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A lifting theorem for locally convex subspaces of L_0

by

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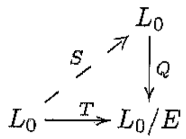
Abstract. We prove that for every closed locally convex subspace E of L_0 and for any continuous linear operator T from L_0 to L_0/E there is a continuous linear operator S from L_0 to L_0 such that $T = QS$ where Q is the quotient map from L_0 to L_0/E.

0. Introduction. Let E be a subspace of $L_0 = L_0[0,1]$, the space of all measurable functions from $[0,1]$ to $\mathbb{R}$. Let T be an operator from L_0 to L_0/E. What conditions on E ensure that we can find an operator S that makes the following diagram commute?



A. Pełczyński was the first to ask if locally convex subspaces E have this property. If E is locally bounded then we can find such an operator (Kalton–Peck [2]). Peck–Starbird [6] showed that this is also true when E is isomorphic to ω, the space of all real sequences. The goal of the present paper is to show that if E is locally convex then we can complete the previous diagram.

We will state some notation. We will let μ represent the standard Lebesgue measure. We also define the map $f \mapsto \|f\|_0$ $(L_0 \to \mathbb{R})$ as

$$\|f\|_0 = \int_0^1 \frac{|f(x)|}{1+|f(x)|}\,dx.$$

This map is an F-norm on L_0, that is,

(i) $\|f\|_0 > 0$ for $f \neq 0$,
(ii) $\|\alpha f\|_0 \le \|f\|_0$ for $|\alpha| \le 1$ and $f \in L_0$,

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(iii) $\lim_{\alpha\to 0}\|\alpha f\|_0 = 0$ for $f\in L_0$,
(iv) $\|f+g\|_0 \le \|f\|_0 + \|g\|_0$ for $f,g\in L_0$.

The map also induces a metric on L_0. The topology induced by the L_0 metric is just the topology of convergence in measure. For $f\in L_0$ we define $\sigma : L_0 \to [0,1]$ by

$$\sigma(f) = \sup_{n\in\mathbb{N}} \|nf\|_0 = \lim_{n\to\infty} \|nf\|_0.$$

By the dominated convergence theorem we can see that $\sigma(f) = \mu(\operatorname{supp} f)$, where $\operatorname{supp} f = \{x : |f(x)| > 0\}$. The F-norm on the quotient space L_0/E is defined in the usual way:

$$\|\gamma\|_{L_0/E} = \inf_{f\in\gamma} \|f\|_0 \quad \text{for all } \gamma\in L_0/E.$$

For a subset A of $[0,1]$ we will let $L_0(A)$ mean the subspace of L_0 consisting of all functions supported on A. We define

$$\|f\|_{L_0(A)} = \|f\cdot\chi_A\|_0,$$

where χ_A is the characteristic function of A.

1. Preliminary lemmas. We have a lifting theorem for locally bounded subspaces (see Theorem 3.6 of [2]) and we will see that locally convex subspaces are in some sense almost locally bounded. The lemmas that follow show us that the "unbounded part" of a locally convex subspace is arbitrarily small. Lemma 1.2 is at the heart of this argument. However, we first need a lemma from Paley and Zygmund [5].

LEMMA 1.1. *Let $\alpha > \beta \ge 0$. If $f \in L_0[0,1]$ such that $\int_0^1 f \ge \alpha$ and $\|f\|_2 = 1$ then*

$$\mu(t : f(t)\ge\beta) \ge (\alpha-\beta)^2.$$

Proof. We have

$$\begin{aligned}\alpha \le \int_0^1 f &= \int_{\{f\ge\beta\}} f + \int_{\{f<\beta\}} f \le \int_0^1 f\cdot I_{\{f\ge\beta\}} + \beta\\ &\le \|f\|_2\cdot\|I_{\{f\ge\beta\}}\|_2 + \beta \quad \text{(Schwarz Inequality)}\\ &= \sqrt{\mu(t: f(t)\ge\beta)} + \beta.\end{aligned}$$

So $\mu(t : f(t)\ge\beta) \ge (\alpha-\beta)^2$. ■

Notice that Rademacher functions do not appear in the statement of the next lemma but they play a key role in the proof. Recall that all the Rademacher functions act on $[0,1]$ and have values in the two-point set $\{-1,1\}$. The first Rademacher function, r_1, is 1 everywhere. The second, r_2, is 1 on $[0,1/2)$ and -1 on $[1/2,1]$; r_3 is 1 on $[0,1/4)$ and $[1/2,3/4)$ but -1 on $[1/4,1/2)$ and $[3/4,1]$; and so on. For convenience we will say that a sequence of functions $(f_i)_{i=1}^\infty$ is *δ-tapering* if $\|2^i\cdot f_i\|_0 \le \delta$ for all $i\ge 1$.

LEMMA 1.2. *Let E be a locally convex subspace of L_0. For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $(f_i)_{i=1}^\infty \subset E$ is δ-tapering then*

$$\mu\Big(\bigcup_{i=1}^\infty \{x : |f_i(x)| > 1\}\Big) \le \varepsilon.$$

Moreover, δ can be chosen to be any positive number such that the closed convex hull of $\{f\in E : \|f\|_0 \le \delta\}$ is contained in $\{f\in L_0 : \|f\|_0 \le \varepsilon/80\}$.

Proof. Consider the following function on $[0,1]$:

$$g(t) = \Big|\Big(\sum_{j=1}^N a_j^2\Big)^{-1/2} \sum_{k=1}^N a_k r_k(t)\Big|,$$

where $a_1,\ldots,a_N\in\mathbb{R}$ and $r_1,\ldots,r_N$ are the first N Rademacher functions. Then from Khinchin's inequality we have

$$\int_0^1 g = \Big(\sum_{j=1}^N a_j^2\Big)^{-1/2}\cdot\Big\|\sum_{k=1}^N a_k\cdot r_k\Big\|_1 \ge \Big(\sum_{j=1}^N a_j^2\Big)^{-1/2}\cdot\frac{1}{2}\Big(\sum_{k=1}^N a_k^2\Big)^{1/2} = \frac{1}{2}.$$

Since the Rademacher functions are orthonormal over $[0,1]$ we have

$$\|g\|_2^2 = \Big\|\sum_{k=1}^N \Big(\frac{a_k}{(\sum_{j=1}^N a_j^2)^{1/2}} r_k\Big)\Big\|_2^2 = \sum_{k=1}^N \Big(\frac{a_k}{(\sum_{j=1}^N a_j^2)^{1/2}}\Big)^2 = 1.$$

We are now ready to use Lemma 1.1 with $\alpha = 1/2$ and $\beta = 1/4$:

$$\mu\Big(t : \Big(\sum_{j=1}^N a_j^2\Big)^{-1/2}\Big|\sum_{k=1}^N a_k r_k(t)\Big| \ge \frac{1}{4}\Big) \ge \Big(\frac{1}{2}-\frac{1}{4}\Big)^2 = \frac{1}{16}.$$

Therefore,

$$(*)\qquad \mu\Big(t : \Big|\sum_{k=1}^N a_k r_k(t)\Big| \ge \frac{1}{4}\Big(\sum_{j=1}^N a_j^2\Big)^{1/2}\Big) \ge \frac{1}{16}$$

for $a_1,\ldots,a_N\in\mathbb{R}$.

Let $\varepsilon > 0$ be given. Since E is locally convex there is a $\delta > 0$ such that the closed convex hull of $\{f\in E : \|f\|_0 \le \delta\}$ is contained in $\{f\in L_0 : \|f\|_0 \le \varepsilon/80\}$. Suppose $(f_i)_{i=1}^\infty \subset E$ is δ-tapering. Then for every $N\ge 0$ we have

$$\frac{\varepsilon}{80} = \int_0^1 \frac{\varepsilon}{80}\,dt \ge \int_0^1 \Big\|\sum_{i=1}^N \frac{1}{2^i} r_i(t) 2^i f_i(x)\Big\|_0\,dt \quad \text{(local convexity)}$$

$$= \int_0^1 \int_0^1 \frac{|\sum_{i=1}^N r_i(t) f_i(x)|}{1 + |\sum_{i=1}^N r_i(t) f_i(x)|}\, dx\, dt$$

$$= \int_0^1 \int_0^1 \frac{|\sum_{i=1}^N f_i(x) r_i(t)|}{1 + |\sum_{i=1}^N f_i(x) r_i(t)|}\, dt\, dx \qquad \text{(Tonelli)}$$

$$\geq \frac{1}{16} \int_0^1 \frac{\frac14 \sqrt{\sum_{i=1}^N f_i(x)^2}}{1 + \frac14 \sqrt{\sum_{i=1}^N f_i(x)^2}}\, dx \qquad \text{(by } (*))$$

So

$$\int_0^1 \frac{\frac14 \sqrt{\sum_{i=1}^N f_i(x)^2}}{1 + \frac14 \sqrt{\sum_{i=1}^N f_i(x)^2}}\, dx \leq 16 \frac{\varepsilon}{80} = \frac{\varepsilon}{5}$$

for all $N \geq 1$. Therefore $\mu(x : (\sum_{i=1}^N f_i(x)^2)^{1/2} > 1) \leq \varepsilon$ for all N. Indeed, suppose not; then

$$\int_0^1 \frac{\frac14 \sqrt{\sum_{i=1}^N f_i(x)^2}}{1 + \frac14 \sqrt{\sum_{i=1}^N f_i(x)^2}}\, dx > \varepsilon\Big(\frac{1/4}{1+1/4}\Big) = \frac{\varepsilon}{5}.$$

This is a contradiction. Thus $\mu(x : \sum_{i=1}^N f_i(x)^2 > 1) \leq \varepsilon$ for all $N \geq 1$. Letting N go to infinity we get $\mu(x : \sum_{i=1}^\infty f_i(x)^2 > 1) \leq \varepsilon$. Finally, since

$$\bigcup_{i=1}^\infty \{x : |f_i(x)| > 1\} \subset \Big\{x : \sum_{i=1}^\infty f_i(x)^2 > 1\Big\}$$

we can conclude that

$$\mu\Big(\bigcup_{i=1}^\infty \{x : |f_i(x)| > 1\}\Big) \leq \varepsilon. \ \blacksquare$$

Lemmas 1.3 and 1.4 find an arbitrarily small set that contains all the "unboundedness" of E.

LEMMA 1.3. *Let E be a locally convex subspace of L_0. Let $\varepsilon > 0$ and find $\delta > 0$ so that the closed convex hull of $\{f \in E : \|f\|_0 \leq \delta\}$ is contained in $\{f \in L_0 : \|f\|_0 \leq \varepsilon/80\}$. Then for any countable collection $((f_i^{(k)})_{i=1}^\infty)_{k=1}^\infty \subset E$ of δ-tapering sequences we have*

$$\mu\Big(\bigcup_{k=1}^\infty \bigcap_{l=1}^\infty \bigcup_{i=l}^\infty \{x : |f_i^{(k)}(x)| > 1\}\Big) \leq \varepsilon.$$

Proof. We will start by considering the first n sequences. Let $N_1 < \ldots < N_{n-1}$. Then

$$(f_i^{(1)})_{i=1}^{N_1} \cup (f_i^{(2)})_{i=N_1+1}^{N_2} \cup \ldots \cup (f_i^{(n-1)})_{i=N_{n-2}+1}^{N_{n-1}} \cup (f_i^{(n)})_{i=N_{n-1}+1}^{\infty}$$

is another δ-tapering sequence. So for all N_{n-1} we have

$$\begin{aligned}\varepsilon &\geq \mu\Big(\bigcup_{i=1}^{N_1}\{x : |f_i^{(1)}(x)| > 1\} \cup \bigcup_{i=N_1+1}^{N_2} \{x : |f_i^{(2)}(x)| > 1\} \cup \ldots \\ &\qquad \cup \bigcup_{i=N_{n-2}+1}^{N_{n-1}} \{x : |f_i^{(n-1)}(x)| > 1\} \cup \bigcup_{i=N_{n-1}+1}^{\infty} \{x : |f_i^{(n)}(x)| > 1\}\Big) \\ &\geq \mu\Big(\bigcup_{i=1}^{N_1}\{x : |f_i^{(1)}(x)| > 1\} \cup \bigcup_{i=N_1+1}^{N_2} \{x : |f_i^{(2)}(x)| > 1\} \cup \ldots \\ &\qquad \cup \bigcup_{i=N_{n-2}+1}^{N_{n-1}} \{x : |f_i^{(n-1)}(x)| > 1\} \cup \bigcap_{l=1}^{\infty}\bigcup_{i=l}^{\infty} \{x : |f_i^{(n)}(x)| > 1\}\Big).\end{aligned}$$

Let N_{n-1} go to infinity to obtain

$$\begin{aligned}\mu\Big(&\bigcup_{i=1}^{N_1}\{x : |f_i^{(1)}(x)| > 1\} \cup \bigcup_{i=N_1+1}^{N_2} \{x : |f_i^{(2)}(x)| > 1\} \cup \ldots \\ &\cup \bigcup_{i=N_{n-2}+1}^{\infty} \{x : |f_i^{(n-1)}(x)| > 1\} \cup \bigcap_{l=1}^{\infty}\bigcup_{i=l}^{\infty} \{x : |f_i^{(n)}(x)| > 1\}\Big) \leq \varepsilon.\end{aligned}$$

Repeat this step $n-2$ times to get

$$\mu\Big(\bigcup_{k=1}^{n} \bigcap_{l=1}^\infty \bigcup_{i=l}^\infty \{x : |f_i^{(k)}(x)| > 1\}\Big) \leq \varepsilon.$$

Let n go to infinity to get the desired conclusion,

$$\mu\Big(\bigcup_{k=1}^\infty \bigcap_{l=1}^\infty \bigcup_{i=l}^\infty \{x : |f_i^{(k)}(x)| > 1\}\Big) \leq \varepsilon. \ \blacksquare$$

In the proof of Lemma 1.4 we use the fact that the space of all Lebesgue measurable subsets of $[0,1]$ is a complete separable metric space. The distance definition is

$$d(A,B) = \mu(A \Delta B),$$

where $A\Delta B$ stands for the symmetric difference $(A\backslash B)\cup(B\backslash A)$. We consider A and B to be identical if $\mu(A\Delta B) = 0$.

LEMMA 1.4. *Let E be a locally convex subspace of L_0. Let $\varepsilon > 0$ and find $\delta > 0$ such that the closed convex hull of $\{f \in E : \|f\|_0 \leq \delta\}$ is contained in*

$\{f \in L_0 : \|f\|_0 \le \varepsilon/80\}$. *Then there is a measurable set* A, $\mu(A) \le \varepsilon$, *such that if* $(f_i)_{i=1}^{\infty} \subset E$ *is any* δ*-tapering sequence then*

$$\mu\Big(\bigcap_{l=1}^{\infty}\bigcup_{i=l}^{\infty}\{x : |f_i(x)| > 1\} \setminus A\Big) = 0.$$

Proof. Let $(f_i^{(t)})_{i=1}^{\infty}$, $t \in T$, be the collection of all sequences in E such that $\|2^i \cdot f_i^{(t)}\|_0 \le \delta$ for all i. T could be an uncountable index set. For each $t \in T$ define

$$A_t = \bigcap_{l=1}^{\infty}\bigcup_{i=l}^{\infty}\{x : |f_i^{(t)}(x)| > 1\}.$$

$(A_t)_{t\in T}$ is a subspace of the separable metric space consisting of all Lebesgue measurable subsets of $[0,1]$. So $(A_t)_{t\in T}$ is separable. Let $(A_{t_j})_{j=1}^{\infty}$ be a countable dense subset. Let

$$A = \bigcup_{j=1}^{\infty} A_{t_j}.$$

By Lemma 1.3, $\mu(A) \le \varepsilon$. Let $\eta > 0$ and $t \in T$ be given. There is a j such that $\mu(A_{t_j} \Delta A_t) < \eta$ since $(A_{t_j})_{j=1}^{\infty}$ is dense in $(A_t)_{t\in T}$. Further,

$$\mu(A_t \setminus A) \le \mu(A_t \setminus A_{t_j}) \le \mu(A_t \Delta A_{t_j}) < \eta.$$

Since $\eta > 0$ is arbitrary, $\mu(A_t \setminus A) = 0$ for all $t \in T$. ■

We are now ready to prove the main theorem. The proof for locally bounded spaces in Kalton–Peck–Roberts [3] was the inspiration for this proof. However, the proofs are quite different in places.

2. The lifting theorem

THEOREM 2.1. *Let* E *be a closed locally convex subspace of* $L_0[0,1]$. *Let* $T : L_0[0,1] \to L_0[0,1]/E$ *be a continuous linear operator. Then there is a unique continuous linear operator* $S : L_0[0,1] \to L_0[0,1]$ *so that* $T = QS$, *where* $Q : L_0[0,1] \to L_0[0,1]/E$ *is the quotient map:*

$$\begin{array}{ccc} & & L_0 \\ & {\scriptstyle S}\nearrow & \downarrow{\scriptstyle Q} \\ L_0 & \xrightarrow{\;T\;} & L_0/E \end{array}$$

Proof. For each $n = 1, 2, \ldots$ find $\delta_n > 0$ so that the closed convex hull of $\{f \in E : \|f\|_0 \le \delta_n\}$ is contained in $\{f \in L_0 : \|f\|_0 \le 1/(80n)\}$, and use Lemma 1.4 to find a measurable set A_n so that

(i) $\mu(A_n) \le 1/n$,

(ii) if $(f_i)_{i=1}^{\infty} \subset E$ and $\|2^i \cdot f_i\|_0 \le \delta_n$ for all i then

$$\mu\Big(\bigcap_{l=1}^{\infty}\bigcup_{i=l}^{\infty}\{x : |f_i(x)| > 1\} \setminus A_n\Big) = 0.$$

Without loss of generality we may assume that $\delta_1 \ge \delta_2 \ge \ldots$, $\delta_n \to 0$, and referring to the construction we can take $A_1 \supset A_2 \supset \ldots$ Since T is continuous, for each δ_n we can find $\varepsilon_n > 0$ so that $\|f\|_0 \le \varepsilon_n \Rightarrow \|Tf\|_{L_0/E} \le \delta_n/6$. Without loss of generality we may also assume that $\varepsilon_1 \ge \varepsilon_2 \ge \ldots$ For each m and $k = 1, \ldots, 2^m$ define

$$\Delta_k^m = \Big[\frac{k-1}{2^m}, \frac{k}{2^m}\Big).$$

Define

$$\chi_k^m = \chi_{\Delta_k^m} \quad \text{for } k = 1, \ldots, 2^m \text{ and } m = 1, 2, \ldots$$

Let $v \in L_0$ be given. Define $S(0) = 0$. So we will assume $v \ne 0$. For the next few pages we will work to define $S(v)$. Define $w_k^m = v \cdot \chi_k^m$ for $k = 1, \ldots, 2^m$ and $m = 1, 2, \ldots$ For the time being we will consider m and k to be fixed and look at w_k^m. Let m_0 be the smallest integer so that $1/2^{m_0} \le \varepsilon_1$, and assume $m \ge m_0$. Let $n(m)$ be the largest integer so that $\varepsilon_{n(m)} \ge 1/2^m$. Since T is continuous we know that $n(m)$ goes to infinity as m goes to infinity unless T is identically 0. For each $i = 1, 2, \ldots$ we can select $g_i \in L_0$ so that $Qg_i = Tw_k^m$ and

$$\|4^i \cdot g_i\|_0 \le \Big(1 + \frac{1}{2i}\Big)\|4^i \cdot Tw_k^m\|_{L_0/E}.$$

If $v = 0$ then $g_i = 0$ for all $i = 1, 2, \ldots$ Note that $\|4^i \cdot w_k^m\|_0 \le 1/2^m \le \varepsilon_{n(m)}$ for all $i = 1, 2, \ldots$ So $\|4^i \cdot Tw_k^m\|_0 \le \delta_{n(m)}/6$ for all $i = 1, 2, \ldots$ Therefore $\sigma(Tw_k^m) \le \delta_{n(m)}/6$. For $j \ge i \ge 1$,

$$\begin{aligned}\|4^i(g_i - g_j)\|_0 &\le \|4^i \cdot g_i\|_0 + \|4^j \cdot g_j\|_0 \\ &\le \Big(2 + \frac{1}{2i} + \frac{1}{2j}\Big) \cdot \sigma(Tw_k^m) \le 3 \cdot \frac{\delta_{n(m)}}{6} < \delta_{n(m)}.\end{aligned}$$

Let $f_i = 2^i(g_i - g_{i+1})$. Then $\|2^i \cdot f_i\|_0 \le \delta_{n(m)}$ for all $i = 1, 2, \ldots$ Therefore

$$\mu\Big(\bigcap_{l=1}^{\infty}\bigcup_{i=l}^{\infty}\{x : |2^i(g_i - g_{i+1})| > 1\} \setminus A_{n(m)}\Big) = 0,$$

that is,

$$\mu\Big(\bigcap_{l=1}^{\infty}\bigcup_{i=l}^{\infty}\Big\{x : |g_i - g_{i+1}| > \frac{1}{2^i}\Big\} \setminus A_{n(m)}\Big) = 0.$$

Let $L(1)=1$, and for each $p=2,3,\dots$ find $L(p)\ge L(p-1)$ such that

$$\mu\Big(\bigcup_{i=L(p)}^{\infty}\Big\{x:|g_i-g_{i+1}|>\frac{1}{2^i}\Big\}\setminus A_{n(m)}\Big)\le\frac{1}{p}.$$

Define

$$B_p=\bigcup_{i=L(p)}^{\infty}\Big\{x:|g_i-g_{i+1}|>\frac{1}{2^i}\Big\}\setminus A_{n(m)},\qquad p=1,2,\dots$$

Observe that $B_1\supset B_2\supset\dots$, and $\mu(\bigcap_{p=1}^\infty B_p)=0$. Suppose $x\notin\bigcap_{p=1}^\infty B_p$ and $x\notin A_{n(m)}$. We will show that $(g_i(x))_{i=1}^\infty$ converges in this case. First, there is a p_x such that $x\notin B_{p_x}$. Therefore $|g_i(x)-g_{i+1}(x)|\le 1/2^i$ for all $i\ge L(p_x)$. Let $\alpha>0$ be given. Find M such that $2/2^M\le\alpha$ and $M\ge L(p_x)$. Suppose $j>i\ge M$. Then

$$\begin{aligned}|g_i(x)-g_j(x)|&\le|g_i(x)-g_{i+1}(x)|\\&\quad+|g_{i+1}(x)-g_{i+2}(x)|+\dots+|g_{j-1}(x)-g_j(x)|\\&\le\frac{1}{2^i}+\frac{1}{2^{i+1}}+\dots+\frac{1}{2^{j-1}}<\frac{2}{2^i}\le\frac{2}{2^M}\le\alpha.\end{aligned}$$

So $(g_i(x))_{i=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$. For all x define

$$g_k^m(x)=\begin{cases}\lim_{i\to\infty}g_i(x), & x\notin\bigcap_{p=1}^\infty B_p\cup A_{n(m)},\\ 0, & x\in\bigcap_{p=1}^\infty B_p\cup A_{n(m)}.\end{cases}$$

g_k^m is the pointwise limit of measurable functions, namely

$$g_i\cdot\chi_{(\bigcap_{p=1}^\infty B_p)^c\cap(A_{n(m)})^c},$$

so g_k^m is measurable. Let $B_k^m=\bigcap_{p=1}^\infty B_p$.

We now remember that k and m were arbitrarily chosen, so for each w_k^m we have defined g_k^m and B_k^m for $k=1,\dots,2^m$ and $m=m_0,m_0+1,\dots$ Let

$$B=\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^m}B_k^m.$$

$\mu(B)=0$ since B is the countable union of sets with zero measure. Define

$$S(v)=\lim_{m\to\infty}\sum_{k=1}^{2^m}g_k^m.$$

It is not immediately clear that $S(v)$ exists. We turn to this question next.

CLAIM. *For almost all $x\notin A_{n(m)}$ $(m\ge m_0)$ we have*

$$g_p^m(x)=g_{2p-1}^{m+1}(x)+g_{2p}^{m+1}(x).$$

Proof of claim. We know that

- $((g_p^m)_i)_{i=1}^\infty\subset T(w_p^m)$ converges in $(A_{n(m)})^c$,
- $((g_{2p-1}^{m+1})_i)_{i=1}^\infty\subset T(w_{2p-1}^{m+1})$ converges in $(A_{n(m+1)})^c\supset(A_{n(m)})^c$,
- $((g_{2p}^{m+1})_i)_{i=1}^\infty\subset T(w_{2p}^{m+1})$ converges in $(A_{n(m+1)})^c\supset(A_{n(m)})^c$, and
- $((g_{2p-1}^{m+1})_i+(g_{2p}^{m+1})_i)_{i=1}^\infty\subset T(w_p^m)$,

because T is additive. (The notation $((g_p^m)_i)_{i=1}^\infty$ simply means the sequence $(g_i)_{i=1}^\infty$ that is associated with w_p^m.) Since $\|4^i(g_p^m)_i\|_0\le(1+1/(2i))\cdot\|4^i\cdot T(w_p^m)\|_{L_0/E}\le2\cdot\sigma(T(w_p^m))\le2\delta_{n(m)}/6$ we have

$$\begin{aligned}&\|4^i((g_p^m)_i-(g_{2p-1}^{m+1})_i-(g_{2p}^{m+1})_i)\|_0\\&\qquad\le\frac{\delta_{n(m)}}{3}+\frac{\delta_{n(m+1)}}{3}+\frac{\delta_{n(m+1)}}{3}\le\delta_{n(m)}.\end{aligned}$$

Therefore

$$\mu\Big(\bigcap_{l=1}^{\infty}\bigcup_{i=l}^{\infty}\{x:|(g_p^m)_i(x)-(g_{2p-1}^{m+1})_i(x)-(g_{2p}^{m+1})_i(x)|>2^{-i}\}\setminus A_{n(m)}\Big)=0.$$

So for almost all $x\in(A_{n(m)})^c$,

$$\lim_{i\to\infty}(g_p^m)_i(x)=\lim_{i\to\infty}((g_{2p-1}^m)_i(x)+(g_{2p}^m)_i(x)).$$

Thus for almost all $x\in(A_{n(m)})^c$,

$$g_p^m(x)=g_{2p-1}^{m+1}(x)+g_{2p}^{m+1}(x),$$

which finishes the proof of the claim.

So the sequence

$$\Big(\sum_{k=1}^{2^m}g_k^m\Big)_{m=1}^\infty$$

remains essentially fixed in $L_0((A_{n(m)})^c)$ for $r\ge m\ge m_0$. So the sequence converges in $L_0(\bigcup_{m=1}^\infty(A_{n(m)})^c)=L_0[0,1]$, and $S(v)$ is well defined.

Next we show that $T=QS$. Consider $m\ge m_0$. Then

$$\Big\|\Big(\sum_{k=1}^{2^m}g_k^m\Big)-S(v)\Big\|_0\le\frac{1}{n(m)},$$

since the two functions are essentially identical except possibly on $A_{n(m)}$ and $\mu(A_{n(m)})\le1/n(m)$. For each k we can find $f_k^m\in T(w_k^m)$ so that

$$\|g_k^m|_{(A_{n(m)})^c}-f_k^m|_{(A_{n(m)})^c}\|_0\le1/4^m.$$

We then have the following inequalities:

$$\Big\|\Big(\sum_{k=1}^{2^m}g_k^m|_{(A_{n(m)})^c}\Big)-\Big(\sum_{k=1}^{2^m}f_k^m|_{(A_{n(m)})^c}\Big)\Big\|_0\le2^m\cdot\frac{1}{4^m}=\frac{1}{2^m},$$

$$\Big\|\sum_{k=1}^{2^m} g_k^m - \sum_{k=1}^{2^m} f_k^m\Big\|_0 \le \frac{1}{2^m} + \frac{1}{n(m)},$$

$$\Big\|S(v) - \sum_{k=1}^{2^m} f_k^m\Big\|_0 \le \frac{1}{2^m} + \frac{2}{n(m)}.$$

Notice that $1/2^m + 1/n(m) \to 0$ as $m \to \infty$. The function $\sum_{k=1}^{2^m} f_k^m$ is an element of $T(v)$. So we can find functions in $T(v)$ that are arbitrarily close to $S(v)$, which means that $S(v) \in T(v)$ since E is closed. That is, $QS(v) = T(v)$.

Next we will show that S is a continuous linear operator. If S is additive and continuous at zero then S must also be homogeneous, and thus linear. So it suffices to show that S is additive and continuous at zero.

S is additive. To see this, let $u, v \in L_0$ and let $\alpha > 0$ be given. Find m so that $\mu(A_{n(m)}) \le \alpha$. (Recall that $\mu(A_{n(m)}) \le 1/n(m)$.) We will consider $v\cdot\chi_k^m$, $u\cdot\chi_k^m$, and $(u+v)\cdot\chi_k^m$ for an arbitrary k between 1 and 2^m. From our earlier construction we have $(f_i)_{i=1}^\infty \subset T(u\cdot\chi_k^m)$ such that $f_i \to S(u\cdot\chi_k^m)$ on $(A_{n(m)})^c$ and

$$\|4^i\cdot f_i\|_0 \le \Big(1+\frac{1}{2i}\Big)\|4^i\cdot T(u\cdot\chi_k^m)\|_{L_0/E},$$

and $(g_i)_{i=1}^\infty \subset T(v\cdot\chi_k^m)$ such that $g_i \to S(v\cdot\chi_k^m)$ on $(A_{n(m)})^c$ and

$$\|4^i\cdot g_i\|_0 \le \Big(1+\frac{1}{2i}\Big)\|4^i\cdot T(v\cdot\chi_k^m)\|_{L_0/E},$$

and $(h_i)_{i=1}^\infty \subset T((u+v)\cdot\chi_k^m)$ such that $h_i \to S((u+v)\cdot\chi_k^m)$ on $(A_{n(m)})^c$ and

$$\|4^i\cdot h_i\|_0 \le \Big(1+\frac{1}{2i}\Big)\|4^i\cdot T((u+v)\cdot\chi_k^m)\|_{L_0/E}.$$

We have $f_i + g_i \in T((u+v)\cdot\chi_k^m)$ for all $i = 1, 2, \ldots$ For $i \ge 1$,

$$\begin{aligned}\|4^i(f_i+g_i) - 4^i\cdot h_i\|_0 &\le \|4^i\cdot f_i\|_0 + \|4^i\cdot g_i\|_0 + \|4^i\cdot h_i\|_0\\ &\le (3+3/(2i))\delta_{n(m)}/6 < \delta_{n(m)}.\end{aligned}$$

Therefore,

$$\mu\Big(\bigcap_{l=1}^\infty\bigcup_{i=l}^\infty\Big\{x : |(f_i+g_i) - h_i| > \frac{1}{2^i}\Big\}\setminus A_{n(m)}\Big) = 0.$$

This implies that (f_i+g_i) and h_i converge to the same function on $(A_{n(m)})^c$. Thus for all $k = 1, \ldots, 2^m$, $S(u\cdot\chi_k^m)+S(v\cdot\chi_k^m) = S((u+v)\cdot\chi_k^m)$ on $(A_{n(m)})^c$. Therefore $S(u)+S(v) = S(u+v)$ on $(A_{n(m)})^c$ and $\|S(u)+S(v)-S(u+v)\|_0 \le \alpha$. Since $\alpha > 0$ was arbitrary we have $S(u)+S(v) = S(u+v)$.

S is continuous at zero. To see this, suppose $(v_j)_{j=1}^\infty$ is a sequence in L_0 such that $v_j \to 0$. Let $\alpha > 0$ be given. Find m so that $1/n(m) \le \alpha$. Our set $A_{n(m)}$ then has measure less than α, and $\delta_{n(m)}$ is a positive number such that the closed convex hull of the $\delta_{n(m)}$-ball in E is contained in the $(\alpha/80)$-ball in L_0. There also is an $\varepsilon_{n(m)} > 0$ so that $\|f\|_0 \le \varepsilon_{n(m)} \Rightarrow \|Tf\|_{L_0/E} \le \delta_{n(m)}/6$, and we have $1/2^m \le \varepsilon_{n(m)}$. Let $j \ge 1$ be given. For each $k = 1, \ldots, 2^m$ there is a sequence

$$(g_{j,i}^{(k)})_{i=1}^\infty \subset T(v_j\cdot\chi_k^m)$$

such that $g_{j,i}^{(k)} \to S(v_j\cdot\chi_k^m)$ on $(A_{n(m)})^c$ as $i \to \infty$ and

$$\|4^i\cdot 4^k\cdot g_{j,i}^{(k)}\|_0 \le \Big(1+\frac{1}{2i}\Big)\|4^i\cdot 4^k\cdot T(v_j\cdot\chi_k^m)\|_{L_0/E}.$$

For each $i = 1, 2, \ldots$ and $k = 1, \ldots, 2^m$ let $f_{i,k} = 2^i\cdot 2^k(g_{j,i}^{(k)} - g_{j,i+1}^{(k)})$. Then $f_{i,k} \in E$ for all i and k and

$$\begin{aligned}\|2^i\cdot 2^k\cdot f_{i,k}\|_0 &= \|4^i\cdot 4^k\cdot(g_{j,i}^{(k)} - g_{j,i+1}^{(k)})\|_0\\ &\le \|4^i\cdot 4^k\cdot g_{j,i}^{(k)}\|_0 + \|4^{i+1}\cdot 4^k\cdot g_{j,i+1}^{(k)}\|_0\\ &\le 3\cdot\sigma(T(v_j\cdot\chi_k^m)) \le \delta_{n(m)}.\end{aligned}$$

Using the technique employed in proving Lemma 1.2 we can conclude that

$$\mu\Big(\bigcup_{k=1}^{2^m}\bigcup_{i=1}^\infty\Big\{x : |g_{j,i}^{(k)} - g_{j,i+1}^{(k)}| > \frac{1}{2^i}\cdot\frac{1}{2^k}\Big\}\Big) \le \alpha.$$

Let the set above be called D (so $\mu(D) \le \alpha$). Find I such that $2/2^I \le \alpha$. Then

$$\|S(v_j\cdot\chi_k^m) - g_{j,I}^{(k)}\|_{L_0((A_{n(m)})^c\cup D^c)} \le \sum_{i=I}^\infty \frac{1}{2^i}\cdot\frac{1}{2^k} = \frac{2}{2^I}\cdot\frac{1}{2^k}.$$

Therefore

$$\Big\|S(v_j) - \sum_{k=1}^{2^m} g_{j,I}^{(k)}\Big\|_{L_0((A_{n(m)})^c\cup D^c)} \le \sum_{k=1}^{2^m}\frac{2}{2^I}\cdot\frac{1}{2^k} < \frac{2}{2^I} \le \alpha,$$

and

$$\Big\|S(v_j) - \sum_{k=1}^{2^m} g_{j,I}^{(k)}\Big\|_0 \le 3\alpha.$$

This is true for any $j \ge 1$. Now

$$\Big\|\sum_{k=1}^{2^m} g_{j,I}^{(k)}\Big\|_0 \le 2\sum_{k=1}^{2^m}\|4^I\cdot 4^k\cdot T(v_j\cdot\chi_k^m)\|_{L_0/E}.$$

Since T is continuous for each k, $\|4^I \cdot 4^k \cdot T(v_j \cdot \chi_k^m)\|_{L_0/E}$ goes to zero as j goes to infinity. Therefore the whole sum goes to zero as j goes to infinity. So $\limsup_{j\to\infty} \|S(v_j)\|_0 \le 3\alpha$. However, $\alpha > 0$ was arbitrary, so $\lim_{j\to\infty} S(v_j) = 0$. That is, S is a continuous linear operator.

Suppose that S' is another continuous linear operator from L_0 to L_0 such that $QS' = T$. Then $Q(S - S') = QS - QS' = T - T = 0$, whence $S-S'$ maps L_0 into the locally convex space E. We conclude that $S=S'$. ∎

The proof of Theorem 2.1 works with a milder assumption on the subspace E. It does not have to be locally convex—the key assumption is only that given a neighborhood V of 0 there is a smaller neighborhood U so that if $x_n \in U$ then $\sum_{n=1}^N 2^{-n}x_n$ is in V for all N (i.e. E is exponentially galbed in the sense of Turpin [8]). We can generalize further by replacing the sequence (2^{-n}) with a strictly positive term sequence (a_n) such that $\sum a_n < \infty$. By a classical result due to Aoki [1] and Rolewicz [7] we know that locally bounded spaces are locally p-convex for some $p > 0$. Also, if U is locally p-convex then $\sum_{n=1}^N 2^{-(n/p)}U \subset U$ for all N. In this way we can see that the generalized result includes locally bounded subspaces of L_0.

We can combine Theorem 2.1 with Kwapień's theorem [4].

THEOREM 2.2. *Let $S : L_0 \to L_0$ be a linear operator. Then*

$$S(f)(x) = \sum_{n=1}^{\infty} g_n(x) f(\sigma_n(x))$$

for every $f \in L_0$, where

(i) *each $\sigma_n : [0,1] \to [0,1]$ is a non-singular measurable map,*
(ii) *each g_n is in L_0,*
(iii) *for almost all x in $[0,1]$, $g_n(x) \ne 0$ for only finitely many n.*

Conversely, every map defined in the above way is a linear operator from L_0 to L_0.

COROLLARY 2.3. *Let E be a closed locally convex subspace of L_0 and Q be the quotient map. Then T is an operator from L_0 to L_0/E if and only if $T = QS$ for some S of the form in Theorem 2.2.*

By following the proof of Theorem 4.1 in [2] we have the following corollary.

COROLLARY 2.4. *Let E and F be closed subspaces of L_0, each of which is either locally convex or locally bounded. Then L_0/E is isomorphic to L_0/F if and only if there is an isomorphism S of L_0 to itself such that $S(E) = F$.*

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