References

[1] A. Atzmon, *Operators which are annihilated by analytic functions and invariant subspaces*, Acta Math. 144 (1980), 27–63.

[2] C. Bennett and J. E. Gilbert, *Homogeneous algebras on the circle: I. Ideals of analytic functions*, Ann. Inst. Fourier (Grenoble) 22 (3) (1972), 1–19.

[3] J. G. Caughran, *Factorization of analytic functions with H^p derivative*, Duke Math. J. 36 (1969), 153–158.

[4] J. Esterle et F. Zouakia, *Rythme de décroissance des itérés de certains opérateurs de l'espace de Hilbert*, J. Operator Theory 21 (1989), 387–396.

[5] I. M. Gelfand, D. A. Raikov and G. E. Shilov, *Commutative Normed Rings*, Chelsea, Bronx, 1964.

[6] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, N.J., 1962.

[7] J.-P. Kahane, *Idéaux primaires fermés dans certaines algèbres de Banach de fonctions analytiques*, in: E. J. Akutowicz (ed.), L'analyse harmonique dans le domaine complexe, Lecture Notes in Math. 336, Springer, Berlin, 1973, 5–14.

[8] Y. Katznelson, *An Introduction to Harmonic Analysis*, Wiley, New York, 1968.

[9] T. V. Pedersen, *Norms of powers in Banach algebras*, Bull. London Math. Soc., to appear.

[10] H. Reiter, *Classical Harmonic Analysis and Locally Compact Groups*, Oxford Univ. Press, London, 1968.

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Analyticity of transition semigroups and closability of bilinear forms in Hilbert spaces

by

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Abstract. We consider a semigroup acting on real-valued functions defined in a Hilbert space H, arising as a transition semigroup of a given stochastic process in H. We find sufficient conditions for analyticity of the semigroup in the $L^2(\mu)$ space, where μ is a gaussian measure in H, intrinsically related to the process. We show that the infinitesimal generator of the semigroup is associated with a bilinear closed coercive form in $L^2(\mu)$. A closability criterion for such forms is presented. Examples are also given.

1. Introduction. Let H be a real Hilbert space with scalar product $\langle , \rangle$. Let $\mathcal{L}(H)$ be the algebra of all bounded, everywhere defined, linear operators in H. We denote the norm in H and in $\mathcal{L}(H)$ by the same symbol $\| \, \|$. Let $\mathcal{B}_b(H)$ be the set of all bounded Borel measurable functions $f : H \to \mathbb{R}$. Let A be the infinitesimal generator of a strongly continuous semigroup e^{tA}, $t \geq 0$, of linear operators in H. Assume $R \in \mathcal{L}(H)$ is a nonnegative operator in H, i.e. $R = R^* \geq 0$ and assume that Q_t given by the formula

$$Q_t = \int_0^t e^{sA} R e^{sA^*} \, ds$$

is a trace class operator (here and in the following, operator-valued integrals converge in the strong operator topology). Then one can define the *transition semigroup*

$$(P_t\phi)(x) = \int_H \phi(y) \mathcal{N}(e^{tA}x, Q_t)(dy), \qquad \phi \in \mathcal{B}_b(H), \tag{1}$$

where $\mathcal{N}(e^{tA}x, Q_t)$ is the gaussian measure in H with mean value $e^{tA}x$ and covariance operator Q_t. In this paper we will study regularity properties of P_t.

A motivation for studying P_t is its well known probabilistic interpretation which we now sketch. Consider the stochastic differential equation in H:

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$$(2)\qquad \begin{cases} dX(t) = AX(t)dt + R^{1/2}dW(t), & t \geq 0, \\ X(0) = x \in H, \end{cases}$$

where $W(t)$ is white noise. Let $X(t,x)$ be its mild solution. Then

$$(P_t\phi)(x) = \mathbb{E}\phi(X(t,x)), \quad \phi \in \mathcal{B}_{\mathrm{b}}(H).$$

Here $\mathbb{E}$ denotes the expected value. We refer to [6] for details.

One can consider P_t as a semigroup in the space of all continuous real-valued bounded functions in H: we refer to [6], [2], [3] for this approach. But other natural choices are possible. If we assume that $\|e^{tA}\| \leq Me^{-\omega t}$ for some constants $M, \omega > 0$ and that $\sup_{t>0} \operatorname{Tr} Q_t < \infty$ then one can define the operator

$$Q_\infty = \int_0^\infty e^{tA}Re^{tA^*}\,dt,$$

which is a trace class operator, and the gaussian measure $\mu = \mathcal{N}(0, Q_\infty)$ is well defined. μ is an *invariant measure* for P_t, i.e.

$$\int_H (P_t\phi)(x)\,\mu(dx) = \int_H \phi(x)\,\mu(dx), \quad \phi \in \mathcal{B}_{\mathrm{b}}(H),\ t \geq 0$$

(see [6]). So it is natural to consider P_t as a semigroup in the space $L^2(\mu)$, the Hilbert space of all Borel measurable functions $f : H \to \mathbb{R}$ which are square integrable with respect to μ, endowed with its usual scalar product:

$$\langle \phi, \psi \rangle_{L^2(\mu)} = \int_H \phi(x)\psi(x)\,\mu(dx), \quad \phi, \psi \in L^2(\mu).$$

It is known that P_t has a unique continuous extension to a strongly continuous contraction semigroup of linear operators in $L^2(\mu)$ (see [5]), which we will still denote by P_t. In the following we will always consider P_t as a semigroup in $L^2(\mu)$. Some additional properties of P_t, such as a more precise characterization of its infinitesimal generator and its domain, as well as applications to nonlinear cases, are studied in [5].

One of the main aims of this paper is to find sufficient conditions for the infinitesimal generator $\mathcal{A}$ of the semigroup P_t to be variational, i.e. to be associated with a bilinear closed coercive form $\mathcal{E}$. Consequently, we will find conditions implying that P_t is an analytic semigroup. For terminology and material on bilinear forms we refer to [8]; also, Sections 2 and 3 contain a short review of the results we need. More explicitly, we will prove that

$$(3)\qquad \mathcal{E}(\phi, \psi) = -\int_H (\mathcal{A}\phi)(x)\psi(x)\,\mu(dx), \quad \phi \in D(\mathcal{A}),\ \psi \in D(\mathcal{E}),$$

where $\mathcal{E}$ is given, at least formally, by

$$(4)\qquad \mathcal{E}(\phi, \psi) = -\int_H \langle \phi_x(x), AQ_\infty\psi_x(x)\rangle\,\mu(dx), \quad \phi, \psi \in D(\mathcal{E}),$$

provided certain assumptions are satisfied. In (4), $D(\mathcal{E}) \subset L^2(\mu)$ is the domain of the form $\mathcal{E}$, and ϕ_x, ψ_x are the Fréchet derivatives of ϕ and ψ, at least if they are sufficiently regular (see Section 3 for precise statements).

The equation (4) is only formal. In order to define properly the form $\mathcal{E}$ we are led to considerations which we think have an interest in themselves and consitute another aim of this paper. $\mathcal{E}$ will be defined as the closure of the form

$$(5)\qquad \mathcal{E}_0(\phi, \psi) = -\int_H \langle \phi_x(x), AQ_\infty\psi_x(x)\rangle\,\mu(dx), \quad \phi, \psi \in D(\mathcal{E}_0) = \mathcal{F}C_{\mathrm{b}}^\infty(H),$$

where $\mathcal{F}C_{\mathrm{b}}^\infty(H)$ is defined as follows. Let $\{e_k\}$ be an orthonormal basis of H consisting of eigenvectors of Q_∞ and let us denote by $C_{\mathrm{b}}^\infty(\mathbb{R}^m)$ the set of all functions $f : \mathbb{R}^m \to \mathbb{R}$ which have continuous and bounded derivatives of all orders. Then we say that $\phi : H \to \mathbb{R}$ belongs to $\mathcal{F}C_{\mathrm{b}}^\infty(H)$ if there exist $m \in \mathbb{N}$ and $f \in C_{\mathrm{b}}^\infty(\mathbb{R}^m)$ such that

$$\phi(x) = f(\langle x, e_1\rangle, \ldots, \langle x, e_m\rangle), \quad x \in H.$$

Closability of infinite-dimensional bilinear forms has been the subject of recent investigation: we refer to [8] (see also Remark 2.3). In Section 2 we will give conditions implying that the form $\mathcal{E}_0$ is closable in $L^2(\mu)$. This will be done through an integration by parts formula and the remark that the symmetric part of $\mathcal{E}_0$ is

$$(6)\qquad \frac{1}{2}(\mathcal{E}_0(\phi, \psi) + \mathcal{E}_0(\psi, \phi)) = \frac{1}{2}\int_H \langle R\phi_x(x), \psi_x(x)\rangle\,\mu(dx), \quad \phi, \psi \in \mathcal{F}C_{\mathrm{b}}^\infty(H).$$

In turn, this depends on the fact that Q_∞ satisfies the (formal) Lyapunov equation

$$AQ_\infty + Q_\infty A^* = -R.$$

Equality (6) allows us to state sufficient conditions for the closability of the form $\mathcal{E}_0$ in terms of A and R, without requiring, in particular, the operator R to be strictly positive (compare also with Remark 2.3).

We finally recall that, even in the infinite-dimensional case, the connection between bilinear forms and transition semigroups (associated with even more general stochastic processes than the solution $X(t)$ of (2)) has been deeply studied: the interested reader can consult for instance [8] and the references given there. One of the major achievements is the characterization of those forms that possess an associated stochastic process. Our point of view is slightly different, since we start from a specific transition semigroup P_t (for which we even have the explicit formula (1)) associated with a given stochastic process $X(t)$, and we prove that P_t can be associated with a form $\mathcal{E}$. However, we think that the situation we are considering is sufficiently general and, in view of (2), typical. Moreover, we obtain an explicit description of the form $\mathcal{E}$. Also we recall that it is an open problem to find necessary and sufficent conditions on A and R in order that P_t is analytic in $L^2(\mu)$.

In the following sections we will list again all our assumptions, but we will keep the same notations already introduced. Additional notations will be used in Sections 2 and 3.

Throughout the paper we consider real Hilbert spaces. Whenever complex scalars are needed (e.g. in order to consider analytic semigroups, or in some examples of Section 4) it is enough to consider the complexification of the space, and we will do this without explicit mention.

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With the exception of Example 4.3 and some minor comments, all the results of this paper have been presented in the preprint [7]. During the preparation of the final manuscript the author learnt that results similar to those of Theorem 3.6 have been independently proved by B. Schmuland [10]: the two results are similar, but neither includes the other.

2. An integration by parts formula and the closability of a symmetric form. The following proposition will be used only to prove Theorem 2.2, but it may have an interest of its own.

PROPOSITION 2.1. *Assume $Q \in \mathcal{L}(H)$, $Q = Q^* \geq 0$, $\operatorname{Tr} Q < \infty$. Let $\{e_k\}$ be a complete orthonormal basis of H such that $Qe_k = \lambda_k e_k$ for some $\lambda_k \geq 0$ and let $\mu = \mathcal{N}(0, Q)$. Then*

(i) *For any $h \in H$ the series*

$$S_h(x) = \sum_{k=1,\,\lambda_k>0}^{\infty} \langle h, e_k\rangle\langle x, e_k\rangle \frac{1}{\sqrt{\lambda_k}} \tag{7}$$

converges in $L^2(\mu)$. Moreover, if $l \in H$ and $h = Q^{1/2}l$ we have

$$S_h(x) = \langle l, x\rangle \quad \mu\text{-a.e.}$$

(ii) *For any $h \in H$ and $\phi, \psi \in \mathcal{F}C_b^\infty(H)$ we have*

$$\int_H \langle \phi_x(x), Q^{1/2}h\rangle \psi(x)\,\mu(dx) = -\int_H \langle \psi_x(x), Q^{1/2}h\rangle \phi(x)\,\mu(dx) + \int_H \phi(x)\psi(x)S_h(x)\,\mu(dx), \tag{8}$$

where $S_h(x)$ is given by (7).

Proof. (i) Recalling that $\langle x, e_k\rangle$ are independent real random variables with law $\mathcal{N}(0, \lambda_k)$ on H, endowed with the probability measure μ, we obtain, for $M > N$,

$$\int_H \Big(\sum_{k=N,\,\lambda_k>0}^{M} \langle h, e_k\rangle\langle x, e_k\rangle \frac{1}{\sqrt{\lambda_k}}\Big)^2 \mu(dx) = \sum_{k=N,\,\lambda_k>0}^{M} \int_H \langle h, e_k\rangle^2 \langle x, e_k\rangle^2 \frac{1}{\lambda_k}\,\mu(dx) \leq \sum_{k=N}^{M} \langle h, e_k\rangle^2 \to 0, \qquad M, N \to \infty.$$

Now observe that if $\lambda_k = 0$, then $\langle x, e_k\rangle$ has law $\mathcal{N}(0,0)$, so that $\langle x, e_k\rangle = 0$ μ-a.e. So if $h = Q^{1/2}l$ we obtain from (7),

$$S_h(x) = \sum_{k=1,\,\lambda_k>0}^{\infty} \langle l, Q^{1/2}e_k\rangle\langle x, e_k\rangle \frac{1}{\sqrt{\lambda_k}} = \sum_{k=1}^{\infty} \langle l, e_k\rangle\langle x, e_k\rangle = \langle l, x\rangle \quad \mu\text{-a.e.}$$

(ii) There exist $m \in \mathbb{N}$ and $f, g \in C_b^\infty(\mathbb{R}^m)$ such that

$$\phi(x) = f(x_1, \ldots, x_m), \quad \psi(x) = g(x_1, \ldots, x_m), \quad x \in H,$$

where we set $x_k = \langle x, e_k\rangle$. Without loss of generality, we can suppose $\lambda_k > 0$ for $k = 1, \ldots, r$ and $\lambda_{r+1} = \ldots = \lambda_m = 0$. So we have $x_{r+1} = \ldots = x_m = 0$, μ-a.e. Denoting by ∂_k the partial derivative with respect to the kth argument, we have

$$\langle \phi_x(x), e_k\rangle = (\partial_k f)(x_1, \ldots, x_m).$$

Now we obtain

$$\int_H \langle \phi_x(x), Q^{1/2}h\rangle \psi(x)\,\mu(dx)$$
$$= \sum_{k=1}^m \int_H \langle \phi_x(x), e_k\rangle\langle Q^{1/2}h, e_k\rangle \psi(x)\,\mu(dx)$$
$$= \sum_{k=1}^m \int_H (\partial_k f)(x_1,\dots,x_m)\langle h, e_k\rangle \sqrt{\lambda_k} g(x_1,\dots,x_m)\,\mu(dx)$$
$$= \sum_{k=1}^r \int_{\mathbb{R}^r} (\partial_k f)(\xi_1,\dots,\xi_r,0,\dots,0) g(\xi_1,\dots,\xi_r,0,\dots,0)\varrho_r(\xi)\,d\xi$$
$$\times \langle h, e_k\rangle\sqrt{\lambda_k},$$

where $\varrho_r(\xi) = (2\pi)^{-r/2}(\lambda_1\dots\lambda_r)^{-1/2}\exp\left(-\frac{1}{2}\sum_{k=1}^r \xi_k^2/\lambda_k\right)$. Integrating by parts we obtain

$$\text{(9)}\qquad \int_H \langle \phi_x(x), Q^{1/2}h\rangle \psi(x)\,\mu(dx)$$
$$= -\sum_{k=1}^r \int_{\mathbb{R}^r} (\partial_k g)(\xi_1,\dots,\xi_r,0,\dots,0) f(\xi_1,\dots,\xi_r,0,\dots,0)\varrho_r(\xi)\,d\xi$$
$$\times \langle h, e_k\rangle\sqrt{\lambda_k}$$
$$+ \sum_{k=1}^r \int_{\mathbb{R}^r} g(\xi_1,\dots,\xi_r,0,\dots,0) f(\xi_1,\dots,\xi_r,0,\dots,0)\varrho_r(\xi)\frac{\xi_k}{\sqrt{\lambda_k}}\,d\xi\,\langle h, e_k\rangle.$$

Denote by I_1 (respectively, I_2) the first (respectively, second) sum on the right-hand side of equality (9). By similar arguments it is easy to see that

$$I_1 = -\int_H \langle \psi_x(x), Q^{1/2}h\rangle \phi(x)\,\mu(dx)$$

and

$$I_2 = \sum_{k=1}^r \int_H \phi(x)\psi(x)\langle x, e_k\rangle\langle h, e_k\rangle\frac{1}{\sqrt{\lambda_k}}\,\mu(dx)$$
$$= \sum_{k=1,\,\lambda_k>0}^m \int_H \phi(x)\psi(x)\langle x, e_k\rangle\langle h, e_k\rangle\frac{1}{\sqrt{\lambda_k}}\,\mu(dx),$$
$$= \int_H \phi(x)\psi(x)S_h(x)\,\mu(dx),$$

where $S_h(x)$ is the same as in (i). The last equality is due to the fact the series

$$\sum_{k=m+1,\,\lambda_k>0}^\infty \langle x, e_k\rangle\langle h, e_k\rangle\frac{1}{\sqrt{\lambda_k}}$$

converges in $L^2(\mu)$ to a random variable independent of $\phi(x)\psi(x)$ and having zero mean value. ∎

We now recall some definitions and fix some additional notations concerning bilinear forms. We keep the approach of [8], to which we refer the reader for further information and details.

Let D be a dense linear subspace of a real Hilbert space $\mathcal{H}$ (in the following we will take $\mathcal{H} = L^2(\mu)$). We denote by $\|\ \|_{\mathcal{H}}$ the norm of $\mathcal{H}$ and by $\langle\ ,\ \rangle_{\mathcal{H}}$ its scalar product. A *form* is a bilinear function $D\times D \to \mathbb{R}$, $(\phi,\psi)\to \mathcal{E}(\phi,\psi)$, satisfying $\mathcal{E}(\phi,\phi)\ge 0$, $\forall\phi\in\mathcal{H}$. D will be called the *domain* $D(\mathcal{E})$ of the form and will be endowed with the norm

$$\text{(10)}\qquad \|\phi\|_{D(\mathcal{E})} = (\mathcal{E}(\phi,\phi) + \|\phi\|_{\mathcal{H}}^2)^{1/2}.$$

A form $\mathcal{E}$ is called *coercive* if there exists $K>0$ satisfying

$$\text{(11)}\qquad |\mathcal{E}(\phi,\psi)| \le K\|\phi\|_{D(\mathcal{E})}\|\psi\|_{D(\mathcal{E})},\qquad \phi,\psi\in D(\mathcal{E}).$$

Note that every *symmetric* form (i.e. satisfying $\mathcal{E}(\phi,\psi) = \mathcal{E}(\psi,\phi)$, $\forall\phi,\psi\in D(\mathcal{E})$) is coercive, since (11) holds with $K=1$, by Cauchy–Schwarz's inequality. We will call a coercive form $\mathcal{E}$ *closed* if $D(\mathcal{E})$, endowed with the norm (10), is a Banach space. We will call a coercive form $\mathcal{E}_0$ *closable* if every sequence $\phi_n\in D(\mathcal{E}_0)$ satisfying

$$\|\phi_n\|_{\mathcal{H}} \to 0,\ n\to\infty,\qquad \|\phi_n-\phi_m\|_{D(\mathcal{E}_0)}\to 0,\ n,m\to\infty,$$

also satisfies

$$\|\phi_n\|_{D(\mathcal{E}_0)} \to 0,\qquad n\to\infty.$$

If this is the case, the completion of $D(\mathcal{E}_0)$ can be naturally identified with a subspace D' of $\mathcal{H}$ and the unique continuous extension of $\mathcal{E}_0$ to D' is a coercive closed form $\mathcal{E}$ in $\mathcal{H}$ with $D' = D(\mathcal{E})$, called the *closure* of $\mathcal{E}_0$. Note that the closedness or closability of a coercive form $\mathcal{E}$ depends only on its *symmetric part* $\frac{1}{2}(\mathcal{E}(\phi,\psi)+\mathcal{E}(\psi,\phi))$.

THEOREM 2.2. *Assume $Q\in\mathcal{L}(H)$, $Q=Q^*\ge 0$ and $\operatorname{Tr} Q<\infty$. Let $\{e_k\}$ be a complete orthonormal basis of H such that $Qe_k=\lambda_k e_k$ for some $\lambda_k\ge 0$ and let $\mu=\mathcal{N}(0,Q)$. Also assume that the operator $R\in\mathcal{L}(H)$ satisfies $R=R^*\ge 0$ and we have*

$$\text{(12)}\qquad R^{1/2}g\in Q^{1/2}(H)$$

for all g in a dense subset G of H. Then the symmetric form

$$(13)\qquad \mathcal{E}_0(\phi,\psi)=\int_H \langle R\phi_x(x),\psi_x(x)\rangle\,\mu(dx),\qquad D(\mathcal{E}_0)=\mathcal{F}C_b^\infty(H),$$

is closable in $L^2(\mu)$.

Proof. Denote by $L^2(\mu,H)$ the Hilbert space of all Borel measurable functions $F:H\to H$ such that $\int_H\|F(x)\|^2\,\mu(dx)<\infty$ endowed with the scalar product $\int_H\langle F(x),G(x)\rangle\,\mu(dx)$, $F,G\in L^2(\mu,H)$. Let T_R be the following operator, with domain $D(T_R)$:

$$(T_R\phi)(x)=R^{1/2}\phi_x(x),\qquad D(T_R)=\mathcal{F}C_b^\infty(H).$$

Since $\mathcal{E}_0(\phi,\phi)=\langle T_R\phi,T_R\phi\rangle_{L^2(\mu,H)}$ the closability of $\mathcal{E}_0$ is equivalent to the closability of T_R as an unbounded operator $L^2(\mu)\to L^2(\mu,H)$. Take $\phi_n\in D(T_R)$ such that $\phi_n\to 0$ in $L^2(\mu)$ and $T_R\phi_n\to F$ in $L^2(\mu,H)$. We have to show that $F=0$. Let $\psi\in\mathcal{F}C_b^\infty(H,H)$, i.e. ψ has the form

$$\psi(x)=\sum_{i=1}^m\psi_i(x)g_i,\qquad \psi_i\in\mathcal{F}C_b^\infty(H),\ x\in H,$$

for some $m\in\mathbb{N}$ and $g_1,\ldots,g_m\in G$. Then

$$\int_H\langle (T_R\phi_n)(x),\psi(x)\rangle\,\mu(dx)=\sum_{i=1}^m\int_H\langle(\phi_n)_x(x),R^{1/2}g_i\rangle\psi_i(x)\,\mu(dx).$$

Since by (12) there exist $h_i\in H$ such that $Q^{1/2}h_i=R^{1/2}g_i$, $i=1,\ldots,m$, we obtain, by (8),

$$\begin{aligned}\int_H\langle (T_R\phi_n)(x),\psi(x)\rangle\,\mu(dx)&\\ =-\sum_{i=1}^m\int_H\langle(\psi_i)_x(x),Q^{1/2}h_i\rangle\phi_n(x)\,\mu(dx)&\\ +\sum_{i=1}^m\int_H\phi_n(x)\psi_i(x)\Big(\sum_{k=1,\,\lambda_k>0}^\infty\langle x,e_k\rangle\langle h_i,e_k\rangle\frac{1}{\sqrt{\lambda_k}}\Big)\,\mu(dx),&\end{aligned}$$

where the series converges in $L^2(\mu)$. Letting $n\to\infty$ we obtain

$$\int_H\langle F(x),\psi(x)\rangle\,\mu(dx)=0,\qquad\forall\psi\in\mathcal{F}C_b^\infty(H,H),$$

so that $F=0$, since $\mathcal{F}C_b^\infty(H,H)$ is dense in $L^2(\mu,H)$. ■

Remark 2.3. The proof of Theorem 2.2 runs along classical lines, since closability of bilinear forms is known to follow from integration by parts. However, we are not aware of any result that trivially leads to Theorem 2.2. Observe that condition (12) does not imply that R is boundedly invertible, or even injective. If $R>0$ and $\operatorname{Ker}Q=0$, then condition (12) is always satisfied, since we can take as G the subspace spanned by $R^{-1/2}(e_k)$, $k\in\mathbb{N}$. However, in this case, the closability of $\mathcal{E}_0$ follows also, for instance, from the results of [8]. There the reader can find generalizations to other situations: operators R depending on $x\in H$ and more general measures on topological vector spaces than a gaussian measure μ on a Hilbert space H. Finally, we remark that a satisfactory extension of Theorem 2.2 would consist in finding necessary and sufficient conditions for the closability of the form (13) in terms of Q and R; however, we do not know whether (12) is also necessary.

3. Analyticity of the transition semigroup. We start this section by fixing some additional notations. We also recall some results about bilinear forms and semigroups, referring the reader to [8] for details.

Suppose $\mathcal{E}$ is a closed coercive form in $\mathcal{H}$. Then one can define a linear operator $\mathcal{A}^{\mathcal{E}}:\mathcal{H}\supset D(\mathcal{A}^{\mathcal{E}})\to\mathcal{H}$ as follows:

$$D(\mathcal{A}^{\mathcal{E}})=\{\phi\in D(\mathcal{E}):\psi\to\mathcal{E}(\phi,\psi)\text{ is continuous with respect to }\|\,\|_{\mathcal{H}}\}$$

and $\mathcal{A}^{\mathcal{E}}\phi$ is the unique element in $\mathcal{H}$ such that

$$\mathcal{E}(\phi,\psi)=-\langle\mathcal{A}^{\mathcal{E}}\phi,\psi\rangle_{\mathcal{H}},\qquad\forall\psi\in D(\mathcal{E}).$$

It can be proved that $\mathcal{A}^{\mathcal{E}}$ is the infinitesimal generator of a strongly continuous analytic semigroup of contractions in $\mathcal{H}$. Moreover, for any $\alpha>0$ and any $\phi\in\mathcal{H}$, the vector $u=(\alpha I-\mathcal{A}^{\mathcal{E}})^{-1}\phi$ (where I denotes the identity operator on H) is the unique solution of the equation

$$(14)\qquad \mathcal{E}(u,\psi)+\alpha\langle u,\psi\rangle_{\mathcal{H}}=\langle\phi,\psi\rangle_{\mathcal{H}},\qquad\forall\psi\in D(\mathcal{E}).$$

In this situation, we will say that the semigroup generated by $\mathcal{A}^{\mathcal{E}}$ is *associated* with the form $\mathcal{E}$. Operators arising in such a way from a closed coercive form will be called *variational.*

The rest of this section is devoted to proving that, under suitable conditions, the transition semigroup P_t mentioned in the introduction is associated with a closed coercive form in $\mathcal{H}=L^2(\mu)$.

In the sequel we will need the following assumptions.

HYPOTHESIS 3.1. (i) *The operator A generates a strongly continuous semigroup in H satisfying $\|e^{tA}\|\le Me^{-\omega t}$ for some $M>0$ and $\omega>0$.*

(ii) $R\in\mathcal{L}(H)$, $R=R^*\ge 0$.

(iii) *Setting*

$$Q_t=\int_0^t e^{sA}Re^{sA^*}\,ds$$

we have

$$\sup_{t>0}\operatorname{Tr}Q_t<\infty.$$

Under Hypothesis 3.1 one can define the transition semigroup P_t defined by equation (1). In the following we set

$$Q_\infty = \int_0^\infty e^{tA} R e^{tA^*}\, dt.$$

Note that Q_∞ is well defined by Hypothesis 3.1(i) and it is a trace class operator by Hypothesis 3.1(iii). Setting $\mu = \mathcal{N}(0, Q_\infty)$, it can be proved that P_t has a unique extension to a strongly continuous semigroup of contractions in $L^2(\mu)$ (see [5]).

HYPOTHESIS 3.2. $Q_\infty(H) \subset D(A)$.

Some sufficient conditions in order that Hypothesis 3.2 hold will be given in Section 5. Note that 3.2 implies that $AQ_\infty \in \mathcal{L}(H)$, by the closed graph theorem.

LEMMA 3.3. *Assume that Hypotheses* 3.1 *and* 3.2 *hold. Then* Q_∞ *satisfies the Lyapunov-type equation*

$$AQ_\infty + (AQ_\infty)^* = -R.$$

Proof. We consider the Yosida approximations of A given by

$$A_n = nA(nI - A)^{-1}, \quad n \in \mathbb{N}.$$

Then it is well known that $A_n \in \mathcal{L}(H)$ and by 3.1(i) there exist $M_0, \omega_0 > 0$ independent of n such that

$$\|e^{tA_n}\| \le M_0 e^{-\omega_0 t}, \quad t \ge 0,\ n \in \mathbb{N} \tag{15}$$

(see for instance [9]). Define

$$Q^{(n)} = \int_0^\infty e^{tA_n} R e^{tA_n^*}\, dt.$$

$Q^{(n)}$ is well defined by (15). Furthermore, we have

$$\begin{aligned} A_n Q^{(n)} + Q^{(n)} A_n^* &= \int_0^\infty (A_n e^{tA_n} R e^{tA_n^*} + e^{tA_n} R e^{tA_n^*} A_n^*)\, dt \\ &= \int_0^\infty \left(\frac{d}{dt} e^{tA_n} R e^{tA_n^*}\right) dt = -R. \end{aligned}$$

It follows that

$$\langle Q^{(n)} x, A_n^* y\rangle + \langle A_n^* x, Q^{(n)} y\rangle = -\langle Rx, y\rangle, \quad x, y \in H. \tag{16}$$

Recall that for $x \in H, y \in D(A^*)$ we have $A_n^* y \to A^* y$, $e^{tA_n} x \to e^{tA} x$, and $e^{tA_n^*} x \to e^{tA^*} x$ as $n \to \infty$ (see [9]). It follows that $Q^{(n)} x \to Q_\infty x$ as $n \to \infty$ for $x \in H$. Choosing $x, y \in D(A^*)$ and letting $n \to \infty$ in (16) we have

$$\langle Q_\infty x, A^* y\rangle + \langle A^* x, Q_\infty y\rangle = -\langle Rx, y\rangle, \quad x, y \in D(A^*).$$

By Hypothesis 3.2, $Q_\infty x, Q_\infty y \in D(A)$ and we obtain

$$\langle AQ_\infty x, y\rangle + \langle x, AQ_\infty y\rangle = -\langle Rx, y\rangle, \quad x, y \in D(A^*). \tag{17}$$

Since $D(A^*)$ is dense in H, (17) holds for every $x, y \in H$. ∎

HYPOTHESIS 3.4. *There exists* $K > 0$ *such that*

$$|\langle x, AQ_\infty y\rangle| \le K \langle Rx, x\rangle^{1/2} \langle Ry, y\rangle^{1/2}, \quad x, y \in H.$$

First note that 3.4 always holds, by Hypothesis 3.2, if we assume $R > 0$. In particular, it holds in the important case $R = I$.

Also note that by Lemma 3.3 and Cauchy–Schwarz's inequality, 3.4 is equivalent to

$$|\langle x, AQ_\infty y\rangle - \langle AQ_\infty x, y\rangle| \le K' \langle Rx, x\rangle^{1/2} \langle Ry, y\rangle^{1/2}, \tag{18}$$

for some $K' > 0$.

Finally, note that 3.4 implies that

$$Q_\infty(H) \subset R^{1/2}(H). \tag{19}$$

In fact, for any $x \in H$,

$$\begin{aligned} \|Q_\infty x\|^2 &= \langle AQ_\infty x, (A^*)^{-1} Q_\infty x\rangle \\ &\le K \langle Rx, x\rangle^{1/2} \langle R(A^*)^{-1} Q_\infty x, (A^*)^{-1} Q_\infty x\rangle^{1/2} \\ &\le K \|R^{1/2} x\|\, \|R^{1/2} (A^*)^{-1}\|\, \|Q_\infty x\| \end{aligned}$$

so that $\|Q_\infty x\| \le K \|R^{1/2}(A^*)^{-1}\|\, \|R^{1/2} x\|$, which implies (19) (see for instance [6, appendix B]).

Now consider the following definition:

$$\mathcal{E}_0(\phi, \psi) = -\int_H \langle \phi_x(x), AQ_\infty \psi_x(x)\rangle\, \mu(dx), \quad D(\mathcal{E}_0) = \mathcal{F}C_b^\infty(H). \tag{20}$$

We claim that, under Hypotheses 3.1, 3.2 and 3.4, $\mathcal{E}_0$ is a coercive form. First note that formula (20) is meaningful by Hypothesis 3.2. Then, by Lemma 3.3, it is easily verified that the symmetric part of $\mathcal{E}_0$ is given by

$$\begin{aligned} &\frac{1}{2}(\mathcal{E}_0(\phi, \psi) + \mathcal{E}_0(\psi, \phi)) \\ &\qquad = \frac{1}{2} \int_H \langle R\phi_x(x), \psi_x(x)\rangle\, \mu(dx), \quad \phi, \psi \in \mathcal{F}C_b^\infty(H). \end{aligned} \tag{21}$$

From (21) it is apparent that $\mathcal{E}_0(\phi, \phi) \ge 0$, $\forall \phi \in D(\mathcal{E}_0)$ and from Hypothesis 3.4 it follows that

$$|\mathcal{E}_0(\phi, \psi)| \le K \mathcal{E}_0(\phi, \phi)^{1/2} \mathcal{E}_0(\psi, \psi)^{1/2}, \quad \phi, \psi \in \mathcal{F}C_b^\infty(H),$$

so that $\mathcal{E}_0$ is a coercive form.

HYPOTHESIS 3.5. *The form $\mathcal{E}_0$ defined in* (20) *is closable in* $L^2(\mu)$.

In the applications, we can apply Theorem 2.2 to the symmetric form (21) in order to check the validity of Hypothesis 3.5. In particular, 3.5 always holds if we suppose $R = I$ or, more generally, $R > 0$ (see Remark 2.3).

We will denote by $\mathcal{E}$ the closure of $\mathcal{E}_0$, by $W_R^{1,2}(\mu)$ its domain, and by $\mathcal{A}^{\mathcal{E}}$ the infinitesimal generator of the semigroup associated with $\mathcal{E}$.

THEOREM 3.6. *Assume that Hypotheses* 3.1, 3.2, 3.4, 3.5 *hold. Then, in the space $L^2(\mu)$, the semigroup associated with the form $\mathcal{E}$ coincides with the transition semigroup P_t. In particular, P_t is analytic.*

Proof. We first remark that the functions

$$\phi_\lambda^{(1)}(x) = \cos\langle\lambda, x\rangle, \quad \phi_\lambda^{(2)}(x) = \sin\langle\lambda, x\rangle, \quad \lambda \in D(A^*),$$

generate $L^2(\mu)$. In fact, if

$$f \in L^2(\mu), \quad \langle f, \phi_\lambda^{(i)}\rangle_{L^2(\mu)} = 0, \quad \lambda \in D(A^*),\ i = 1, 2,$$

then

$$\int_H f(x) e^{i\langle\lambda,x\rangle}\,\mu(dx) = 0, \quad \lambda \in D(A^*). \tag{22}$$

Then (22) holds for all $\lambda \in H$, so that, setting $\nu(dx) = f(x)\mu(dx)$, (22) implies that the characteristic function of the measure ν vanishes, so that $\nu = 0$ and $f = 0$, μ-a.e. Therefore in order to prove the theorem it is enough to show that

$$(\alpha I - \mathcal{A})^{-1}\phi_\lambda^{(i)} = (\alpha I - \mathcal{A}^{\mathcal{E}})^{-1}\phi_\lambda^{(i)}, \quad \alpha > 0,\ \lambda \in D(A^*),\ i = 1, 2. \tag{23}$$

We limit ourselves to proving the equality (23) for the case $i = 1$, since the other case is analogous. Note that

$$(P_t\phi_\lambda^{(1)})(x) = \Re\Big(\int_H e^{i\langle\lambda,x\rangle}\,\mathcal{N}(e^{tA}x, Q_t)(dx)\Big) = \Re e^{i\langle e^{tA}x,\lambda\rangle - \frac12\langle Q_t\lambda,\lambda\rangle},$$

where $\Re$ denotes the real part. It follows that, setting

$$v(x) = \int_0^\infty \cos\langle e^{tA}x, \lambda\rangle e^{-t\alpha - \frac12\langle Q_t\lambda,\lambda\rangle}\,dt, \tag{24}$$

we obtain

$$((\alpha I - \mathcal{A})^{-1}\phi_\lambda^{(1)})(x) = \int_0^\infty e^{-t\alpha}(P_t\phi_\lambda^{(1)})(x)\,dt = v(x).$$

Set $u = (\alpha I - \mathcal{A}^{\mathcal{E}})^{-1}\phi_\lambda^{(1)}$. In order to prove (23), we have to show that $u = v$. Recalling (14), we see that u is the unique solution of

$$\mathcal{E}(u, \psi) + \alpha\langle u, \psi\rangle_{L^2(\mu)} = \langle\phi_\lambda^{(1)}, \psi\rangle_{L^2(\mu)}, \quad u, \psi \in W_R^{1,2}(\mu).$$

So it is sufficient to prove that v belongs to $W_R^{1,2}(\mu)$ and

$$\mathcal{E}(v, \psi) + \alpha\langle v, \psi\rangle_{L^2(\mu)} = \langle\phi_\lambda^{(1)}, \psi\rangle_{L^2(\mu)}, \quad \psi \in W_R^{1,2}(\mu). \tag{25}$$

Since $\mathcal{F}C_b^\infty(H)$ is dense in $W_R^{1,2}(\mu)$ by definition, in (25) we can suppose $\psi \in \mathcal{F}C_b^\infty(H)$. Let $\{e_n\}$ be an orthonormal basis of H consisting of eigenvectors of Q_∞ and let P_n denote the orthogonal projection in H onto the subspace spanned by $\{e_1, \ldots, e_n\}$. Define $v_n(x) = v(P_n x)$. Since

$$\langle (v_n)_x(x), h\rangle = -\int_0^\infty \sin\langle e^{tA}P_n x, \lambda\rangle e^{-t\alpha - \frac12\langle Q_t\lambda,\lambda\rangle}\langle h, P_n e^{tA^*}\lambda\rangle\,dt,$$

it is easily seen that $\{v_n\} \subset W_R^{1,2}(\mu)$ is a Cauchy sequence for the norm of $W_R^{1,2}(\mu)$ and that $\|v_n - v\|_{L^2(\mu)} \to 0$ as $n \to \infty$. By definition $v \in W_R^{1,2}(\mu)$ and $\|v_n - v\|_{W_R^{1,2}(\mu)} \to 0$. So we can let $n \to \infty$ in the equality

$$\mathcal{E}(v_n, \psi) = \mathcal{E}_0(v_n, \psi) = \int_H \Big\langle AQ_\infty\psi_x(x), \int_0^\infty \sin\langle e^{tA}P_n x, \lambda\rangle e^{-t\alpha - \frac12\langle Q_t\lambda,\lambda\rangle} P_n e^{tA^*}\lambda\,dt\Big\rangle \mu(dx),$$

obtaining

$$\mathcal{E}(v, \psi) = \int_H \Big\langle AQ_\infty\psi_x(x), \int_0^\infty \sin\langle e^{tA}x, \lambda\rangle e^{-t\alpha - \frac12\langle Q_t\lambda,\lambda\rangle} e^{tA^*}\lambda\,dt\Big\rangle \mu(dx).$$

By Fubini's theorem,

$$\mathcal{E}(v, \psi) = \int_0^\infty e^{-t\alpha - \frac12\langle Q_t\lambda,\lambda\rangle}\int_H \langle\psi_x(x), Q_\infty e^{tA^*}A^*\lambda\rangle \sin\langle e^{tA}x, \lambda\rangle\,\mu(dx)\,dt. \tag{26}$$

We now integrate by parts. Since the function $x \to \sin\langle e^{tA}P_n x, \lambda\rangle$ belongs to $\mathcal{F}C_b^\infty(H)$ we can apply Proposition 2.1 to obtain

$$\int_H \langle\psi_x(x), Q_\infty e^{tA^*}A^*\lambda\rangle \sin\langle e^{tA}P_n x, \lambda\rangle\,\mu(dx) = -\int_H \cos\langle e^{tA}P_n x, \lambda\rangle\langle P_n e^{tA^*}\lambda, Q_\infty A^* e^{tA^*}\lambda\rangle\psi(x)\,\mu(dx) + \int_H \langle x, e^{tA^*}A^*\lambda\rangle\psi(x)\sin\langle e^{tA}P_n x, \lambda\rangle\,\mu(dx).$$

Letting $n \to \infty$ gives

$$\int_H \langle \psi_x(x), Q_\infty e^{tA^*} A^*\lambda\rangle \sin\langle e^{tA}x,\lambda\rangle\,\mu(dx)$$
$$= -\int_H \cos\langle e^{tA}x,\lambda\rangle \langle e^{tA^*}\lambda, Q_\infty A^* e^{tA^*}\lambda\rangle \psi(x)\,\mu(dx)$$
$$+\int_H \langle x, e^{tA^*}A^*\lambda\rangle \psi(x) \sin\langle e^{tA}x,\lambda\rangle\,\mu(dx).$$

Since, by Lemma 3.3,

$$\langle e^{tA^*}\lambda, Q_\infty A^* e^{tA^*}\lambda\rangle = -\tfrac{1}{2}\langle Re^{tA^*}\lambda, e^{tA^*}\lambda\rangle,$$

we obtain from (26) and another application of Fubini's theorem

$$\mathcal{E}(v,\psi) + \alpha\langle v,\psi\rangle_{L^2(\mu)}$$
$$= \int_H \psi(x) \int_0^\infty e^{-t\alpha-\frac{1}{2}\langle Q_t\lambda,\lambda\rangle} \big(\tfrac{1}{2}\langle Re^{tA^*}\lambda, e^{tA^*}\lambda\rangle \cos\langle e^{tA}x,\lambda\rangle$$
$$+ \langle x, e^{tA^*}A^*\lambda\rangle \sin\langle e^{tA}x,\lambda\rangle + \alpha\cos\langle e^{tA}x,\lambda\rangle\big)\,dt\,\mu(dx).$$

Now observe that

$$\frac{d}{dt}\langle Q_t\lambda,\lambda\rangle = \langle Re^{tA^*}\lambda, e^{tA^*}\lambda\rangle, \quad \frac{d}{dt}\langle e^{tA}x,\lambda\rangle = \langle e^{tA}x, A^*\lambda\rangle,$$

and we finally obtain

$$\mathcal{E}(v,\psi) + \alpha\langle v,\psi\rangle_{L^2(\mu)}$$
$$= -\int_H \psi(x) \int_0^\infty \frac{d}{dt}(e^{-t\alpha-\frac{1}{2}\langle Q_t\lambda,\lambda\rangle}\cos\langle e^{tA}x,\lambda\rangle)\,dt\,\mu(dx)$$
$$= \int_H \psi(x)\cos\langle x,\lambda\rangle\,\mu(dx) = \int_H \psi(x)\phi_\lambda^{(1)}(x)\,\mu(dx).$$

Thus (25) is proved, and so is the theorem. ∎

Remark 3.7. Assume that Hypothesis 3.1 holds; then we do not know whether Hypotheses 3.2, 3.4 and 3.5 are also necessary for the infinitesimal generator $\mathcal{A}$ of P_t to be variational. Theorem 3.6 gives sufficient conditions for $\mathcal{A}$ to be variational, but it could be interesting also to find conditions implying only that P_t is analytic, which is a weaker condition in general. Example 4.3 shows that assuming only Hypotheses 3.1 and 3.2 is not sufficient to have analyticity of P_t even in case $\dim H < \infty$.

In the finite-dimensional case the hypotheses of Theorem 3.6 become simpler. We do not restate them explicitly, but we limit ourselves to stating as an instance the following result, corresponding to the case $R = I$. Examples 4.2 and 4.3 deal with the finite-dimensional case.

COROLLARY 3.8. *Assume $H = \mathbb{R}^n$ and A is an $n\times n$ real stable matrix* (i.e. satisfying $\|e^{tA}\| \le Me^{-\omega t}$ for some $M > 0$ and $\omega > 0$). *Define*

$$(27)\qquad Q_t = \int_0^t e^{sA}e^{sA^*}\,ds, \quad Q_\infty = \int_0^\infty e^{sA}e^{sA^*}\,ds, \quad \mu = \mathcal{N}(0,Q_\infty),$$
$$(P_t\phi)(x) = \int_{\mathbb{R}^n} \phi(y)\,\mathcal{N}(e^{tA}x, Q_t)(dy), \qquad \phi\in\mathcal{B}_b(\mathbb{R}^n).$$

Then P_t extends to an analytic semigroup in $L^2(\mu)$ and its infinitesimal generator is variational.

4. Examples

EXAMPLE 4.1 (The deterministic case). Assume A satisfies Hypothesis 3.1(i) and $R = 0$. Then $Q_\infty = 0$ and μ is Dirac measure. So $\mathcal{E} = 0$ and $\mathcal{A} = 0$. Indeed, for $\phi\in\mathcal{B}_b(H)$,

$$(P_t\phi)(x) = \int_H \phi(y)\,\mathcal{N}(e^{tA}x, 0)(dy) = \phi(e^{tA}x), \qquad \phi\in\mathcal{B}_b(H),$$

and $\phi(e^{tA}x) = \phi(0)$, μ-a.e., so that $\frac{d}{dt}P_t\phi = 0$.

EXAMPLE 4.2. We take $H = \mathbb{R}^2$ and the operators A and R given by the matrices

$$A = \begin{pmatrix} -2 & 1\\ -1 & 0\end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix}.$$

Then Hypotheses 3.1 and 3.2 are clearly satisfied and an easy computation shows that

$$Q_\infty = \frac{1}{2}\begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix}.$$

So it follows that

$$AQ_\infty = Q_\infty A^* = -\tfrac{1}{2}R.$$

By (18), hypothesis 3.4 holds and the form $\mathcal{E}_0$ given by (20) is a symmetric form which is closable by Theorem 2.2, since (12) is satisfied (in fact, $R = 2Q_\infty$). In this case, although A is not symmetric, the generator $\mathcal{A}$ of the transition semigroup P_t is self-adjoint (this is possible since $R\ne I$: compare with [12]). In [5], among other results, it is shown that $\mathcal{A}$ is the closure of a differential operator, defined on a properly chosen set of smooth functions on H, for which an explicit expression is found. As a consequence, for $\phi\in\mathcal{F}C_b^\infty(\mathbb{R}^2)$, we have

$$(\mathcal{A}\phi)(x_1,x_2) = \frac{1}{2}\Big(\frac{\partial^2\phi}{\partial x_1^2} + 2\frac{\partial^2\phi}{\partial x_1\partial x_2} + \frac{\partial^2\phi}{\partial x_2^2}\Big) + (x_2 - 2x_1)\frac{\partial\phi}{\partial x_1} - x_1\frac{\partial\phi}{\partial x_2}.$$

EXAMPLE 4.3. We take $H = \mathbb{R}^2$. Let $\{e_1, e_2\}$ be the canonical basis and let the operators A and R be given by the matrices

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then Hypotheses 3.1 and 3.2 hold but 3.4 does not, since in this case

$$Q_\infty = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

We want to show that P_t is not analytic in $L^2(\mu)$, with $\mu = \mathcal{N}(0, Q_\infty)$. First we remark that P_t extends by linearity to $L^2_{\mathbb{C}}(\mu)$, the Hilbert space of μ-square summable complex-valued functions, with the usual inner product. Analyticity would imply (see [9]) that there exists $C > 0$ such that

$$(28)\qquad \left\| \frac{d}{dt} P_t\phi \right\|_{L^2_{\mathbb{C}}(\mu)} \le Ct^{-1}\|\phi\|_{L^2_{\mathbb{C}}(\mu)}, \quad 0 < t \le 1, \ \phi \in L^2_{\mathbb{C}}(\mu).$$

Take $\phi_\nu(x) = e^{i\langle x,\nu\rangle}$, with $\nu \in \mathbb{R}^2$. Then

$$(P_t\phi)(x) = e^{i\langle \nu, e^{tA}x\rangle - \frac{1}{2}\langle Q_t\nu,\nu\rangle},$$

$$\left|\left(\frac{d}{dt}P_t\phi\right)(x)\right|^2 = e^{-\langle Q_t\nu,\nu\rangle}\left(\langle \nu, Ae^{tA}x\rangle^2 + \tfrac{1}{4}\langle e^{tA}Re^{tA^*}\nu,\nu\rangle\right),$$

$$\left\|\frac{d}{dt}P_t\phi\right\|^2_{L^2_{\mathbb{C}}(\mu)} \ge e^{-\langle Q_t\nu,\nu\rangle}\int_H \langle \nu, Ae^{tA}x\rangle^2\,\mu(dx) = e^{-\langle Q_t\nu,\nu\rangle}\langle A^*e^{tA^*}\nu, Q_\infty A^*e^{tA^*}\nu\rangle.$$

Since ϕ_ν has unit norm, (28) implies that

$$(29)\qquad t^2e^{-\langle Q_t\nu,\nu\rangle}\langle A^*e^{tA^*}\nu, Q_\infty A^*e^{tA^*}\nu\rangle \le C^2, \quad 0 < t \le 1, \ \nu \in \mathbb{R}^2.$$

An easy computation shows that

$$Q_t = \begin{pmatrix} \frac{1}{4}(1-e^{-2t}) - \frac{1}{2}e^{-2t}(t^2+t) & \frac{1}{4}(1-e^{-2t}) - \frac{1}{2}te^{-2t} \\ \frac{1}{4}(1-e^{-2t}) - \frac{1}{2}te^{-2t} & \frac{1}{2}(1-e^{-2t}) \end{pmatrix}.$$

Setting $\nu = e_1\big(\frac{1}{4}(1-e^{-2t}) - \frac{1}{2}e^{-2t}(t^2+t)\big)^{-1/2}$ and evaluating the left-hand side of (29) we obtain

$$\frac{1}{4}e^{-2t-1}(1-2t+2t^2)\frac{t^2}{\frac{1}{4}(1-e^{-2t}) - \frac{1}{2}e^{-2t}(t^2+t)} \le C^2, \quad 0 < t \le 1.$$

Letting $t \to 0$ we get a contradiction.

EXAMPLE 4.4. Suppose G is a skew-adjoint (in general unbounded) operator in H, i.e. $G^* = -G$. Take $R = I$ and $A = G^2 + G - \omega I$, with domain $D(A) = D(G^2) = \{x \in D(G) : Gx \in D(G)\}$ and $\omega > 0$ (if G is boundedly invertible, we can take $A = G^2 + G$ in the rest of this example). Recall that,

by Stone's theorem, G is the infinitesimal generator of a strongly continuous group of unitary operators e^{tG} in H. Since $G^2 = -GG^* \le 0$, it turns out that A is the infinitesimal generator of the semigroup $e^{tA} = e^{tG^2}e^{tG}e^{-\omega t}$ (this follows by applying e.g. [9. Cor. 5.5] and remarking that e^{tG^2} and e^{tG} commute). So Hypotheses 3.1(i),(ii) hold. Assume now in addition that Hypothesis 3.1(iii) holds. By commutativity and since $e^{tG^*} = e^{-tG}$ we obtain

$$Q_\infty = \int_0^\infty e^{tG^2}e^{tG}e^{-\omega t}e^{tG^2}e^{tG^*}e^{-\omega t}\,dt = \int_0^\infty e^{2tG^2}e^{-2\omega t}\,dt = \frac{1}{2}(\omega I - G^2)^{-1}.$$

Therefore $Q_\infty(H) = D(G^2) = D(A)$ and Hypothesis 3.2 is verified. Since $R = I$, Hypotheses 3.4 and 3.5 hold, and we can conclude that $\mathcal{A}$ is variational, by Theorem 3.6.

EXAMPLE 4.5 (The commutative case). Assume A and R satisfy Hypothesis 3.1. Define

$$Q_1 = \int_0^\infty e^{tA}e^{tA^*}\,dt.$$

In addition, assume

$$(30)\qquad RA^{-1} = A^{-1}R, \quad Q_1(H) \subset D(A).$$

Then we have $Q_\infty = RQ_1 = Q_1R$ and Hypothesis 3.2 holds. Next, using again the commutativity condition in (30), we obtain for $x, y \in H$,

$$\begin{aligned} |\langle AQ_\infty x, y\rangle| &= |\langle RAQ_1x, y\rangle| = |\langle R^{1/2}AQ_1x, R^{1/2}y\rangle| \\ &\le \|R^{1/2}AQ_1x\|\,\|R^{1/2}y\| = \|AQ_1R^{1/2}x\|\,\|R^{1/2}y\| \\ &\le \|AQ_1\|\,\|R^{1/2}x\|\,\|R^{1/2}y\|, \end{aligned}$$

which implies Hypothesis 3.4. Finally, in order to verify Hypothesis 3.5, we check the closability of (21) by using Theorem 2.2. We additionally assume

$$(31)\qquad \operatorname{Ker} Q_\infty = 0.$$

Then we take an orthonormal basis $\{e_k\}$ of H such that $Q_\infty e_k = \lambda_k e_k$ for suitable numbers $\lambda_k > 0$. We take as G the linear span of $\{e_k\}$ and we see that (12) holds, since

$$R^{1/2}e_k = \lambda_k^{-1}R^{1/2}Q_\infty e_k = \lambda_k^{-1}Q_\infty R^{1/2}e_k \in Q_\infty^{1/2}(H).$$

Therefore we obtain the following

COROLLARY 4.6. *Assume that Hypothesis* 3.1 *holds and assume that* (30), (31) *hold. Then the operator $\mathcal{A}$ is variational.*

The previous passages also show that the conclusion of Corollary 4.6 still holds if we assume Hypotheses 3.1, 3.5, and (30).

EXAMPLE 4.7. Assume $R = I$ and suppose that Hypothesis 3.1 holds. Moreover, assume that

(i) $A^{-1}(A^*)^{-1} = (A^*)^{-1}A^{-1}$,
(ii) A generates an analytic semigroup in H,
(iii) we have

$$(H, D(A))_{\theta,2} = (H, D(A^*))_{\theta,2}, \tag{32}$$

for some $\theta > 0$, where $(H, D(A))_{\theta,2}$ denotes the real interpolation spaces (see [1]) between H and the domain of A (endowed with the norm $\|x\| + \|Ax\|$). By the results of [4, Th. 3.7, Rem. 3.8, Th. 3.14], the operator

$$A + A^* : H \supset D(A) \cap D(A^*) \to H$$

is boundedly invertible and generates an analytic semigroup. It follows that $Q_\infty = (A + A^*)^{-1}$, so that Hypothesis 3.2 holds. Therefore $\mathcal{A}$ is variational. For a discussion of (32) we refer the reader to [11]; in particular, (32) holds if e^{tA} is a semigroup of contractions.

EXAMPLE 4.8. Assume that Hypotheses 3.1, 3.4 and 3.5 hold; assume in addition that

(i) A generates an analytic semigroup in H,
(ii) R is a bounded linear operator from $D((-A^*)^{\theta_1})$ to $D((-A)^{\theta_2})$ for some θ_1, θ_2 with $0 \le \theta_1 < \theta_2 \le 1$.

Then $\mathcal{A}$ is variational. In fact, we only have to verify Hypothesis 3.2. Since we have (see [9])

$$\|(-A)^{1-\theta_2} e^{tA}\| \le Ct^{-1+\theta_2} e^{-\omega t}, \qquad \|(-A^*)^{\theta_1} e^{tA^*}\| \le Ct^{-\theta_1} e^{-\omega t},$$

for some $C > 0$ (depending on θ_1, θ_2) and all $t > 0$, it follows that

$$\begin{aligned}\|Ae^{tA}Re^{tA^*}\| &= \|(-A)^{1-\theta_2} e^{tA} (-A)^{\theta_2} R (-A^*)^{-\theta_1} (-A^*)^{\theta_1} e^{tA^*}\| \\ &\le Ct^{-1+\theta_2} e^{-\omega t} \|(-A)^{\theta_2} R (-A^*)^{-\theta_1}\| t^{-\theta_1} e^{-\omega t},\end{aligned}$$

which is integrable on $(0, \infty)$. So by the closedness of A it follows that $Q_\infty(H) \subset D(A)$ and

$$AQ_\infty = \int_0^\infty Ae^{tA}Re^{tA^*}\, dt.$$

References

[1] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer, 1976.
[2] S. Cerrai, *A Hille-Yosida theorem for weakly continuous semigroups*, preprint, Scuola Normale Superiore di Pisa, 1993.
[3] S. Cerrai and F. Gozzi, *Strong solutions of Cauchy problems associated to weakly continuous semigroups*, preprint, Scuola Normale Superiore di Pisa, 1993.

[4] G. Da Prato and P. Grisvard, *Sommes d'opérateurs linéaires et équations différentielles opérationnelles*, J. Math. Pures Appl. 54 (1975), 305–387.
[5] G. Da Prato and J. Zabczyk, *Regular densities of invariant measures in Hilbert spaces*, preprint, Scuola Normale Superiore di Pisa, 1993.
[6] —, —, *Stochastic Equations in Infinite Dimensions*, Encyclopedia Math. Appl. 44, Cambridge University Press, 1992.
[7] M. Fuhrman, *Analyticity of transition semigroups and closability of bilinear forms in Hilbert spaces*, preprint, Dipartimento di Matematica, Politecnico di Milano, n. 105/p, ottobre 1993.
[8] Z. M. Ma and M. Röckner, *Dirichlet Forms*, Springer, 1992.
[9] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, 1983.
[10] B. Schmuland, *Non-symmetric Ornstein–Uhlenbeck processes in Banach space via Dirichlet forms*, Canad. J. Math. 45 (1993), 1324–1338.
[11] A. Yagi, *Coïncidence entre des espaces d'interpolation et des domaines de puissances fractionnaires d'opérateurs*, C. R. Acad. Sci. Paris Sér. I Math. 299 (1984), 173–176.
[12] J. Zabczyk, *Symmetric solutions of semilinear stochastic equations*, in: Stochastic Partial Differential Equations and Applications, G. Da Prato and L. Tubaro (eds.), Lecture Notes in Math. 1390, Springer, 1989, 237–256.

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