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Some results about Beurling algebras with applications to operator theory

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Abstract. We prove that certain maximal ideals in Beurling algebras on the unit disc have approximate identities, and show the existence of functions with certain properties in these maximal ideals. We then use these results to prove that if T is a bounded operator on a Banach space X satisfying $\|T^n\| = O(n^\beta)$ as $n \to \infty$ for some $\beta \geq 0$, then

$$\sum_{n=1}^{\infty} \frac{\|(1-T)^n x\|}{\|(1-T)^{n-1}x\|}$$

diverges for every $x \in X$ such that $(1-T)^{[\beta]+1}x \neq 0$.

1. Introduction. For a power bounded operator T on a Banach space X it was proved in [9] that $\sum_{n=1}^{\infty} \|(1-T)^n x\|/\|(1-T)^{n-1}x\|$ diverges for every $x \in X$ with $Tx \neq x$. (The Hilbert space case was proved in [4].) This was done by proving certain results about the algebra $\mathcal{A}^+$ of analytic functions with absolutely convergent Taylor series on the closed unit disc. When we are studying operators which only satisfy the weaker condition $\|T^n\| = O(n^\beta)$ as $n \to \infty$ for some $\beta > 0$, it seems natural to work with the Beurling algebras $\mathcal{A}_\beta^+$ (see below) instead of $\mathcal{A}^+$. In Section 2 we prove that, although the maximal ideal $\mathcal{M}_\beta^+$ in $\mathcal{A}_\beta^+$ does not have a bounded approximate identity for $\beta > 0$, it does have a sequential approximate identity for $0 < \beta < 1$, and that it satisfies a similar condition for $\beta \geq 1$. The results of Bennett and Gilbert ([2]) then enable us to describe the closed primary ideals in $\mathcal{A}_\beta^+$. In Section 3 we estimate the norms in $\mathcal{A}_\beta^+$ of the infinite products used in [4] and [9], and in Section 4 we use the results of Sections 2 and 3 to prove the existence of certain functions in $\mathcal{A}_\beta^+$, which we then use to deduce the main result.

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1.1. *Beurling algebras.* Denote the open unit disc in $\mathbb{C}$ by Δ and the unit circle by $\mathbb{T}$. Let $\mathcal{A}(\bar{\Delta})$ be the disc algebra of functions analytic in Δ and continuous on $\bar{\Delta}$ equipped with the supremum norm $\|f\|_\infty = \sup\{|f(z)| : z \in \bar{\Delta}$ $(f \in \mathcal{A}(\bar{\Delta}))$ and let $\mathcal{M}_1$ be the maximal ideal $\{f \in \mathcal{A}(\bar{\Delta}) : f(1) = 0\}$. For $\beta \geq 0$, define the *Beurling algebra* $\mathcal{A}_\beta^+$ as the subalgebra of $\mathcal{A}(\bar{\Delta})$ of functions whose Taylor coefficients at 0,

$$\widehat{f}(n) = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) e^{-int}\, dt, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

satisfy

$$\|f\|_{\mathcal{A}_\beta^+} = \sum_{n=0}^{\infty} |\widehat{f}(n)|(1+n)^\beta < \infty.$$

It is not hard to see that $\mathcal{A}_\beta^+$ with the norm $\|\cdot\|_{\mathcal{A}_\beta^+}$ is a semisimple Banach algebra with character space $\bar{\Delta}$. Let $\mathcal{M}_\beta^+$ be the maximal ideal $\{f \in \mathcal{A}_\beta^+ : f(1) = 0\}$ in $\mathcal{A}_\beta^+$ and denote the function $z \mapsto z$ by α. Then α generates $\mathcal{A}_\beta^+$ so $\mathcal{M}_\beta^+ = \overline{(1-\alpha)\mathcal{A}_\beta^+}$.

Also, for $m \in \mathbb{N}$, let $\mathcal{A}^m(\bar{\Delta}) = \{f \in \mathcal{A}(\bar{\Delta}) : f', \ldots, f^{(m)} \in \mathcal{A}(\bar{\Delta})\}$. With the norm $\|f\|_{\mathcal{A}^m(\bar{\Delta})} = \sum_{j=0}^m \|f^{(j)}\|_\infty$ (for $f \in \mathcal{A}^m(\bar{\Delta})$), $\mathcal{A}^m(\bar{\Delta})$ becomes a Banach algebra.

2. Approximate identities and the ideal structure in Beurling algebras

2.1. *Approximate identities.* Recall that a subset $\mathcal{E}$ of a commutative Banach algebra $\mathcal{B}$ is called an *approximate identity* for $\mathcal{B}$ if, for every finite set $b_1, \ldots, b_n \in \mathcal{B}$ $(n \in \mathbb{N})$ and $\varepsilon > 0$, there exists $e \in \mathcal{E}$ such that

$$\|b_j e - b_j\| < \varepsilon \quad \text{for } j = 1, \ldots, n.$$

If $\mathcal{B}$ has an approximate identity, then obviously $\mathcal{B} = \overline{\mathcal{B}^{[2]}}$, where $\mathcal{B}^{[2]} := \{ab : a, b \in \mathcal{B}\}$. If $\mathcal{E}$ is bounded, then it is called a *bounded approximate identity*. It is well known that $\mathcal{B}$ has a bounded approximate identity if and only if there exists a bounded subset $\mathcal{E}$ of $\mathcal{B}$ such that, for every $b \in \mathcal{B}$ and $\varepsilon > 0$, there exists $e \in \mathcal{E}$ such that $\|be - b\| < \varepsilon$. If $\mathcal{B}$ has a bounded approximate identity, then Cohen's factorization theorem states that there is factorization in $\mathcal{B}$, i.e. that $\mathcal{B} = \mathcal{B}^{[2]}$. We say that a sequence (e_n) in $\mathcal{B}$ is a *sequential approximate identity* for $\mathcal{B}$ if $be_n \to b$ as $n \to \infty$ for all $b \in \mathcal{B}$. If the sequence (e_n) is bounded, then we call it a *sequential bounded approximate identity.*

In this section we will be concerned with approximate identities in the maximal ideals $\mathcal{M}_\beta^+$ $(\beta \geq 0)$. If $\beta \geq 1$, then it is easily seen that $\mathcal{A}_\beta^+ \subseteq \mathcal{A}^1(\bar{\Delta})$ and thus $\overline{(\mathcal{M}_\beta^+)^{[2]}} \subseteq \{f \in \mathcal{A}_\beta^+ : f(1) = f'(1) = 0\} \subsetneq \mathcal{M}_\beta^+$, so $\mathcal{M}_\beta^+$ does not have an approximate identity. The same is true for $0 < \beta < 1$, but we will need the following two results to prove it.

LEMMA 2.1. *Let $0 < \beta \leq 1$. Then $\mathcal{M}_\beta^+ \subseteq (1-\alpha)^\beta \mathcal{M}_1$.*

Proof. Let $n \in \mathbb{N}$. Since

$$\left|\frac{z^n - 1}{(1-z)^\beta}\right| \leq \begin{cases} \dfrac{n|1-z|}{|1-z|^\beta} \leq n^\beta & \text{for } z \in \bar{\Delta} \text{ with } |1-z| \leq 1/n, \\ \dfrac{2}{|1-z|^\beta} \leq 2n^\beta & \text{for } z \in \bar{\Delta} \text{ with } |1-z| \geq 1/n \end{cases}$$

we have

$$\left\|\frac{\alpha^n - 1}{(1-\alpha)^\beta}\right\|_\infty \leq 2n^\beta \quad \text{for } n \in \mathbb{N}.$$

Now let $f \in \mathcal{M}_\beta^+$ and note that $f = \sum_{n=1}^\infty \widehat{f}(n)\,(\alpha^n - 1)$. It thus follows that

$$g = \sum_{n=1}^\infty \widehat{f}(n)\, \frac{\alpha^n - 1}{(1-\alpha)^\beta}$$

converges in $\mathcal{A}(\bar{\Delta})$ and that $(1-\alpha)^\beta g = f$. ■

LEMMA 2.2. *Let $0 < t < 1$. Then $(1-\alpha)^t \in \mathcal{M}_\beta^+$ if and only if $\beta < t$.*

Proof. It follows from the previous lemma that $(1-\alpha)^t \notin \mathcal{M}_\beta^+$ for $\beta \geq t$. Conversely, for $n \in \mathbb{N}$, we have

$$((1-\alpha)^t)^\frown(n) = \frac{((1-\alpha)^t)^{(n)}(0)}{n!} = -\frac{t}{n}\,\frac{1-t}{1}\,\frac{2-t}{2}\cdots\frac{n-1-t}{n-1}.$$

Since $\sum_{j=1}^{n-1} \log((j-t)/j) \leq -\sum_{j=1}^{n-1}(t/j) \leq -t\log n$ it thus follows that

$$|((1-\alpha)^t)^\frown(n)| \leq (t/n)\exp(-t\log n) = tn^{-(1+t)} \quad \text{for } n \in \mathbb{N}$$

and the result follows. ■

PROPOSITION 2.3. *For $\beta > 0$ the maximal ideal $\mathcal{M}_\beta^+$ does not have a bounded approximate identity.*

Proof. We have already mentioned the case $\beta \geq 1$. Let $0 < \beta < 1$ and choose t such that $\beta < t < 2\beta$. By the previous lemma we have $(1-\alpha)^t \in \mathcal{M}_\beta^+$, but on the other hand it follows from Lemma 2.1 that $(\mathcal{M}_\beta^+)^{[2]} \subseteq (1-\alpha)^{2\beta}\mathcal{M}_1$. Hence $(1-\alpha)^t \notin (\mathcal{M}_\beta^+)^{[2]}$, so the result follows from Cohen's factorization theorem. ■

We will soon turn our attention to unbounded approximate identities, but first we need the following two lemmas. For $\beta \geq 0$ and $m \leq [\beta]$, let

$I_m = \{f \in \mathcal{A}_\beta^+ : f(1) = \ldots = f^{(m)}(1) = 0\}$ (it is easily seen that $\mathcal{A}_\beta^+ \subseteq \mathcal{A}^m(\bar{\Delta})$ if and only if $m \le \beta$). We will discuss ideal structures at the end of the section and right now we will only need the following.

LEMMA 2.4. *For $\beta \ge 0$ we have $I_{[\beta]} = \overline{(1-\alpha)^{[\beta]+1}\mathcal{A}_\beta^+}$.*

Proof. It is clear that $(1-\alpha)^{[\beta]+1} \in I_{[\beta]}$. Conversely, let $f \in I_{[\beta]}$ and choose a sequence $(p_n)_{n\in\mathbb{N}}$ of polynomials such that $p_n \to f$ in $\mathcal{A}_\beta^+$ as $n \to \infty$. Since the injection $\mathcal{A}_\beta^+ \hookrightarrow \mathcal{A}^{[\beta]}(\bar{\Delta})$ is continuous we have $p_n^{(j)}(1) \to 0$ as $n \to \infty$ for $j = 0, \ldots, [\beta]$, from which it follows that $q_n = p_n - \sum_{j=0}^{[\beta]} (1/j!)\, p_n^{(j)}(1)\,(\alpha-1)^j \to f$ in $\mathcal{A}_\beta^+$ as $n \to \infty$. Furthermore, $q_n^{(j)}(1) = 0$ for $j = 0, \ldots, [\beta]$, and as q_n is a polynomial this implies that $(1-\alpha)^{[\beta]+1}$ divides q_n and thus $q_n \in (1-\alpha)^{[\beta]+1}\mathcal{A}_\beta^+$ for $n \in \mathbb{N}$. ■

The following result is well known but we include it for the sake of completeness.

LEMMA 2.5. *Let $n \in \mathbb{N}$ and let p be a polynomial of degree at most $n-1$. Then*

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}p(j) = 0.$$

Proof. For $m = 0, \ldots, n-1$ we have

$$0 = ((1-\alpha)^n)^{(m)}(1) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}j(j-1)\ldots(j-m+1).$$

Since $\{1, \alpha, \alpha(\alpha-1), \ldots, \alpha(\alpha-1)\ldots(\alpha-(n-2))\}$ span the space of polynomials of degree at most $n-1$ the result follows. ■

For $k, m \in \mathbb{N}_0$, let

$$g_{m,k} = \binom{m+k+1}{k+1}^{-1}\sum_{j=0}^{m}\binom{m+k-j}{k}\alpha^j$$

and note that $g_{m,k}(1) = 1$. Also, it is not hard to see that $\|g_{m,[\beta]}\|_{\mathcal{A}_\beta^+} \to \infty$ as $m \to \infty$ if $\beta > 0$. The proof of the following result is rather technical, but it should be noted that it becomes much simpler when $0 \le \beta < 1$ (i.e. when $k = 0$). Hence a direct and fairly simple proof of the fact that $\mathcal{M}_\beta^+$ has a sequential approximate identity when $0 \le \beta < 1$ (i.e. of Corollary 2.7) can be obtained by letting $k = 0$ in the proof below.

PROPOSITION 2.6. *Let $\beta \ge 0$ and let $k = [\beta]$. Then $fg_{m,k} \to 0$ in $\mathcal{A}_\beta^+$ as $m \to \infty$ for $f \in I_k$.*

Proof. We split the proof into three parts:

(i) $(1-\alpha)^{k+1}g_{m,k} \to 0$ *in $\mathcal{A}_\beta^+$ as $m \to \infty$.* Let $m \in \mathbb{N}_0$. Since

$$((1-\alpha)^{k+1}g_{m,k})^\frown(n) = \sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j}(\alpha^j g_{m,k})^\frown(n)$$
$$= \binom{m+k+1}{k+1}^{-1}\sum_{j=\max\{0,n-m\}}^{\min\{k+1,n\}}(-1)^j\binom{k+1}{j}\binom{m+k-n+j}{k}$$

for $n \in \mathbb{N}_0$ we deduce that $((1-\alpha)^{k+1}g_{m,k})^\frown(n) = 0$ for $n \ge m+k+2$, and since $\binom{m+k-n+j}{k}$ is a polynomial of degree k in j with $n-m-1, \ldots, n-m-k$ as zeros it follows from the previous lemma that $((1-\alpha)^{k+1}g_{m,k})^\frown(n) = 0$ for $k+1 \le n \le m+k$. Furthermore,

$$((1-\alpha)^{k+1}g_{m,k})^\frown(n) = \binom{m+k+1}{k+1}^{-1}q_n(m) \quad \text{for } 0 \le n \le k,$$

where q_n is a polynomial of degree k and

$$((1-\alpha)^{k+1}g_{m,k})^\frown(m+k+1) = \binom{m+k+1}{k+1}^{-1}(-1)^{k+1}.$$

Hence

$$\|(1-\alpha)^{k+1}g_{m,k}\|_{\mathcal{A}_\beta^+} = \binom{m+k+1}{k+1}^{-1}\Big(\sum_{n=0}^{k}|q_n(m)|\,(1+n)^\beta + (m+k+2)^\beta\Big),$$

so since $\beta < 1+k$ and since $\binom{m+k+1}{k+1}$ is a polynomial of degree $k+1$ in m it follows that $(1-\alpha)^{k+1}g_{m,k} \to 0$ in $\mathcal{A}_\beta^+$ as $m \to \infty$.

(ii) *There exists a constant c_β such that $\|fg_{m,k}\|_{\mathcal{A}_\beta^+} \le c_\beta\|f\|_{\mathcal{A}_\beta^+}$ for $f \in I_k$ and $m \in \mathbb{N}_0$.* Let $f \in I_k$ and let $m \in \mathbb{N}_0$. Then

$$(fg_{m,k})^\frown(n) = \begin{cases} \binom{m+k+1}{k+1}^{-1}\sum_{j=0}^{n}\binom{m+k-n+j}{k}\widehat{f}(j) & \text{for } n \le m, \\ \binom{m+k+1}{k+1}^{-1}\sum_{j=n-m}^{n}\binom{m+k-n+j}{k}\widehat{f}(j) & \text{for } n \ge m+1. \end{cases}$$

Let $0 \le n \le m$ and write

$$\binom{m+k-n+j}{k} = \sum_{l=0}^{k}d_l\, j^l, \quad j \in \mathbb{N}_0,$$

for some constants d_l, $l = 0, \ldots, k$. Since $f^{(l)}(1) = 0$ for $l = 0, \ldots, k$ it follows that $\sum_{j=0}^{\infty}\widehat{f}(j)\,p(j) = 0$ for every polynomial p of degree at most k.

Hence

$$\left|\sum_{j=0}^{n}\binom{m+k-n+j}{k}\widehat{f}(j)\right| = \left|\sum_{j=n+1}^{\infty}\binom{m+k-n+j}{k}\widehat{f}(j)\right|$$
$$\le \sum_{l=0}^{k} d_l \sum_{j=n+1}^{\infty} |\widehat{f}(j)|\, j^l$$
$$\le \sum_{l=0}^{k} d_l\,(1+n)^{l-\beta}\,\|f\|_{\mathcal{A}_\beta^+}$$
$$= \binom{m+k-n+1+n}{k}(1+n)^{-\beta}\,\|f\|_{\mathcal{A}_\beta^+},$$

so it follows that

$$\sum_{n=0}^{m} |(fg_{m,k})^{\widehat{}}(n)|(1+n)^{\beta} \le (m+1)\binom{m+k+1}{k+1}^{-1}\binom{m+k+1}{k}\|f\|_{\mathcal{A}_\beta^+}$$
$$= (k+1)\,\|f\|_{\mathcal{A}_\beta^+}.$$

Furthermore,

$$\sum_{n=m+1}^{\infty} |(fg_{m,k})^{\widehat{}}(n)|(1+n)^{\beta}$$
$$\le \binom{m+k+1}{k+1}^{-1}\sum_{n=m+1}^{\infty}\sum_{j=n-m}^{n}\binom{m+k-n+j}{k}|\widehat{f}(j)|\,(1+n)^{\beta}$$
$$\le \binom{m+k+1}{k+1}^{-1}\sum_{j=1}^{\infty}\sum_{n=\max\{m+1,j\}}^{m+j}\binom{m+k-n+j}{k}(1+n)^{\beta}\,|\widehat{f}(j)|.$$

For $1\le j\le m$ we have

$$\sum_{n=m+1}^{m+j}\binom{m+k-n+j}{k}(1+n)^{\beta} \le j\binom{k+j-1}{k}(1+m+j)^{\beta}$$
$$\le j^{k+1}\,(1+2m)^{\beta} \le j^{\beta}\,(1+2m)^{k+1},$$

and for $j\ge m+1$ we have

$$\sum_{n=j}^{m+j}\binom{m+k-n+j}{k}(1+n)^{\beta} \le (m+1)\binom{m+k}{k}(2j)^{\beta}.$$

Since $\binom{m+k+1}{k+1}$ is a polynomial of degree $k+1$ in m it thus follows that

$$\|fg_{m,k}\|_{\mathcal{A}_\beta^+} \le c_\beta\,\|f\|_{\mathcal{A}_\beta^+}$$

for some constant c_β.

(iii) *Conclusion.* Define bounded linear operators G_m on I_k by $G_m f = fg_{m,k}$ for $f\in I_k$ and $m\in\mathbb{N}_0$. Then $G_m(1-\alpha)^{k+1}\to 0$ as $m\to\infty$ by (i) and $\|G_m\|\le c_\beta$ for $m\in\mathbb{N}_0$ by (ii), so it follows from Lemma 2.4 that $G_m f\to 0$ as $m\to\infty$ for all $f\in I_k$. ■

COROLLARY 2.7. *Let $0\le\beta<1$. Then $(1-g_{m,0})_{m\in\mathbb{N}}$ is a sequential approximate identity for $\mathcal{M}_\beta^+$. It is bounded if and only if $\beta=0$.*

2.2. *The ideal structure in $\mathcal{A}_\beta^+$.* For a closed ideal I in $\mathcal{A}_\beta^+$, let $h(I)=\{z\in\bar{\Delta} : f(z)=0 \text{ for all } f\in I\}$ (the *hull* of I) and let Q_I be the greatest common divisor of the inner factors of non-zero functions in I ([6, p. 85]). Furthermore, let $\mathcal{A}_\beta=\{f(e^{it})=\sum_{n=-\infty}^{\infty}\widehat{f}(n)\,e^{int} : \sum_{n=-\infty}^{\infty}|\widehat{f}(n)|\,(1+|n|)^{\beta}<\infty\}$ (the Beurling algebra on $\mathbb{T}$) and let $I^{\mathcal{A}_\beta}$ be the closed ideal in $\mathcal{A}_\beta$ generated by I. For a closed set $E\subseteq\mathbb{T}$, let

$$I_\beta^+(E)=\{f\in\mathcal{A}_\beta^+ : f=0 \text{ on } E\}.$$

For $0\le\beta<1$ it follows from Corollary 2.7 that $\mathcal{A}_\beta^+$ satisfies the "analytic Ditkin condition" ([2]). Also, from [10, V.3.3, V.3.5] and [8, p. 230] it follows that countable closed sets are of synthesis for $\mathcal{A}_\beta$, so we can apply a result of Bennett and Gilbert ([2, Theorem A]; see also [1, Proposition 7] where Atzmon obtained slightly stronger results using a different method of proof) to obtain the following result.

THEOREM 2.8. *Let $0\le\beta<1$. If I is a closed ideal in $\mathcal{A}_\beta^+$ with $h(I)\cap\mathbb{T}$ countable, then*

$$I=Q_I\mathcal{A}(\bar{\Delta})\cap I_\beta^+(h(I)\cap\mathbb{T}).$$

Similarly, Proposition 2.6 implies that $\mathcal{A}_\beta^+$ satisfies the "[β]-analytic Ditkin condition" ([2]) for $\beta\ge 0$, so we deduce the following from [2, Theorem B].

THEOREM 2.9. *Let $\beta\ge 0$. If I is a closed ideal in $\mathcal{A}_\beta^+$ with $h(I)\cap\mathbb{T}$ finite, then*

$$I=Q_I\mathcal{A}(\bar{\Delta})\cap I^{\mathcal{A}_\beta}.$$

(Note that we obviously cannot write $I_\beta^+(h(I)\cap\mathbb{T})$ instead of $I^{\mathcal{A}_\beta}$; just take $\beta=1$ and $I=I_1$.)

Let $f\in\mathcal{A}(\bar{\Delta})$ with $f\ne 0$ and let $f=B_fS_fF_f$ be the factorization of f as a product of a Blaschke product B_f, a singular inner function S_f and an outer function $F_f\in\mathcal{A}(\bar{\Delta})$ (see e.g. [6, p. 67]). Let μ_f be the positive singular measure on $\mathbb{T}$ that defines S_f ([6, p. 66]). Then $\varrho(f):=\max\{a\ge 0 : \psi_a f \text{ is bounded on } \Delta\}=\mu_f(\{1\})$. Also, for $I\subseteq\mathcal{A}(\bar{\Delta})$, define $\varrho(I)=\inf\{\varrho(f) : f\in I,\ f\ne 0\}$. We are now ready to describe the structure of the closed primary ideals in $\mathcal{A}_\beta^+$. (Recall that an ideal is called primary if it is contained in exactly one maximal ideal.) For $\beta = 0$ the following result was proved by Kahane ([7]).

PROPOSITION 2.10. *Let $\beta \ge 0, N = [\beta]$ and let I be a closed primary ideal in $\mathcal{A}_\beta^+$ contained in the maximal ideal $\mathcal{M}_\beta^+$ (i.e. with $h(I) = \{1\}$) and let $a = \varrho(I)$. Then*

(i) *If $a = 0$, then there exists $0 \le n \le N$ such that $I = I_n$.*
(ii) *If $a > 0$, then $I = I_{a,N} := \psi_{-a}\mathcal{A}(\bar{\Delta}) \cap I_N$.*
(iii) *Furthermore, $\mathcal{M}_\beta^+ = I_0 \supsetneqq I_1 \supsetneqq \ldots \supsetneqq I_N \supsetneqq I_{a,N} \supsetneqq I_{b,N}$ for $b > a > 0$.*

Proof. By the previous theorem $I = Q_I\mathcal{A}(\bar{\Delta}) \cap I^{\mathcal{A}_\beta} \cap \mathcal{A}_\beta^+$ and it follows from [5, Theorem 1, p. 216] that there exists $0 \le n \le N$ such that $I^{\mathcal{A}_\beta} \cap \mathcal{A}_\beta^+ = I_n$. Write $Q_I = B_I S_I$, where B_I is a Blaschke product and S_I a singular inner function. As $h(I) = \{1\}$ we deduce that $B_I = 1$. Let $f \in I$ and let $f = B_f S_f F_f$ be the factorization of f. Since S_f does not extend continuously to any point in supp μ_f ([6, pp. 68–69]) it follows that $f = F_f = 0$ on supp μ_f. Also, S_f/S_I is bounded so supp $\mu_I \subseteq$ supp μ_f (where μ_I is the positive singular measure on $\mathbb{T}$ defining S_I). Hence supp $\mu_I \subseteq h(I) = \{1\}$, and since $\mu_I(\{1\}) = \inf\{\mu_f(\{1\}) : f \in I, f \ne 0\} = a$ we deduce that $S_I = \psi_{-a}$. Hence

$$I = \psi_{-a}\mathcal{A}(\bar{\Delta}) \cap I_n$$

and (i) follows immediately.

Now assume that $a > 0$ and let $f \in \psi_{-a}\,\mathcal{A}(\bar{\Delta}) \cap I_0$. For $n = 0, \ldots, N$, let $f^{(n)} = B_{f^{(n)}} S_{f^{(n)}} F_{f^{(n)}}$ be the factorization of $f^{(n)}$. From a result of Caughran ([3]) we deduce that $S_{f^{(n)}}/S_f$ is bounded (i.e. that $\mu_{f^{(n)}} \ge \mu_f$). Since S_f/ψ_{-a} is bounded it thus follows that so is $S_{f^{(n)}}/\psi_{-a}$. Hence $1 \in$ supp $\mu_{f^{(n)}}$, so we deduce as before that $f^{(n)}(1) = 0$. Therefore $f \in \psi_{-a}\mathcal{A}(\bar{\Delta}) \cap I_N$ and (ii) follows.

For (iii), note that $(1-\alpha)^{n+1} \in I_n \setminus I_{n+1}$ for $n = 0, \ldots, N-1$ and that $(1-\alpha)^{N+1} \in I_N \setminus I_{a,N}$ for $a > 0$. Also, it follows from a result of Hardy (see [6, p. 70] or [9, Lemma 3.3]) that $\mathcal{A}^1(\bar{\Delta}) \subseteq \mathcal{A}^+$ (actually it can be shown that $\mathcal{A}^1(\bar{\Delta}) \subseteq \mathcal{A}_\gamma^+$ if and only if $\gamma < 1/2$) and from this we deduce that $\mathcal{A}^{N+2}(\bar{\Delta}) \subseteq \mathcal{A}_\beta^+$. Since $\psi_{-a}' = -2a(1-\alpha)^{-2}\psi_{-a}$ we thus have $\psi_{-a}(1-\alpha)^{2N+5} \in I_{a,N} \setminus I_{b,N}$ for $b > a > 0$. ∎

COROLLARY 2.11. *Let $\beta \ge 0$ and let $N = [\beta]$. Let f be an outer function in $\mathcal{M}_\beta^+$ and assume that f has no other zeros than $z = 1$. Let $m = \max\{0 \le n \le N : f \in I_n\}$. Then $\overline{f\mathcal{A}_\beta^+} = I_m$.*

3. Norms of infinite products.

Let $w = (w_n)_{n\in\mathbb{N}_0}$ be a sequence of strictly positive numbers and assume that $D_w := \sum_{n=1}^\infty w_{n-1}/w_n < \infty$.

With $\varepsilon_n = w_{n-1}/w_n$ for $n \in \mathbb{N}$ it follows from the proof of the second part of [4, Théorème 3.1] that

$$g_w(z) = w_0 \prod_{k=1}^\infty \frac{1-z}{1+\varepsilon_k - z}$$

converges uniformly on every compact subset of ${}_1\Pi := \{z \in \mathbb{C} : \operatorname{Re} z < 1\}$ and defines an element $g_w \in \bigcap_{n=0}^\infty (1-\alpha)^n \mathcal{A}(\bar{\Delta})$ with

$$\left\| \frac{g_w}{(1-\alpha)^n} \right\|_\infty \le w_n \quad \text{for } n \in \mathbb{N}_0. \tag{1}$$

We first prove the following.

LEMMA 3.1. *With the above notation, g_w is an outer function.*

Proof. For $m \in \mathbb{N}$, let

$$h_m(z) = \prod_{k=m}^\infty \frac{1-z}{1+\varepsilon_k - z} \quad \text{for } z \in {}_1\Pi.$$

Since $h_m \to 1$ uniformly on every compact subset of $\bar{\Delta} \setminus \{1\}$ as $m \to \infty$ it follows that $(h_m)_{m\in\mathbb{N}}$ is a bounded approximate identity for $\mathcal{M}_1$ and therefore that

$$\mathcal{M}_1 = \overline{\bigcup_{m\in\mathbb{N}} h_m \mathcal{M}_1}.$$

On the other hand, g/h_k is an outer function with $z = 1$ as its only zero, so it follows from the Rudin–Beurling ideal theory ([6, p. 85]) that $\overline{h_k\mathcal{M}_1} = \overline{g\mathcal{M}_1}$ for $k \in \mathbb{N}$. Hence

$$\mathcal{M}_1 = \overline{g\mathcal{M}_1},$$

which, again by the Rudin–Beurling ideal theory, implies that g is outer. ∎

Let $\mathcal{W}$ be the set of all sequences $w = (w_n)_{n\in\mathbb{N}_0}$ of strictly positive numbers satisfying $D_w < \infty$ and $w_{n-1} \le w_n$ for $n \in \mathbb{N}$. The following result will enable us to reduce the proof of Theorem 4.1 to the case where $u \in \mathcal{W}$ and by restricting ourselves to sequences in $\mathcal{W}$ we avoid some technical difficulties.

LEMMA 3.2. *Let $(v_n)_{n\in\mathbb{N}_0}$ be a sequence of strictly positive numbers with $\sum_{n=1}^\infty v_{n-1}/v_n < \infty$. Then there exists $w \in \mathcal{W}$ satisfying $0 < w_n \le v_n$ and $w_n/w_{n+1} \le v_n/v_{n+1}$ for $n \in \mathbb{N}_0$.*

Proof. Choose $N \in \mathbb{N}_0$ such that $v_n \le v_{n+1}$ for $n \ge N$. Let $a = \sup\{v_{n-1}/v_n : n = 1, \ldots, N\}$. If $a \le 1$, then just set $w_n = v_n$ for $n \in \mathbb{N}_0$. If $a > 1$, then set $w_n = a^{(n-N)}v_n$ for $n = 0, \ldots, N$ and $w_n = v_n$ for $n \ge N+1$. Then $w_{n-1}/w_n = a^{-1}v_{n-1}/v_n \le \min\{1, v_{n-1}/v_n\}$ for $n = 1, \ldots, N$ and hence for $n \in \mathbb{N}$, and $w_n \le v_n$ for $n \in \mathbb{N}_0$. ∎

For $w \in \mathcal{W}$ we will now prove that $g_w \in \bigcap_{n=0}^{\infty}(1-\alpha)^n \mathcal{A}^m(\bar{\Delta})$ for $m \in \mathbb{N}_0$ and obtain bounds for $\|g_w/(1-\alpha)^n\|_{\mathcal{A}^m(\bar{\Delta})}$ for $m, n \in \mathbb{N}_0$.

LEMMA 3.3. *For $w \in \mathcal{W}$ and $m, n \in \mathbb{N}_0$ we have $g_w^{(m)}/(1-\alpha)^n \in \mathcal{A}(\bar{\Delta})$. Furthermore, there exist polynomials p_m of degree m with non-negative coefficients such that*

$$\left\|\frac{g_w^{(m)}}{(1-\alpha)^n}\right\|_\infty \le p_m(D_w)\, w_{n+2m} \quad \text{for } w \in \mathcal{W} \text{ and } m, n \in \mathbb{N}.$$

Proof. Let $w \in \mathcal{W}$. Note that the result holds for $m = 0$ and $n \in \mathbb{N}_0$ by (1). Furthermore, an easy induction exercise shows that, for $m \in \mathbb{N}$, we have

$$g_w^{(m)}(z) = g_w(z) \sum_{r=1}^{m} (1-z)^{-r} \sum_{s=1}^{r} \sum_{\substack{1 \le j_1 \le \ldots \le j_s \\ j_1+\ldots+j_s = m+s-r}} b_{m,r,\underline{j}}\, f_{w,j_1}(z) \ldots f_{w,j_s}(z)$$

for $z \in {}_1\Pi$, where the $b_{m,r,\underline{j}}$'s are constants and where

$$f_{w,j}(z) = \sum_{k=1}^{\infty} \varepsilon_k/(1+\varepsilon_k - z)^j \quad \text{for } z \in {}_1\Pi \text{ and } j \in \mathbb{N}.$$

Since $|1+\varepsilon_k - z| \ge |1-z|$ we have $|f_{w,j}(z)| \le D_w |1-z|^{-j}$ for $z \in {}_1\Pi$ and $j \in \mathbb{N}$, so it follows from (1) that $g_w^{(m)}/(1-\alpha)^n \in \mathcal{A}(\bar{\Delta})$ with

$$\begin{aligned}
\left\|\frac{g_w^{(m)}}{(1-\alpha)^n}\right\|_\infty &\le \sum_{r=1}^{m}\sum_{s=1}^{r} \sum_{\substack{1 \le j_1 \le \ldots \le j_s \\ j_1+\ldots+j_s \\ = m+s-r}} |b_{m,r,\underline{j}}|\, D_w^s \left\|\frac{g_w}{(1-\alpha)^{n+r+j_1+\ldots+j_s}}\right\|_\infty \\
&\le \sum_{r=1}^{m}\sum_{s=1}^{r} \sum_{\substack{1 \le j_1 \le \ldots \le j_s \\ j_1+\ldots+j_s \\ = m+s-r}} |b_{m,r,\underline{j}}|\, D_w^s\, w_{n+r+m+s-r} \\
&\le \sum_{r=1}^{m}\sum_{s=1}^{r} \sum_{\substack{1 \le j_1 \le \ldots \le j_s \\ j_1+\ldots+j_s \\ = m+s-r}} |b_{m,r,\underline{j}}|\, D_w^s w_{n+2m} \quad \text{for } m, n \in \mathbb{N}_0
\end{aligned}$$

as required. ■

LEMMA 3.4. *For $w \in \mathcal{W}$ and $m, n \in \mathbb{N}_0$ we have $g_w/(1-\alpha)^n \in \mathcal{A}^m(\bar{\Delta})$. Furthermore, there exist polynomials q_m and ϱ_m of degree m with non-negative coefficients such that*

$$\left\|\frac{g_w}{(1-\alpha)^n}\right\|_{\mathcal{A}^m(\bar{\Delta})} \le \varrho_m(n)\, q_m(D_w)\, w_{n+2m} \quad \text{for } w \in \mathcal{W} \text{ and } m, n \in \mathbb{N}_0.$$

Proof. Let $w \in \mathcal{W}$ and $m, n \in \mathbb{N}_0$. For $r \in \mathbb{N}_0$ and $z \in {}_1\Pi$ we have

$$\left(\frac{g_w}{(1-\alpha)^n}\right)^{(r)}(z) = \sum_{l=0}^{r} \binom{r}{l} g_w^{(r-l)}(z)(n+l-1)(n+l-2)\ldots n(1-z)^{-(n+l)}.$$

The previous lemma thus implies that $g_w/(1-\alpha)^n \in \mathcal{A}^m(\bar{\Delta})$ with

$$\begin{aligned}
\left\|\frac{g_w}{(1-\alpha)^n}\right\|_{\mathcal{A}^m(\bar{\Delta})} &= \sum_{r=0}^{m} \left\|\left(\frac{g_w}{(1-\alpha)^n}\right)^{(r)}\right\|_\infty \\
&\le \sum_{r=0}^{m}\sum_{l=0}^{r} \binom{r}{l} (n+l-1)(n+l-2)\ldots n p_{r-l}(D_w)\, w_{n+l+2(r-l)} \\
&\le (n+m-1)(n+m-2)\ldots n \left(\sum_{r=0}^{m}\sum_{l=0}^{r} \binom{r}{l} p_{r-l}(D_w)\right) w_{n+2m}
\end{aligned}$$

as required. ■

4. The main results

THEOREM 4.1. *Let $\beta \ge 0$ and $(u_n)_{n\in\mathbb{N}_0}$ be a sequence of strictly positive numbers with $\sum_{n=1}^{\infty} u_{n-1}/u_n < \infty$. Then there exists $g \in \bigcap_{n=0}^{\infty}(1-\alpha)^n \mathcal{A}_\beta^+$ such that*

(i) $$\overline{g\mathcal{A}_\beta^+} = I_{[\beta]} = \overline{(1-\alpha)^{[\beta]+1}\mathcal{A}_\beta^+},$$

(ii) $$\left\|\frac{g}{(1-\alpha)^n}\right\|_{\mathcal{A}_\beta^+} \le u_n \quad \textit{for } n \in \mathbb{N}_0.$$

Proof. Let $m = [\beta]+2$ so that $\mathcal{A}^m(\bar{\Delta}) \subseteq \mathcal{A}_\beta^+$ with $\|f\|_{\mathcal{A}_\beta^+} \le c\|f\|_{\mathcal{A}^m(\bar{\Delta})}$ for $f \in \mathcal{A}^m(\bar{\Delta})$ for some constant $c>0$ (see the proof of Proposition 2.10(iii)). With ϱ_m and q_m as in Lemma 3.4, let $v_{n+2m} = u_n/\varrho_m(n)$ for $n \in \mathbb{N}_0$ and $v_n = 1$ for $0 \le n \le 2m-1$. Note that $D_v < \infty$ and let $\tilde{v}_n = v_n/(c\, q_m(D_v))$ for $n \in \mathbb{N}_0$. By Lemma 3.2 there exists a sequence $w \in \mathcal{W}$ with $0 < w_n \le \tilde{v}_n$ for $n \in \mathbb{N}_0$ and $D_w \le D_{\tilde{v}} = D_v$ and it follows from Lemma 3.4 that

$$\begin{aligned}
\left\|\frac{g_w}{(1-\alpha)^n}\right\|_{\mathcal{A}_\beta^+} &\le c\left\|\frac{g_w}{(1-\alpha)^n}\right\|_{\mathcal{A}^m(\bar{\Delta})} \\
&\le c\,\varrho_m(n)\, q_m(D_w)\, w_{n+2m} \le c\,\varrho_m(n)\, q_m(D_v)\, \tilde{v}_{n+2m} \\
&= u_n \quad \text{for } n \in \mathbb{N}_0.
\end{aligned}$$

Finally, since g_w is outer and has no other zeros than $z = 1$ we deduce from Corollary 2.11 and Lemma 2.4 that

$$\overline{g_w\mathcal{A}_\beta^+} = I_{[\beta]} = \overline{(1-\alpha)^{[\beta]+1}\mathcal{A}_\beta^+},$$

which finishes the proof. ■

By imitating the method used by Esterle and Zouakia ([4]) we obtain a series of corollaries.

THEOREM 4.2. *Let X be a Banach space and let $T \in \mathcal{B}(X)$. Assume that $\|T^n\| = O(n^\beta)$ as $n \to \infty$ and that $(1-T)^{[\beta]+1}x \neq 0$ for some $\beta \geq 0$ and $x \in X$. Then*

(i) *For every sequence $(u_n)_{n\in\mathbb{N}_0}$ of strictly positive numbers with $\sum_{n=1}^{\infty} u_{n-1}/u_n < \infty$ there exists $\lambda > 0$ such that*

$$\|(1-T)^n x\| \geq \frac{\lambda}{u_n} \quad \text{for } n \in \mathbb{N}_0.$$

(ii) *The series*

$$\sum_{n=1}^{\infty} \frac{\|(1-T)^n x\|}{\|(1-T)^{n-1}x\|}$$

is divergent.

Proof. Define $\varphi : \mathcal{A}_\beta^+ \to \mathcal{B}(X)$ by

$$\varphi(f) = \sum_{n=0}^{\infty} \widehat{f}(n) T^n, \quad f \in \mathcal{A}_\beta^+.$$

Then φ is a continuous homomorphism (with $\|\varphi\| = \sup_{n\in\mathbb{N}} \|T^n\|/(1+n)^\beta$) and $\varphi(\alpha) = T$. Let the sequence $(u_n)_{n\in\mathbb{N}_0}$ be given and let g be as in Theorem 4.1. Then

$$\begin{aligned}\|\varphi(g)x\| &\leq \left\|\varphi\left(\frac{g}{(1-\alpha)^n}\right)\right\| \|\varphi((1-\alpha)^n)x\| \\ &\leq \|\varphi\|\, u_n \|(1-T)^n x\| \quad \text{for } n \in \mathbb{N}_0.\end{aligned}$$

Since $(1-\alpha)^{[\beta]+1} \in \overline{g\mathcal{A}_\beta^+}$ and since $0 \neq (1-T)^{[\beta]+1}x = \varphi((1-\alpha)^{[\beta]+1})x$ it follows that $\varphi(g)x \neq 0$. This proves (i) with $\lambda = \|\varphi(g)x\|/\|\varphi\|$. The proof of (ii) is now the same as that of [4, Théorème 4.1.2°]. ■

Remark. Theorem 4.2 is in some sense the best possible result as the following example shows. Let $\beta \geq 0$ and $m = [\beta]$. Let $\widetilde{\mathcal{A}} = \mathcal{A}_\beta^+/I_\beta$ and let $a \mapsto M_a : \widetilde{\mathcal{A}} \to \mathcal{B}(\widetilde{\mathcal{A}})$ be the isometric embedding defined by $M_a b = ab$ for $a, b \in \widetilde{\mathcal{A}}$. With $T = M_{\alpha+I_\beta}$ we have $\|T^n\| = \|\alpha^n + I_\beta\|_{\widetilde{\mathcal{A}}} \leq \|\alpha^n\|_{\mathcal{A}_\beta^+} = O(n^\beta)$ as $n \to \infty$ and $(1-T)^m \neq 0$, whereas $(1-T)^{m+1} = 0$. To obtain a visually more appealing example, note that the continuous homomorphism $\varphi : \mathcal{A}^m(\overline{\Delta}) \to M_{m+1}(\mathbb{C})$ given by

$$\varphi(f) = \begin{pmatrix} f(1) & f'(1) & \frac{1}{2!}f''(1) & \dots & \frac{1}{m!}f^{(m)}(1) \\ 0 & f(1) & f'(1) & \dots & \frac{1}{(m-1)!}f^{(m-1)}(1) \\ 0 & 0 & f(1) & \dots & \frac{1}{(m-2)!}f^{(m-2)}(1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & f(1) \end{pmatrix}$$

induces an injective continuous homomorphism $\widetilde{\varphi} : \widetilde{\mathcal{A}} \to M_{m+1}(\mathbb{C})$. Hence, with

$$S = \widetilde{\varphi}(\alpha + I_k) = \varphi(\alpha) = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

we have $\|S^n\| = O(n^m) = O(n^\beta)$ as $n \to \infty$ and $(1-S)^m \neq 0$, whereas $(1-S)^{m+1} = 0$.

The proof of Theorem 4.2 carries over almost directly to yield the following.

COROLLARY 4.3. *Let $\mathcal{B}$ be a unital Banach algebra. Let $b \in \mathcal{B}$ and assume that $\|b^n\| = O(n^\beta)$ as $n \to \infty$ and that $(1-b)^{[\beta]+1} \neq 0$ for some $\beta \geq 0$. Then*

(i) *For every sequence $(u_n)_{n\in\mathbb{N}_0}$ of strictly positive numbers satisfying $\sum_{n=1}^{\infty} u_{n-1}/u_n < \infty$ there exists $\lambda > 0$ such that*

$$\|(1-b)^n x\| \geq \frac{\lambda}{u_n} \quad \text{for } n \in \mathbb{N}_0.$$

(ii) *The series*

$$\sum_{n=1}^{\infty} \frac{\|(1-b)^n x\|}{\|(1-b)^{n-1}x\|}$$

is divergent.

Remark. After the submission of this paper it was pointed out to us that Esterle and Zarrabi in a recent preprint *Local properties of powers of operators* have given a short proof of a result which is slightly stronger than our Theorem 4.2(ii). However, their result does not imply Theorem 4.2(i) and our method of proof, using the existence of certain functions in Beurling algebras, seems to be of independent interest.

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Analyticity of transition semigroups and closability of bilinear forms in Hilbert spaces

by

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Abstract. We consider a semigroup acting on real-valued functions defined in a Hilbert space H, arising as a transition semigroup of a given stochastic process in H. We find sufficient conditions for analyticity of the semigroup in the $L^2(\mu)$ space, where μ is a gaussian measure in H, intrinsically related to the process. We show that the infinitesimal generator of the semigroup is associated with a bilinear closed coercive form in $L^2(\mu)$. A closability criterion for such forms is presented. Examples are also given.

1. Introduction. Let H be a real Hilbert space with scalar product $\langle .,. \rangle$. Let $\mathcal{L}(H)$ be the algebra of all bounded, everywhere defined, linear operators in H. We denote the norm in H and in $\mathcal{L}(H)$ by the same symbol $\|\ \|$. Let $\mathcal{B}_b(H)$ be the set of all bounded Borel measurable functions $f : H \to \mathbb{R}$. Let A be the infinitesimal generator of a strongly continuous semigroup e^{tA}, $t \geq 0$, of linear operators in H. Assume $R \in \mathcal{L}(H)$ is a nonnegative operator in H, i.e. $R = R^* \geq 0$ and assume that Q_t given by the formula

$$Q_t = \int_0^t e^{sA} R e^{sA^*}\, ds$$

is a trace class operator (here and in the following, operator-valued integrals converge in the strong operator topology). Then one can define the *transition semigroup*

$$(P_t\phi)(x) = \int_H \phi(y)\mathcal{N}(e^{tA}x, Q_t)(dy), \qquad \phi \in \mathcal{B}_b(H), \tag{1}$$

where $\mathcal{N}(e^{tA}x, Q_t)$ is the gaussian measure in H with mean value $e^{tA}x$ and covariance operator Q_t. In this paper we will study regularity properties of P_t.

A motivation for studying P_t is its well known probabilistic interpretation which we now sketch. Consider the stochastic differential equation in H:

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