \in B(\mu_0, \varepsilon - \delta).$$

Then $|\mu_{n_k} - \lambda_{n_k}| < \varepsilon$, contradicting the inequality in (1).

In the case when $\mathscr{B}$ is a C^*-algebra, several strong continuity properties hold for $\sigma_{\mathscr{B}}(\cdot)$ when the elements involved are selfadjoint. We prove these properties now.

For E and F nonempty subsets of $\mathbb{C}$, let $\delta(E, F) = \max(d_1, d_2)$ where $d_1 = \sup\{d(\lambda, F) : \lambda \in E\}$ and $d_2 = \sup\{d(\mu, E) : \mu \in F\}$.

THEOREM 11. *Assume $\mathscr{B}$ is a closed $*$-subalgebra of $B(H)$, H a Hilbert space. Assume $T = T^* \in \mathscr{A}_{\mathscr{B}}$ and $W = W^* \in \mathscr{B}$. Then*

$$\delta(\sigma_{\mathscr{B}}(T), \sigma_{\mathscr{B}}(T + W)) \leq \|W\|_{\mathscr{B}}.$$

Proof. Assume $d(\lambda, \sigma_{\mathscr{B}}(T)) > \|W\|_{\mathscr{B}}$, and set $R(\lambda) = (\lambda - T)^{-1} \in \mathscr{B}$. The spectrum of $R(\lambda)$ in the C^*-algebra $\mathscr{B}$ is $\{(\lambda - \mu)^{-1} : \mu \in \sigma_{\mathscr{B}}(T)\} \cup \{0\}$ when $T \notin \mathscr{B}$, and this set without $\{0\}$ when $T \in \mathscr{B}$. As $R(\lambda)$ is a normal element of the C^*-algebra $\mathscr{B}$, we have

$$\|R(\lambda)\|_{\mathscr{B}} = \sup\{|\lambda - \mu|^{-1} : \mu \in \sigma_{\mathscr{B}}(T)\} = d(\lambda, \sigma_{\mathscr{B}}(T))^{-1}.$$

Therefore $1 > \|R(\lambda)\|_{\mathscr{B}} \|W\|_{\mathscr{B}}$, and so, $I - R(\lambda)W$ is invertible in $\mathscr{B}$. Then $I - R(\lambda)W = R(\lambda)[\lambda - (T + W)]$, and thus, $\lambda - (T + W)$ has a left inverse in $\mathscr{B}$. By a similar argument it also has a right inverse in $\mathscr{B}$. This proves

$$\lambda \in \sigma_{\mathscr{B}}(T + W) \Rightarrow d(\lambda, \sigma_{\mathscr{B}}(T)) \leq \|W\|_{\mathscr{B}}.$$

Now applying this implication with $T + W$ in place of T and $-W$ in place of W, we have

$$\lambda \in \sigma_{\mathscr{B}}(T) \Rightarrow d(\lambda, \sigma_{\mathscr{B}}(T + W)) \leq \|W\|_{\mathscr{B}}.$$

This proves $\delta(\sigma_{\mathscr{B}}(T), \sigma_{\mathscr{B}}(T + W)) \leq \|W\|_{\mathscr{B}}$.

We need the following note.

Note. Let H be a Hilbert space, and assume that $\mathscr{B}$ is a closed $*$-subalgebra of $B(H)$. For $T \in \mathscr{A}_{\mathscr{B}}$, $\sigma_{\mathscr{B}}(T) = \sigma(T)$ (the spectrum of T as an operator).

Proof. It is clear that $\sigma(T) \subseteq \sigma_{\mathscr{B}}(T)$. We now prove the reverse inclusion. Fix $\lambda_0 \in \text{res}_{\mathscr{B}}(T)$. Assume $\mu \notin \sigma(T)$, $\mu \neq \lambda_0$. By direct computation,

$$(1) \qquad [(\lambda_0 - \mu)^{-1} - (\lambda_0 - T)^{-1}]^{-1} = (\lambda_0 - \mu) + (\lambda_0 - \mu)^2(\mu - T)^{-1}.$$

Thus, $(\lambda_0 - \mu)^{-1} \notin \sigma((\lambda_0 - T)^{-1})$. Since $(\lambda_0 - T)^{-1} \in \mathscr{B}$, as is well known, $\sigma((\lambda_0 - T)^{-1}) = \sigma_{\mathscr{B}}((\lambda_0 - T)^{-1})$. Therefore $(\lambda_0 - \mu)^{-1} \in \text{res}_{\mathscr{B}}((\lambda_0 - T)^{-1})$. Then it follows from (1) that $(\mu - T)^{-1} \in \mathscr{B}$.

The proof of the next theorem is a paraphrase of the proof of [K, Theorem 1.14, p. 431].

THEOREM 12. *Assume $\mathscr{B}$ is a closed $*$-subalgebra of $B(H)$. Assume $\{T_n\} \subseteq \mathscr{A}_{\mathscr{B}}$, $T_n^* = T_n$ for all n, $T_n \to T = T^*$ (GS). If $\lambda_0 \in \sigma_{\mathscr{B}}(T)$ and U is an open set with $\lambda_0 \in U$, then there exists N such that*

$$n \geq N \Rightarrow \sigma_{\mathscr{B}}(T_n) \cap U \textit{ is nonempty.}$$

Proof. Since T_n and T are selfadjoint, it follows from the Note that $\sigma_{\mathscr{B}}(T_n) \subseteq \mathbb{R}$ and $\sigma_{\mathscr{B}}(T) \subseteq \mathbb{R}$. Fix $\delta > 0$ such that $\{\mu : |\lambda_0 - \mu| \leq 3\delta\} \subseteq U$. Now by Proposition 3,

$$\|((\lambda_0 + i\delta) - T_n)^{-1} - ((\lambda_0 + i\delta) - T)^{-1}\|_{\mathscr{B}} \to 0.$$

Since these inverses are normal elements of the C^*-algebra $\mathscr{B}$,

$$\|((\lambda_0+i\delta)-T)^{-1}\|_{\mathscr{B}} = \sup\{|\lambda_0+i\delta-\mu|^{-1} : \mu \in \sigma_{\mathscr{B}}(T)\} = \delta^{-1}.$$

Therefore there exists N such that for $n \geq N$,

$$\|((\lambda_0+i\delta)-T_n)^{-1}\|_{\mathscr{B}} = \sup\{|(\lambda_0+i\delta)-\mu|^{-1} : \mu \in \sigma_{\mathscr{B}}(T_n)\} > (2\delta)^{-1}.$$

Thus, there exists $\mu_n \in \sigma_{\mathscr{B}}(T_n)$ with $|(\lambda_0+i\delta)-\mu_n| \leq 2\delta$ for $n \geq N$. This implies $|\lambda_0-\mu_n| \leq 3\delta$ for $n \geq N$.

[B1] and [B2] contain a number of examples of interesting algebras of bounded operators where the theory of affiliated operators has been investigated. Now we look briefly at the interpretation of GS-convergence relative to one of these.

EXAMPLE I (GS-convergence relative to a Jörgens algebra). Let X and Y be Banach spaces on which there is defined a bounded nondegenerate bilinear form $\langle x,y\rangle$, $x \in X$, $y \in Y$. Let $\mathscr{B}$ be the algebra of all $T \in B(X)$ such that there exists $T^\dagger \in B(Y)$ with

$$\langle Tx,y\rangle = \langle x,T^\dagger y\rangle \quad (x \in X,\ y \in Y).$$

Then $\mathscr{B}$ is a Banach algebra of operators with norm $\|T\|_{\mathscr{B}} = \max(\|T\|, \|T^\dagger\|)$ (where the norms on the right are the usual operator norms). $\mathscr{B}$ is called a *Jörgens algebra*. Jörgens algebras are useful in the study of linear integral operators; see [J] and [KR]. Closed operators with domain in X which are affiliated with $\mathscr{B}$ are studied in [B1]. Assume T is such an operator, and $D(T) \subseteq X$ has the property $\{y \in Y : \langle x,y\rangle = 0 \text{ for all } x \in D(T)\} = \{0\}$. As shown in [B1], there exists a closed operator $T^\dagger$ with $D(T^\dagger) \subseteq Y$ having the property

$$\langle Tx,y\rangle = \langle x,T^\dagger y\rangle \quad (x \in D(T),\ y \in D(T^\dagger)).$$

By [B1, Theorem 14], $\lambda \in \operatorname{res}_{\mathscr{B}}(T)$ if and only if $(\lambda-T)^{-1} \in B(X)$ and $(\lambda-T^\dagger)^{-1} \in B(Y)$. Now it is easy to see that if $\{T_n\} \subseteq \mathscr{A}_{\mathscr{B}}$, $T \in \mathscr{A}_{\mathscr{B}}$, then $T_n \to T$ (GS) if and only if there exists N such that $\lambda \in \operatorname{res}_{\mathscr{B}}(T) \cap \operatorname{res}_{\mathscr{B}}(T_n)$ for $n \geq N$, and $\|(\lambda-T_n)^{-1}-(\lambda-T)^{-1}\| \to 0$ and $\|(\lambda-T_n^\dagger)^{-1}-(\lambda-T^\dagger)^{-1}\| \to 0$ as $N \leq n \to \infty$. When $T_n \to T$ (GS) relative to $\mathscr{B}$, then all of the results in this section apply to the spectral theory of T relative to $\mathscr{B}$.

3. Generalized convergence in LMC-algebras. Let $\mathscr{A}$ be a complex algebra with identity such that the topology of $\mathscr{A}$ is determined by a set $\{p_\delta : \delta \in D$, where $(D,\preceq)$ is a directed set$\}$ of algebra seminorms with the property $\delta \preceq \gamma \Rightarrow p_\delta \leq p_\gamma$. Assume $\mathscr{A}$ is complete in the sense that a sequence $\{a_n\} \subseteq \mathscr{A}$ which is Cauchy in all of the seminorms, p_δ, $\delta \in D$, converges to some $a \in \mathscr{A}$. Also, we assume that if $a \in \mathscr{A}$ and $p_\delta(a) = 0$ for all $\delta \in D$, then $a = 0$.

Convergence of a sequence $\{a_n\} \subseteq \mathscr{A}$ to $a \in \mathscr{A}$, which we denote by $a_n \to a\ (\mathscr{A})$, is often too weak a notion of convergence to be useful in problems involving spectral analysis relative to $\mathscr{A}$. Let $\operatorname{Inv}(\mathscr{A})$ be the group of invertible elements of $\mathscr{A}$, and for $a \in \mathscr{A}$ let $\sigma(a) = \{\lambda \in \mathbb{C} : (\lambda-a) \notin \operatorname{Inv}(\mathscr{A})\}$. Again, these algebraic notions are often not very useful from the point of view of analysis relative to $\mathscr{A}$. Now assume that $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ is a Banach algebra with $\mathscr{B}$ a subalgebra of $\mathscr{A}$ such that

$$p_\delta(b) \leq \|b\|_{\mathscr{B}} \quad \text{for all } \delta \in D \text{ and all } b \in \mathscr{B}.$$

We make the standing assumption that the identity of $\mathscr{A}$ is in $\mathscr{B}$. In many examples the Banach algebra $\mathscr{B}$ and the corresponding notions,

$$\begin{aligned}\operatorname{Inv}_{\mathscr{B}}(\mathscr{A}) &\equiv \{a \in \mathscr{A} : a \in \operatorname{Inv}(\mathscr{A}) \text{ and } a^{-1} \in \mathscr{B}\},\\ \sigma_{\mathscr{B}}(a) &\equiv \{\lambda \in \mathbb{C} : \lambda - a \notin \operatorname{Inv}_{\mathscr{B}}(\mathscr{A})\},\\ \operatorname{res}_{\mathscr{B}}(a) &\equiv \mathbb{C} \setminus \sigma_{\mathscr{B}}(a),\end{aligned}$$

are useful in problems involving analysis. One interesting type of convergence associated with $\mathscr{B}$ is the following:

$$a_n \to a\ (\mathscr{B}) \quad \text{if } a_n - a \in \mathscr{B} \text{ for all } n \text{ sufficiently large and } \|a_n-a\|_{\mathscr{B}} \to 0.$$

We call this type of convergence of a sequence *$\mathscr{B}$-convergence*.

We give two examples to further illustrate these ideas.

EXAMPLE II. Let $\mathscr{A}$ be as above, and in addition, assume $\mathscr{A}$ has an involution $*$ and $p_\delta(a*a) = p_\delta(a)^2$ for all $a \in \mathscr{A}$, $\delta \in D$. Following Inoue [I], we call such an algebra a *locally C^*-algebra*. We assume in this case that the Banach algebra $\mathscr{B}$ is a C^*-algebra. A particular example we refer to often in this section is the following. Let Ω be a locally compact Hausdorff space, and let $C(\Omega)$ be the algebra of all complex-valued continuous functions on Ω. Let D be the set of all nonempty compact subsets of Ω directed by inclusion. For $\delta \in D$, $f \in C(\Omega)$, let

$$p_\delta(f) = \sup\{|f(\omega)| : \omega \in \delta\}.$$

Let $\mathscr{B}$ be $BC(\Omega)$, the algebra of all bounded continuous functions on Ω, and $\|f\|_{\mathscr{B}} = \sup\{|f(\omega)| : \omega \in \Omega\}$. Clearly, $\operatorname{Inv}_{\mathscr{B}}(C(\Omega)) = \{f \in C(\Omega) : |f(\omega)|$ is bounded away from zero on $\Omega\}$, and $\sigma_{\mathscr{B}}(f)$ is the closure of the range of f. Also, $f_n \to f\ (\mathscr{B})$ exactly when $f_n \to f$ uniformly on Ω.

Locally C^*-algebras have been extensively studied; see for example [A], [I], [F], [S].

EXAMPLE III. Let $\mathscr{A}$ be the algebra of all infinite complex lower triangular matrices. Thus a matrix $T = \{t_{jk}\}_{j\geq 1,\, k\geq 1}$ is in $\mathscr{A}$ when $t_{jk} = 0$

whenever $k > j$. For $n \geq 1$ and $T \in \mathscr{A}$, let

$$p_n(T) = \max_{1\leq j\leq n} \Big(\sum_{k=1}^{n} |t_{jk}|\Big).$$

Let $\mathscr{B}$ be the algebra of all bounded linear operators on the Banach space c_0; see [TL, Theorem 6.3, p. 221]. In general, a matrix T determines a closed operator $\overline{T}$ on c_0 by defining $D(\overline{T}) = \{\{a_k\} \in c_0 : T(\{a_k\}) \in c_0\}$, and $\overline{T}(\{a_k\}) = \{\sum_{k=1}^{j} t_{jk}a_k\}_{j\geq 1}$. In this case, $\mathrm{Inv}_{\mathscr{B}}(\mathscr{A})$ is the set of those $T \in \mathscr{A}$ such that $\overline{T}$ has a bounded inverse on c_0, and for $T \in \mathscr{A}$, $\sigma_{\mathscr{B}}(T)$ is the usual spectrum of the closed operator $\overline{T}$. A sequence $T_n \to T$ ($\mathscr{B}$) when $T - T_n$ determines a bounded operator on c_0 for all n sufficiently large and the operator norm of $T - T_n$ goes to zero.

In order to apply the previous results on (GS)-convergence to the algebra $\mathscr{A}$, we define some natural closed operators determined by elements $a \in \mathscr{A}$.

Definition 13. For $a \in \mathscr{A}$, define $D(L_a) = \{b \in \mathscr{B} : ab \in \mathscr{B}\}$; $L_a(b) = ab$ for $b \in D(L_a)$; $D(R_a) = \{b \in \mathscr{B} : ba \in \mathscr{B}\}$; $R_a(b) = ba$ for $b \in D(R_a)$.

Note that $b \to L_b$ is an isometric algebra isomorphism of $\mathscr{B}$ onto the closed subalgebra $\{L_b : b \in \mathscr{B}\}$ of $B(\mathscr{B})$. We again denote this subalgebra of operators by $\mathscr{B}$. Similarly, $\{R_b : b \in \mathscr{B}\}$ can be completely identified with $\mathscr{B}$.

Now we prove a basic result concerning the operators L_a and R_a.

Theorem 14. (1) *For all $a \in \mathscr{A}$, L_a and R_a are closed operators on $\mathscr{B}$.*

(2) *If $\lambda \in \mathrm{res}_{\mathscr{B}}(L_a) \cap \mathrm{res}_{\mathscr{B}}(R_a)$, then $\lambda \in \mathrm{res}_{\mathscr{B}}(a)$.*

(3) *If $\mathrm{res}_{\mathscr{B}}(a)$ is nonempty, then $\mathrm{res}_{\mathscr{B}}(a) = \mathrm{res}_{\mathscr{B}}(L_a) = \mathrm{res}_{\mathscr{B}}(R_a)$.*

Proof. First we show that L_a is closed; the proof that R_a is closed is similar. Assume $\{b_n\} \subseteq D(L_a)$ and $b, c \in \mathscr{B}$ with

$$\|b_n - b\|_{\mathscr{B}} \to 0 \quad \text{and} \quad \|L_a(b_n) - c\|_{\mathscr{B}} \to 0.$$

By definition $\{ab_n\} \subseteq \mathscr{B}$, and we have $\|ab_n - c\|_{\mathscr{B}} \to 0$. Therefore for every seminorm p_δ, $\delta \in D$,

$$p_\delta(ab_n - ab) \to 0 \quad \text{and} \quad p_\delta(ab_n - c) \to 0.$$

This implies $ab = c$. Thus $b \in D(L_a)$ and $L_a(b) = c$.

Now assume $\lambda \in \mathrm{res}_{\mathscr{B}}(L_a) \cap \mathrm{res}_{\mathscr{B}}(R_a)$. Then there are $c, d \in \mathscr{B}$ such that L_c is the inverse of the closed operator $\lambda - L_a$ and R_d is the inverse of $\lambda - R_a$. Since $1 \in \mathscr{B}$, we have

$$1 = (\lambda - L_a)L_c(1) = (\lambda - a)c; \qquad 1 = (\lambda - R_a)R_d(1) = d(\lambda - a).$$

Therefore $\lambda \in \mathrm{res}_{\mathscr{B}}(a)$.

Assume $\lambda \in \mathrm{res}_{\mathscr{B}}(a)$, so $(\lambda - a)^{-1} \in \mathscr{B}$. Set $c = (\lambda - a)^{-1}$. Clearly,

$$L_c(\lambda - L_a)(b) = c(\lambda - a)b = b \quad (b \in D(L_a)).$$

Also,

$$L_c(b) = cb \in D(L_a), \quad \text{and} \quad (\lambda - L_a)L_c(b) = (\lambda - a)cb = b \quad (b \in \mathscr{B}).$$

Thus L_c is an inverse in $\mathscr{B}$ for $\lambda - L_a$. Similarly, R_c is an inverse in $\mathscr{B}$ for $\lambda - R_a$. This proves $\mathrm{res}_{\mathscr{B}}(a) \subseteq \mathrm{res}_{\mathscr{B}}(L_a) \cap \mathrm{res}_{\mathscr{B}}(R_a)$.

Conversely, assume $\lambda \in \mathrm{res}_{\mathscr{B}}(L_a)$. Then there exists $c \in \mathscr{B}$ such that L_c is the inverse of the operator $\lambda - L_a$. This implies

$$1 = (\lambda - L_a)L_c(1) = (\lambda - a)c.$$

Fix $\lambda_0 \in \mathrm{res}_{\mathscr{B}}(a)$, and set $d = (\lambda_0 - a)^{-1} \in \mathscr{B}$. Now $d \in D(L_a)$, and therefore,

$$c(\lambda - a)d = L_c(\lambda - L_a)d = d.$$

Multiplying this equality on the right by $\lambda_0 - a$, we have $c(\lambda - a) = 1$. Thus, $\lambda \in \mathrm{res}_{\mathscr{B}}(a)$. This proves $\mathrm{res}_{\mathscr{B}}(a) = \mathrm{res}_{\mathscr{B}}(L_a)$. The proof that $\mathrm{res}_{\mathscr{B}}(a) = \mathrm{res}_{\mathscr{B}}(R_a)$ is similar.

Definition 15. Let $\mathscr{A}$ be a LMC-algebra with topology defined by $\{p_\delta : \delta \in D\}$. Assume that $\mathscr{B}$ is a Banach algebra, $\mathscr{B}$ is a subalgebra of $\mathscr{A}$, and for all $\delta \in D$, $p_\delta(b) \leq \|b\|_{\mathscr{B}}$ for all $b \in \mathscr{B}$. A sequence $\{a_n\} \subseteq \mathscr{A}$ *converges to $a \in \mathscr{A}$ in the generalized sense* (*relative to $\mathscr{B}$*) if there exist $\lambda_0 \in \mathrm{res}_{\mathscr{B}}(a)$ and N such that $\lambda_0 \in \mathrm{res}_{\mathscr{B}}(a_n)$ for $n \geq N$, and

$$\|(\lambda_0 - a_n)^{-1} - (\lambda_0 - a)^{-1}\|_{\mathscr{B}} \to 0 \quad \text{as } N \leq n \to \infty.$$

Theorem 16. *Assume $\{a_n\} \subseteq \mathscr{A}$, $a \in \mathscr{A}$. Then $a_n \to a$ (GS) is equivalent to:*

(i) *$\mathrm{res}_{\mathscr{B}}(L_a) \cap \mathrm{res}_{\mathscr{B}}(R_a)$ is nonempty; and*

(ii) *$L_{a_n} \to L_a$ (GS); and*

(iii) *$R_{a_n} \to R_a$ (GS).*

Proof. Assume $a_n \to a$ (GS). Fix $\lambda \in \mathrm{res}_{\mathscr{B}}(a)$ and N such that $\lambda \in \mathrm{res}_{\mathscr{B}}(a_n)$ for $n \geq N$, and $\|(\lambda - a_n)^{-1} - (\lambda - a)^{-1}\|_{\mathscr{B}} \to 0$ as $N \leq n \to \infty$. Set $c_n = (\lambda - a_n)^{-1}$, $n \geq N$, and $c = (\lambda - a)^{-1}$. By Theorem 14(3), $\lambda \in \mathrm{res}_{\mathscr{B}}(L_{a_n}) \cap \mathrm{res}_{\mathscr{B}}(R_{a_n})$ and $\lambda \in \mathrm{res}_{\mathscr{B}}(L_a) \cap \mathrm{res}_{\mathscr{B}}(R_a)$. Clearly, for $n \geq N$, $\|(\lambda - L_{a_n})^{-1} - (\lambda - L_a)^{-1}\|_{\mathscr{B}} = \|c_n - c\|_{\mathscr{B}} \to 0$. Similarly, $\|(\lambda - R_{a_n})^{-1} - (\lambda - R_a)^{-1}\|_{\mathscr{B}} \to 0$. This proves that (i), (ii), and (iii) hold.

Conversely, assume (i), (ii), and (iii) are true. Choose $\lambda \in \mathrm{res}_{\mathscr{B}}(L_a) \cap \mathrm{res}_{\mathscr{B}}(R_a)$. Applying Proposition 3, we see that there exists N such that for $n \geq N$, $\lambda \in \mathrm{res}_{\mathscr{B}}(L_{a_n}) \cap \mathrm{res}_{\mathscr{B}}(R_{a_n})$ and $\|(\lambda - L_{a_n})^{-1} - (\lambda - L_a)^{-1}\|_{\mathscr{B}} \to 0$ and $\|(\lambda - R_{a_n})^{-1} - (\lambda - R_a)^{-1}\|_{\mathscr{B}} \to 0$ as $N \leq n \to \infty$. By Theorem 14(2),

$\lambda \in \text{res}_{\mathscr{B}}(a_n)$ for $n \geq N$. Let $c_n = (\lambda - a_n)^{-1}$ for $n \geq N$ and $c = (\lambda - a)^{-1}$. Then

$$\|c_n - c\|_{\mathscr{B}} = \|(\lambda - L_{a_n})^{-1} - (\lambda - L_a)^{-1}\|_{\mathscr{B}} \to 0 \quad \text{as } N \leq n \to \infty.$$

This proves $a_n \to a$ (GS).

When restated in the terminology of LMC-algebras, all of the results in §2 hold in this context. The restatement is usually straightforward. For illustrative purposes, we restate two results.

RESTATEMENT OF THEOREM 6 (for the case of an LMC-algebra $\mathscr{A}$). *Assume $a_n \to a$ (GS). Let Γ be a compact subset of $\mathbb{C}$ with $\Gamma \subseteq \text{res}_{\mathscr{B}}(a)$.*

(1) *There exists N such that for all $n \geq N$, $\Gamma \subseteq \text{res}_{\mathscr{B}}(a_n)$.*

(2) *Let N be as in (1). Then $\|(\lambda - a_n)^{-1} - (\lambda - a)^{-1}\|_{\mathscr{B}} \to 0$ as $N \leq n \to \infty$ uniformly in $\lambda \in \Gamma$.*

Now assume that $\mathscr{A}$ is a locally C^*-algebra. We always assume in this situation that $\mathscr{B}$ is a C^*-algebra. Theorems 11 and 12 from §2 have restatements in this context.

RESTATEMENT OF THEOREM 11 (for the case where $\mathscr{A}$ is a locally C^*-algebra and $\mathscr{B}$ is a C^*-algebra). *Assume $t = t^* \in \mathscr{A}$ and $w = w^* \in \mathscr{B}$. Then*

$$\delta(\sigma_{\mathscr{B}}(t), \sigma_{\mathscr{B}}(t+w)) \leq \|w\|_{\mathscr{B}}.$$

There are two approaches to proving the modified results for LMC-algebras (such as the restatements given above). First, one can apply the propositions from §2 to the closed operators L_a and R_a, and use Theorems 14 and 16. Secondly, one can modify the proofs given in §2 directly, replacing statements such as $T_n \to T$ (GS) by $a_n \to a$ (GS), $\sigma_{\mathscr{B}}(T)$ by $\sigma_{\mathscr{B}}(a)$, etc. Either approach works.

Turning to analysis in an LMC-algebra $\mathscr{A}$ relative to a Banach subalgebra $\mathscr{B}$, for $\{a_n\} \subseteq \mathscr{A}$ and $a \in \mathscr{A}$, there are three notions of convergence of $\{a_n\}$ to a:

$$a_n \to a\ (\mathscr{A}); \quad a_n \to a\ (\mathscr{B}); \quad a_n \to a\ (\text{GS}).$$

Now we compare these concepts of convergence, and give some examples.

Assume $a_n \to a$ (GS). Fix $\lambda_0 \in \text{res}_{\mathscr{B}}(a)$ such that $\lambda_0 - a_n$ are in $\text{Inv}_{\mathscr{B}}(\mathscr{A})$ for all n and $\|(\lambda_0 - a_n)^{-1} - (\lambda_0 - a)^{-1}\|_{\mathscr{B}} \to 0$. For each $\delta \in D$ it follows that $p_\delta((\lambda_0 - a_n)^{-1} - (\lambda_0 - a)^{-1}) \to 0$. Fix δ and choose N such that for $n \geq N$,

$$p_\delta((\lambda_0 - a_n)^{-1} - (\lambda_0 - a)^{-1}) p_\delta(\lambda_0 - a) \leq 1/2.$$

Following the computation in [BD, Lemma 5, p. 11], we have for $n \geq N$,

$$p_\delta(a_n - a) = p_\delta((\lambda_0 - a_n) - (\lambda_0 - a)) \leq 2p_\delta((\lambda_0 - a)^{-1} - (\lambda_0 - a_n)^{-1}).$$

Thus $p_\delta(a_n - a) \to 0$ for each $\delta \in D$, so $a_n \to a$ ($\mathscr{A}$).

The argument above shows that $a_n \to a$ (GS) $\Rightarrow a_n \to a$ ($\mathscr{A}$). It is easy to find examples where the converse fails. Let $\mathscr{A} = C(\mathbb{R})$, $\mathscr{B} = BC(\mathbb{R})$. Let f be the zero function. Choose any sequence $\{f_n\} \subseteq \mathscr{A}$ with $f_n(x) = 0$ for all $x \notin [n, n+1]$ and such that 1 is in the range of f_n for all n. Clearly $f_n \to f$ ($\mathscr{A}$) and $1 \notin \sigma_{\mathscr{B}}(f)$. But $1 - f_n \notin \text{Inv}(\mathscr{A})$ for any n, so by Proposition 3, $f_n \not\to f$ (GS).

Turning to $\mathscr{B}$-convergence, assume $a_n \to a\,(\mathscr{B})$ and $\sigma_{\mathscr{B}}(a) \neq \mathbb{C}$. Fix $\lambda_0 \in \text{res}_{\mathscr{B}}(a)$. Then $(\lambda_0 - a_n) \to (\lambda_0 - a)$ ($\mathscr{B}$), and by Proposition 2, there exists N such that

$$n \geq N \Rightarrow (\lambda_0 - a_n)^{-1} \in \text{Inv}_{\mathscr{B}}(\mathscr{A})$$

and as $n \geq N$, $n \to \infty$,

$$\|(\lambda_0 - a_n)^{-1} - (\lambda_0 - a)^{-1}\|_{\mathscr{B}} \to 0.$$

Thus in the case when $\sigma_{\mathscr{B}}(a) \neq \mathbb{C}$, $a_n \to a$ (GS). This proves that, in general, convergence ($\mathscr{B}$) is stronger than convergence (GS). There is an exceptional case which we now illustrate with an example. Let $\mathscr{A} = C(\Omega)$, $\mathscr{B} = BC(\Omega)$ where $\Omega = \mathbb{C} \setminus \{0\}$, and let $f(z) = z$ for $z \in \Omega$. Define $f_n \in C(\Omega)$ by

$$f_n(z) = \begin{cases} z & \text{if } |z| \geq n^{-1}, \\ z|z|^{-1}n^{-1} & \text{if } 0 < |z| \leq n^{-1}. \end{cases}$$

It is easy to see that $f - f_n \in BC(\Omega)$ and $f_n \to f$ ($\mathscr{B}$). But $f_n \not\to f$ (GS) since $\sigma_{\mathscr{B}}(f) = \mathbb{C}$.

Now assume $b \in \mathscr{B}$ and $a_n \to b$ (GS). By definition there exists $\lambda_0 \in \text{res}_{\mathscr{B}}(b)$ such that $\|(\lambda_0 - a_n)^{-1} - (\lambda_0 - b)^{-1}\|_{\mathscr{B}} \to 0$. Since $(\lambda_0 - b)^{-1} \in \text{Inv}_{\mathscr{B}}(\mathscr{A})$, it follows from Proposition 2 that $(\lambda_0 - a_n)^{-1} \in \text{Inv}_{\mathscr{B}}(\mathscr{A})$ for all n sufficiently large. Therefore $a_n \in \mathscr{B}$ for all large n and $\|a_n - b\|_{\mathscr{B}} \to 0$.

The restatements of both Proposition 4(4) and Theorem 5 involve situations where $a_n \to a$ (GS), $b \in \mathscr{B}$, and $a_n + b \to a + b$ (GS). Now we note that even in the best of circumstances it may be true that $a_n \to a$ (GS), $b \in \mathscr{B}$, but $a_n + b \not\to a + b$ (GS). Consider the following example. Let $\Omega = \mathbb{C} \setminus \{0\}$. We work in the commutative locally C^*-algebra $C(\Omega)$. Let $B = BC(\Omega)$. Define f and b in $C(\Omega)$ by

$$f(z) = \begin{cases} z & \text{if } |z| \geq 1, \\ z/|z| & \text{if } 0 < |z| \leq 1, \end{cases} \qquad b(z) = -z/|z| \quad \text{for } z \in \Omega.$$

Then $b \in \mathscr{B}$, and

$$f(z) + b(z) = \begin{cases} z(1 - |z|^{-1}) & \text{if } |z| > 1, \\ 0 & \text{if } 0 < |z| \leq 1. \end{cases}$$

Let $\mu \in \mathbb{C} \setminus \{0\}$ be arbitrary, and let $r = |\mu| > 0$. Let $z_0 = (r+1)r^{-1}\mu$, and note that $|z_0| = (r+1)r^{-1}r = r + 1 > 1$. Then

$$f(z_0) + b(z_0) = z_0(1 - |z_0|^{-1}) = (r+1)r^{-1}\mu(1 - (r+1)^{-1}) = \mu.$$

It follows that $\sigma_{\mathscr{B}}(f+b)=\mathbb{C}$. For each $n\ge 1$, let

$$f_n(z)=\begin{cases} nz|z|^{-1} & \text{if } |z|\ge n,\\ z & \text{if } 1\le |z|\le n,\\ z|z|^{-1} & \text{if } 0<|z|\le 1.\end{cases}$$

It is easily verified that f^{-1}, $f_n^{-1}\in BC(\Omega)$ for all n, and $\|f_n^{-1}-f^{-1}\|_{\mathscr{B}}\to 0$ as $n\to\infty$. Therefore $f_n\to f$ (GS). But $f_n+b\not\to f+b$ (GS) as $\sigma_{\mathscr{B}}(f+b)=\mathbb{C}$.

The restatement of Theorem 9 asserts the local upper semicontinuity of $\sigma_{\mathscr{B}}(\cdot)$ when $a_n\to a$ (GS). It is easy to see that $\mathscr{A}$-convergence is too weak to imply local upper semicontinuity. For example, let $\mathscr{A}=C(\mathbb{C})$, $B=BC(\mathbb{C})$, and set $f_n(z)=n^{-1}z$ for $n\ge 1$, $f(z)\equiv 0$. Then $f_n\to f$ ($\mathscr{A}$), but $\sigma_{\mathscr{B}}(f)=\{0\}$ and $\sigma_{\mathscr{B}}(f_n)=\mathbb{C}$, $n\ge 1$. Also, GS-convergence does not imply upper semicontinuity. For example, let $\mathscr{A}=C([1,\infty))$, $\mathscr{B}=BC([1,\infty))$, and set $f_n(x)=(1+in^{-1})x$ for $n\ge 1$, $f(x)=x$. Then $f_n\to f$ (GS). Let

$$U=\{z\in\mathbb{C}: d(z,[1,\infty))<1/2\},$$

and note that $\sigma_{\mathscr{B}}(f)=[1,\infty)\subseteq U$. For each n, $n+i=f_n(n)\in\sigma_{\mathscr{B}}(f_n)$ but $n+i\notin U$.

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