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Received April 7, 1992 (2926)
Revised version January 30, 1995

STUDIA MATHEMATICA 115 (1) (1995)

Algebras of real analytic functions: Homomorphisms and bounding sets

by

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Abstract. This article deals with bounding sets in real Banach spaces E with respect to the functions in $\mathcal{A}(E)$, the algebra of real analytic functions on E, as well as to various subalgebras of $\mathcal{A}(E)$. These bounding sets are shown to be relatively weakly compact and the question whether they are always relatively compact in the norm topology is reduced to the study of the action on the set of unit vectors in l_∞ of the corresponding functions in $\mathcal{A}(l_\infty)$. These results are achieved by studying the homomorphisms on the function algebras in question, an idea that is also reversed in order to obtain new results for the set of homomorphisms on these algebras.

In this paper we are interested in subsets of a real Banach space on which different classes of functions are bounded. In [3] it is shown that if a subset B of a real Banach space E has the property that each C^∞-function on E is bounded on B, then B is relatively compact. As the continuous polynomials $\mathcal{P}(E)$ on E are bounded on bounded sets the focus of interest in this article is on the algebras between $\mathcal{P}(E)$ and $\mathcal{A}(E)$, the algebra of real analytic functions on E. Let $A(E)$ denote such an algebra. Then we say that a set in E is A-bounding if all functions in $A(E)$ are bounded on it.

A main theme in this paper is the close interplay between the homomorphisms on $A(E)$ and the A-bounding sets. Using appropriate properties of the homomorphisms on the algebra $\mathcal{R}(E)$ of rational forms of elements in $\mathcal{P}(E)$, we deduce that the $\mathcal{R}$-bounding sets are always relatively compact in the weak topology of the Banach space E. With this result at hand, we show that the problem of the $\mathcal{A}$-bounding sets being relatively compact in arbitrary Banach spaces E is reduced to the study of the behaviour of the real analytic functions on l_∞ acting on the set of unit vectors in l_∞. Further we show that the $\mathcal{R}$-bounding and the relatively norm compact

1991 *Mathematics Subject Classification*: 46E25, 46J10.
Research of the second author partially supported by DGICYT grant PB-0044 (Spain).

sets coincide in super-reflexive Banach spaces. We also show that for each infinite-dimensional Banach space E there are a homomorphism ϕ on $\mathcal{P}(E)$ and a polynomial $p \in \mathcal{P}(E)$ such that $\phi(p) \notin p(E)$, which improves a result of [12]. In the same spirit we obtain an example of an inverse-closed algebra that fails to be sequentially evaluating.

Another algebra studied in this context is the subalgebra $\mathcal{AE}(E)$ of $\mathcal{A}(E)$ consisting of the convergent power series in E. Since the complexification of $\mathcal{AE}(E)$ is the algebra $H(\widetilde{E})$ of holomorphic functions on the complexification $\widetilde{E}$ of E, the characterization of holomorphic bounding sets in l_∞ due to Dineen and Josefson gives us information about the homomorphisms on $\mathcal{AE}(E)$. For the algebra $\mathcal{RAE}(E)$ of rational forms of functions in $\mathcal{AE}(E)$ we conclude that the set of unit vectors in l_∞ is $\mathcal{RAE}$-bounding and that $\mathcal{RAE}(E)$ is sequentially evaluating.

1. Preliminaries. In this paper we assume the existence of a mapping $E \mapsto A(E) \subset C(E)$ from the set of all real Banach spaces to the set of real function algebras of continuous real functions. We require that the duals E' are contained in $A(E)$ and also that for any Banach spaces E and F and continuous affine maps $T : E \to F$, we have $f \circ T \in A(E)$ whenever $f \in A(F)$. Let $\operatorname{Hom} A(E)$ denote the set of all non-zero real homomorphisms on $A(E)$. We say that $A(E)$ is *single-set evaluating* if for each $\phi \in \operatorname{Hom} A(E)$ and every $f \in A(E)$ we have $\phi(f) \in f(E)$. If $A(E)$ is single-set evaluating, then for each $\phi \in \operatorname{Hom} A(E)$ and finite set $\{f_1, \ldots, f_n\}$ in $A(E)$, there is a point $a \in E$ such that $\phi(f_i) = f_i(a)$ for all $i = 1, \ldots, n$. It is also clear that every inverse-closed algebra $A(E)$, i.e. one with $1/f \in A(E)$ whenever $f \in A(E)$ and $f(x) \neq 0$ for all $x \in E$, is single-set evaluating. An algebra $A(E)$ is *sequentially evaluating* if for each $\phi \in \operatorname{Hom} A(E)$ and each sequence (f_n) in $A(E)$ there is a point $a \in E$ with $\phi(f_n) = f_n(a)$ for all n.

We denote by $\mathcal{P}_{\mathrm{f}}(E)$ the algebra of all continuous polynomials of finite type on E; that is, the algebra generated by E'. The space of all continuous n-homogeneous polynomials on E is denoted by $\mathcal{P}(^nE)$. Let $\mathcal{P}(E)$ denote the algebra of all continuous polynomials on E and denote by $\mathcal{R}(E)$ the algebra of all rational functions on E, i.e. functions of the form p/q, where $p, q \in \mathcal{P}(E)$ and $q(x) \neq 0$ for every $x \in E$. A function $f : E \to \mathbb{R}$ is said to be *real analytic* on E if it can be represented in a neighbourhood of each point in E by a convergent power series. The algebra $\mathcal{A}(E)$ of all real analytic functions on E is a proper vector subspace of the algebra $C^\infty(E)$ of all infinitely differentiable real-valued functions on E in the continuous Fréchet sense. Note that $\mathcal{R}(E) \subset \mathcal{A}(E)$ and that they are inverse-closed algebras.

Let $\mathcal{AE}(E)$ denote the set of all functions $f : E \to \mathbb{R}$ such that there exists a sequence $(p_n) \in \mathcal{P}(^nE)$ with $f(x) = \sum_{n\in\mathbb{N}} p_n(x)$ for $x \in E$. Then

$\mathcal{AE}(E)$ is an algebra with $\mathcal{P}(E) \subset \mathcal{AE}(E) \subset \mathcal{A}(E)$, where the last inclusion follows from Theorem 5.2 of [4].

We denote by $\mathcal{R}A(E)$ the smallest inverse-closed algebra which contains $A(E)$. Hence every element in $\mathcal{R}A(E)$ is of the form f/g, where $f, g \in A(E)$ and $g(x) \neq 0$ for all $x \in E$. Certainly $\mathcal{R}(E) = \mathcal{RP}(E)$.

EXAMPLE 1. For each Banach space E the algebra $\mathcal{RAE}(E)$ is a proper subalgebra of $\mathcal{A}(E)$. Indeed, if $f \in \mathcal{AE}(\mathbb{R})$, then $f(x) = \sum_{n=0}^{\infty} \alpha_n x^n$ extends to an entire function $\widetilde{f}(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ on $\mathbb{C}$. Now, if $f, g \in \mathcal{AE}(\mathbb{R})$ and $g(x) \neq 0$ on $\mathbb{R}$ we know that $\widetilde{g}$ has only countably many zeros in $\mathbb{C}$. Then f/g extends to a meromorphic complex function $\widetilde{f}/\widetilde{g}$ with only countably many singularities. Consider, for instance, $h(x) = \log(1 + x^2)$; then $h \in \mathcal{A}(\mathbb{R})$. But every complex extension of h has an uncountable number of singularities in $\mathbb{C}$ and therefore $h \notin \mathcal{RAE}(\mathbb{R})$. By composing with elements in the dual E' this process extends to every Banach space E.

2. Bounding sets. Let $A(E)$ be an algebra on a Banach space E. A subset B of E is said to be *A-bounding* if $\sup_{x\in B} |f(x)| < \infty$ for all $f \in A(E)$.

Remark. The definition above certainly makes sense also if $A(E)$ is merely a class of functions on E and not necessarily an algebra. However, an A-bounding set is also bounding with respect to the functions in the algebra generated by $A(E)$. Therefore it is convenient to stick to algebras $A(E)$ in the definition of A-bounding sets in E.

Since we always require the algebra $A(E)$ to contain the dual E' and $A(E)$ is contained in $C(E)$, every relatively compact set is A-bounding and every A-bounding set is bounded. Let $E_{A(E)}$ be the set E endowed with the weakest topology making all $f \in A(E)$ continuous. One of the motivations for studying A-bounding sets is the fact that if the A-bounding sets in E are relatively compact, then E and $E_{A(E)}$ have the same convergent sequences; that is, $x_n \to x$ in E if and only if $f(x_n) \to f(x)$ for all $f \in A(E)$.

The following concept, originating from Grothendieck, provides a most efficient tool for investigating A-bounding sets. We say that two sets $B \subset E$ and $M \subset \mathbb{R}^E$ have the *interchangeable double limit property* if for every pair of sequences (f_m) in M and (x_n) in B the double limits $\lim_n \lim_m f_m(x_n)$ and $\lim_m \lim_n f_m(x_n)$ are equal, provided that all involved limits exist.

THEOREM 1. *Let E be a Banach space. Then every $\mathcal{R}$-bounding subset of E is relatively $\sigma(E, E')$-compact.*

Proof. Let B be an $\mathcal{R}$-bounding subset of E. We first claim that B has the interchangeable double limit property with the polar B_E° of the unit ball B_E. The polar B_E° is $\sigma(E', E)$-compact by the Alaoglu–Bourbaki theorem.

Take sequences (x_n) in B and (l_m) in B_E°. By Tikhonov's theorem B is relatively compact in $\operatorname{Hom}\mathcal{R}(E)$, where $\operatorname{Hom}\mathcal{R}(E)$ is embedded in $\mathbb{R}^{\mathcal{R}(E)}$ as a closed subspace. Let $\phi \in \operatorname{Hom}\mathcal{R}(E)$ and $l \in E'$ be cluster points of the sequences (x_n) and (l_m) respectively. Choose a sequence $(\alpha_m) \in \mathbb{R}^+$ such that the sums

$$f(x) = \sum_{m=1}^{\infty} \alpha_m (l_m(x) - \phi(l_m))^2 \quad \text{and} \quad g(x) = \sum_{m=1}^{\infty} \frac{\alpha_m}{m}(l_m(x) - \phi(l_m))^2$$

are pointwise convergent and therefore belong to $\mathcal{P}(E)$ by the Banach–Steinhaus theorem. Since $\mathcal{R}(E)$ is single-set evaluating, for given $N \in \mathbb{N}$, there exists $a \in E$ such that $\phi(f) = f(a)$, $\phi(g) = g(a)$, $\phi(l) = l(a)$ and $\phi(l_i) = l_i(a)$ for $i = 1, \ldots, N$ (consider the function $(f - \phi(f))^2 + (g - \phi(g))^2 + (l - \phi(l))^2 + (l_1 - \phi(l_1))^2 + \ldots + (l_N - \phi(l_N))^2$ in $\mathcal{R}(E)$). Then

$$\phi(f) = \sum_{m=N+1}^{\infty} \alpha_m (l_m(a) - \phi(l_m))^2 \quad \text{and} \quad \phi(g) = \sum_{m=N+1}^{\infty} \frac{\alpha_m}{m}(l_m(a) - \phi(l_m))^2.$$

Therefore $0 \le N\phi(g) \le \phi(f)$, and hence $\phi(l_m) = l_m(a)$ for each m and $\phi(l) = l(a)$. If all limits involved exist, then

$$\begin{aligned}\lim_m \lim_n l_m(x_n) &= \lim_m \phi(l_m) = \lim_m l_m(a) = l(a) = \phi(l)\\ &= \lim_n l(x_n) = \lim_n \lim_m l_m(x_n),\end{aligned}$$

and our claim is proved.

By the Eberlein–Grothendieck theorem [11, p. 15] a bounded subset of a Banach space E is relatively $\sigma(E, E')$-compact if and only if it has the interchangable double limit property with the polar B_E° of B_E. Therefore B is relatively weakly compact, and the theorem is proved.

Corollary 2. *Let E be a Banach space. If E has the Dunford–Pettis property, then the $\mathcal{R}$-bounding and the relatively $\sigma(E, E')$-compact subsets of E coincide.*

Proof. Let $B \subset E$ be weakly compact and assume that E has the Dunford–Pettis property. Then, by Theorem 7.1 of [6], the restriction of any continuous polynomial on E to a weakly compact set is weakly continuous. Therefore $\inf_{x\in B} |q(x)| > 0$ for every $q \in \mathcal{P}(E)$ such that $q(x) \neq 0$ for each $x \in E$. Hence, every rational function on E is bounded on B.

Examples of Banach spaces with the Dunford–Pettis property are $C(K)$ for any compact K and $L^1(\mu)$. This class is also closed under formation of preduals, if they exist, and hence the important Banach spaces $c_0(\Gamma)$, $l_1(\Gamma)$ and $l_\infty(\Gamma)$ all have this property.

It follows from Theorem 1 that for Banach spaces E with the Schur property, every $\mathcal{R}$-bounding set is relatively compact in E. In our next theorem we show that this is also true if E is a super-reflexive Banach space.

By Theorem 2.4 of [13], $E = \operatorname{Hom}\mathcal{R}(E)$ for E separable. Therefore every $\mathcal{R}$-bounding set in E is relatively compact in $E_{\mathcal{P}(E)}$. The weak topology is angelic and any finer regular topology on E is also angelic [11, p. 31]. Hence the regular topology of $E_{\mathcal{P}(E)}$ is angelic. This means that each $\mathcal{R}$-bounding set in E is relatively sequentially compact in $E_{\mathcal{P}(E)}$. If we assume, in addition, that E is a Λ-space [6] (i.e. weak-polynomial convergence for sequences implies norm convergence, since a sequence (x_n) in E is weak-polynomial convergent to x if and only if $p(x_n - x) \to p(0)$ for every $p \in \mathcal{P}(E)$), then we may state

Lemma 3. *In separable Λ-spaces all the $\mathcal{R}$-bounding sets are relatively compact.*

Theorem 6.3 of [6] says that the separable space l_p is a Λ-space for $1 \le p < \infty$.

In [7] a Banach space E is defined to be in the class $\mathcal{W}_p$ $(1 < p < \infty)$ whenever for each bounded sequence (x_n) in E there exist a $x \in E$ and a subsequence (x_{n_k}) such that $\sum_{k=1}^{\infty} |l(x_{n_k} - x)|^p < \infty$ for all $l \in E'$. Using a convenient characterization of super-reflexivity Castillo and Sanchez proved in [7] that every super-reflexive Banach space is in the class $\mathcal{W}_p$ for some p $(1 < p < \infty)$. Recall that a Banach space is super-reflexive if and only if its dual is super-reflexive. The spaces $L^p(\mu)$ are super-reflexive for $1 < p < \infty$ and any measure μ.

Theorem 4. *Assume that E' is in the class $\mathcal{W}_p$ for some p $(1 < p < \infty)$ (e.g. that E is super-reflexive). Then every $\mathcal{R}$-bounding set is relatively compact in E.*

Proof. Assume that there is an $\mathcal{R}$-bounding set in E that is not relatively compact. Then this set is not precompact and therefore there is in E an $\mathcal{R}$-bounding sequence (x_n) with $\|x_n - x_m\| > 2\varepsilon$ for some $\varepsilon > 0$ whenever $n \neq m$. By passing to subsequences if necessary, we can assume that (x_n) converges weakly to some $x \in E$, that $\|x_n - x\| > \varepsilon$ for all n, and that $(x_n - x)$ is $\mathcal{R}$-bounding in E. By Bessaga–Pełczyński's selection principle [10] we can find a basic subsequence (y_n) of the sequence $(x_n - x)$. Let F denote the closed subspace of E spanned by (y_n). Every $y \in F$ can be represented as $y = \sum_{n=1}^{\infty} l_n(y) y_n$, where the sequence $(l_n) \subset F'$ is biorthogonal to (y_n). Since the sum y converges and $\|y_n\| > \varepsilon$ it follows that $l_n(y) \to 0$ for each $y \in F$. Hence (l_n) is a bounded sequence by the Principle of Uniform Boundedness. Each l_n can be extended to a functional $\widehat{l}_n \in E'$ with the same norm, and therefore $(\widehat{l}_n)$ is a bounded sequence in E'. By the assumption there exist some p $(1 < p < \infty)$, some $l \in E'$ and a subsequence $(\widehat{l}_{n_k}) \subset E'$ such that

$$\sum_{k=1}^{\infty} |(\widehat{l}_{n_k} - l)(x)|^p < \infty \quad \text{for all } x \in E.$$

Since $(\widehat{l}_{n_k} - l)(x) \to 0$ for each $x \in E$ and $l_n(y) \to 0$ for each $y \in F$, we see that $l(y) = 0$ for all $y \in F$. We also obtain a well-defined linear map $T : E \to l_p$, $x \mapsto ((\widehat{l}_{n_k} - l)(x))_k$. The operator T is continuous according to the Banach–Steinhaus theorem. Since $(T(y_n))$ is relatively compact in l_p by Lemma 3, and

$$\|T(y_{n_j})\| = \Big(\sum_{k=1}^{\infty} |(l_{n_k} - l)(y_{n_j})|^p\Big)^{1/p} = 1,$$

we have obtained a contradiction.

Remark. For the slightly smaller class of super-reflexive Banach spaces E of non-measurable cardinality there is another way of obtaining the result in Theorem 4. Indeed, by Theorem 2.4 of [13], $E = \operatorname{Hom} \mathcal{R}(E)$ for these Banach spaces E. Since every super-reflexive space is a Λ-space by Theorem 1 of [15], the result follows in the same way as in the proof of Lemma 3.

A subset $B \subset E$ is said to be *limited* if each $\sigma(E', E)$-null sequence (l_n) converges to zero uniformly on B. This concept translates to the language of bounding sets. Indeed, if $W^*(E) = \{\sum_{n=1}^{\infty} (l_n)^n : l_n \to 0 \text{ in } \sigma(E', E)\}$ then obviously $B \subset E$ is limited if and only if B is W^*-bounding. Furthermore, since $W^*(E) \subset \mathcal{AE}(E) \subset \mathcal{A}(E)$ we have the relation

$$\mathcal{A}\text{-bounding} \Rightarrow \mathcal{AE}\text{-bounding} \Rightarrow \text{limited}.$$

The limited and the relatively compact sets in E coincide when E is isomorphic to a subspace of $C(K)$, where K is a compact, sequentially compact Hausdorff space [10]. All WCG spaces have this property as well as every weak Asplund space. Hence, for large classes of Banach spaces E every limited set is $\mathcal{R}$-bounding. Note also that Bourgain and Diestel [5] proved that in Banach spaces with no copy of l_1 limited sets are relatively weakly compact.

In [23] Schlumprecht has constructed a complex Banach space E which contains a subset that is limited in E but not bounding with respect to holomorphic functions on E. A close examination of this example shows that in the real case we obtain a Banach space E such that

$$\text{limited} \not\Rightarrow \mathcal{AE}\text{-bounding}.$$

The following examples give some information about the difference between limited and $\mathcal{R}$-bounding sets in certain Banach spaces E.

The original Tsirelson space T' is a reflexive space with an unconditional basis such that every continuous polynomial on T' is automatically weakly continuous on weakly compact subsets of T' (see [1]). Thus we can find an $\mathcal{R}$-bounding set in T' that is not limited. Also, since c_0 has the Dunford–Pettis property, and, since there are non-compact but weakly compact sets in c_0, already c_0 provides examples of $\mathcal{R}$-bounding subsets that are not limited. Hence in general

$$\mathcal{R}\text{-bounding} \not\Rightarrow \text{limited}.$$

It is also possible to find a Banach space E and a limited subset that is not $\mathcal{R}$-bounding. Indeed, by Phillips' lemma the closed unit ball B_{c_0} of c_0 viewed in l_∞ is limited. By Corollary 2, the subsets of l_∞ are $\mathcal{R}$-bounding if and only if they are relatively weakly compact. Hence $B_{c_0} \subset l_\infty$ is a limited set that is not $\mathcal{R}$-bounding. Thus also

$$\text{limited} \not\Rightarrow \mathcal{R}\text{-bounding}.$$

Our next theorem shows that the set of unit vectors in l^∞ is, in fact, a sufficient set for testing whether the A-bounding sets in an arbitrary Banach space are always relatively compact or not.

THEOREM 5. *Let $A(E)$ be an algebra on a Banach space E that contains the algebra $\mathcal{R}(E)$. Then each A-bounding set is relatively compact in E if there exists some function in $A(l_\infty)$ that is unbounded on the set of unit vectors in l_∞.*

Proof. Suppose, contrary to the statement, that there exists some function $f \in A(l_\infty)$ with $\sup_n |f(e_n)| = \infty$ and some A-bounding set B in a Banach space E that is not relatively compact. Thus B is not precompact but, by Theorem 1, relatively weakly compact. As in the proof of Theorem 4, we therefore find an A-bounding sequence (x_n) in E that converges weakly to zero such that $\|x_n\| > \varepsilon$ for all n and for some $\varepsilon > 0$. Again, using the theorem of Bessaga–Pełczyński, we obtain a subsequence (y_n) of (x_n) which is a basic sequence. Set $F = \overline{[y_n : n \in \mathbb{N}]}$ and let $(l_n) \subset F'$ be the bounded biorthogonal sequence associated with (y_n). By the Hahn–Banach theorem there is an extension $\widehat{l}_n$ of each l_n to E with $\|\widehat{l}_n\| = \|l_n\|$. Define a mapping $T : E \to l_\infty$ given by $T(x) = (\widehat{l}_n(x))_n$. It is obvious that T is well defined, continuous and linear. Furthermore, $T(y_n) = e_n$ for each n. Thus the composed map $f \circ T \in A(E)$ satisfies

$$\sup_n |(f \circ T)(y_n)| = \sup_n |f(e_n)| = \infty,$$

which is a contradiction since (y_n) is A-bounding.

Theorem 5 is directly applicable to the algebra $C^\infty(E)$ of all C^∞-functions on E in the usual Fréchet sense. Indeed, the function $f : l^\infty \to \mathbb{R}$

which assigns to an arbitrary point $x = (x_1, x_2, \ldots) \in l^\infty$ the value

$$f(x) = \eta(x_1) + 2\mu(x_1)\eta(x_2) + 3\mu(x_1)\mu(x_2)\eta(x_3) + \ldots$$
$$\ldots + k\mu(x_1)\mu(x_2)\ldots\mu(x_{k-1})\eta(x_k) + \ldots,$$

where η and μ are non-negative C^∞-functions on $\mathbb{R}$ such that

$$\eta(t) = \begin{cases} 1 & \text{for } t = 1, \\ 0 & \text{for } t \le \frac{3}{4}, \end{cases} \qquad \mu(t) = \begin{cases} 1 & \text{for } t = 0, \\ 0 & \text{for } t \ge \frac{1}{4}, \end{cases}$$

is locally finite and thus an element in $C^\infty(l^\infty)$. By construction, $f(e_n) = n$ for all n. Therefore we have

COROLLARY 6. *In every Banach space all the C^∞-bounding sets are relatively compact.*

With the use of complexification and the deep results for holomorphically bounding sets in the complex Banach space l_∞ due to Dineen and Josefson we next show that the functions in the algebra $\mathcal{RAE}(l_\infty)$ are all bounded on the set of unit vectors in the real space l_∞. In particular, we show that there is an $\mathcal{AE}$-bounding subset of l_∞ that is not relatively compact. To prove this statement we need the next lemma that follows from Proposition 8.2 (4°) of [4].

LEMMA 7. *Let E be a Banach space and let $\widetilde{E}$ be the complexification of E. Then every $f \in \mathcal{AE}(E)$ may be extended to a holomorphic function $\widetilde{f} \in H(\widetilde{E})$.*

Josefson [18] showed that each function in $H(\widetilde{l}_\infty)$ is bounded on each bounded set contained in $\widetilde{c}_0$. Hence, if $\widetilde{f} = \sum_{n=0}^\infty \widetilde{p}_n$ is the extended holomorphic function of $f = \sum_{n=0}^\infty p_n \in \mathcal{AE}(l_\infty)$, the restriction $\widetilde{f}|_{\widetilde{c}_0}$ is a holomorphic function of bounded type on $\widetilde{c}_0$. Thus its Taylor series at the origin $\sum_{n=0}^\infty \widetilde{p}_n$ converges uniformly on each bounded subset of $\widetilde{c}_0$. When restricted to the bounded subsets of c_0, we therefore get the following result.

PROPOSITION 8. *Every function $f = \sum_{n=0}^\infty p_n \in \mathcal{AE}(l_\infty)$ converges uniformly on each bounded subset of c_0. In particular, every bounded set in c_0 is an $\mathcal{AE}$-bounding subset of l_∞.*

By means of the Phillips lemma, there is, as stated before, a limited set in l_∞ that is not $\mathcal{R}$-bounding. With Proposition 8 at hand, this result is sharpened to the subclass of $\mathcal{AE}$-bounding sets; that is,

$$\mathcal{AE}\text{-bounding} \not\Rightarrow \mathcal{R}\text{-bounding}.$$

This is, however, not an improvement, since a closer investigation of the limited and the $\mathcal{AE}$-bounding sets in l_∞ shows that they coincide. Indeed, let $B \subset l_\infty$ be a limited set. Since l_∞ is continuously embedded in $\widetilde{l}_\infty$, B is limited in $\widetilde{l}_\infty$. By Theorem 1 of [18] every limited set in $\widetilde{l}_\infty$ is bounding with respect to the holomorphic functions on $\widetilde{l}_\infty$. Therefore the statement follows from Lemma 7.

COROLLARY 9. *Every weakly compact set in c_0 is $\mathcal{RAE}$-bounding in l^∞, in particular, the unit vectors $\{e_n : n \in \mathbb{N}\}$ form an $\mathcal{RAE}$-bounding set in l_∞.*

Proof. Let K be a weakly compact subset of c_0 and let $\varepsilon > 0$. Take $f \in \mathcal{AE}(l_\infty)$. For some N, we have $\sup_{x\in K} |f(x) - \sum_{n=0}^N p_n(x)| < \varepsilon/3$ by Proposition 8. Put $p = \sum_{n=0}^N p_n$; then $p \in \mathcal{P}(l_\infty)$. Since l_∞ has the Dunford–Pettis property it follows from Theorem 7.1 of [6] that $p|_K$ is $\sigma(l_\infty, (l_\infty)')$-continuous. Then so is $f|_K$. Indeed, since K is $\sigma(l_\infty, (l_\infty)')$-metrizable, it suffices to show that $f|_K$ is sequentially continuous. Let $x_k \to x$ weakly in K. Then there is a $k_0 \in \mathbb{N}$ such that

$$|f(x_k) - f(x)| \le |f(x_k) - p(x_k)| + |p(x_k) - p(x)| + |p(x) - f(x)| < \varepsilon$$

for $k \ge k_0$. Hence $f(x_k) \to f(x)$, and thus $f|_K$ is weakly continuous. Now the statement follows by the same argument as in the proof of Corollary 2.

Remark. The set B_{c_0} cannot be $\mathcal{RAE}$-bounding in l_∞, since it is not even $\mathcal{R}$-bounding.

3. Evaluating properties of homomorphisms. The principal interest in this section is to investigate the evaluating properties of homomorphisms, such as the single-set and sequential evaluating properties as well as their complete reduction to point evaluations, defined on the algebras $\mathcal{P}_f(E)$, $\mathcal{RP}_f(E)$, $\mathcal{P}(E)$, $\mathcal{R}(E)$, $\mathcal{AE}(E)$, $\mathcal{RAE}(E)$ and $\mathcal{A}(E)$ for various classes of Banach spaces E.

Although the algebra $\mathcal{AE}(\mathbb{R})$ is not inverse-closed, it is single-set evaluating. In fact, every homomorphism on $\mathcal{AE}(\mathbb{R})$ is even a point evaluation. Indeed, let $\phi \in \operatorname{Hom} \mathcal{AE}(\mathbb{R})$ and take $f \in \mathcal{AE}(\mathbb{R})$. For the function $p \in \mathcal{AE}(\mathbb{R})$, where $p(x) = x$ for all $x \in \mathbb{R}$, set $\alpha = \phi(p)$. Expand f in a Taylor series at α. Then $f(x) = f(\alpha) + (x - \alpha)g(x)$, where $g \in \mathcal{AE}(\mathbb{R})$. Hence $\phi(f) = f(\alpha) + 0 \cdot \phi(g) = f(\alpha)$.

In the next example we consider the algebra $\mathcal{AE}_b(E)$ of all functions $f = \sum_{n\in\mathbb{N}} p_n \in \mathcal{AE}(E)$ such that $\|p_n\|^{1/n} \to 0$ as $n \to \infty$. By taking into account the Josefson–Nissenzweig theorem [10], we observe that $\mathcal{AE}(E) = \mathcal{AE}_b(E)$ precisely when E is finite-dimensional. The complex analog of $\mathcal{AE}_b(E)$ is the algebra of all holomorphic functions of bounded type. Using appropriate continuity properties of its homomorphisms, we deduce that they are point evaluations for certain Banach spaces E.

EXAMPLE 2. Let E be a reflexive Banach space such that $\mathcal{P}_f(^nE)$ is dense in $\mathcal{P}(^nE)$ with respect to the norm topology for every n. It is easily seen that

$$\mathcal{AE}_{\rm b}(E)=\Big\{f=\sum_{n=0}^{\infty}p_n\in\mathcal{AE}(E):\sum_{n=0}^{\infty}\sup_{\|x\|\le k}|p_n(x)|<\infty \text{ for } k=1,2,\ldots\Big\}$$

with $\mathcal{P}(E)\subset\mathcal{AE}_{\rm b}(E)\subset\mathcal{AE}(E)$ and that every function in $\mathcal{AE}_{\rm b}(E)$ is bounded on bounded subsets of E. The space $\mathcal{AE}_{\rm b}(E)$ endowed with the topology generated by the norms $(\|\cdot\|_k)_{k=1}^{\infty}$, where $\|f\|_k=\sum_{n=0}^{\infty}\|p_n\|k^n$, is a Fréchet algebra. Since every homomorphism on a real Fréchet algebra is continuous [14], any $\phi\in\operatorname{Hom}\mathcal{AE}_{\rm b}(E)$ is a point evaluation on $\mathcal{P}_{\rm f}(E)$ by reflexivity of E and hence also on $\mathcal{AE}_{\rm b}(E)$ by density.

The assumptions on E in Example 2 are satisfied e.g. when E is the original Tsirelson space T' or a finite-dimensional space.

Proposition 10. *Let $\phi\in\operatorname{Hom}\mathcal{R}(E)$ and let (p_n) be a sequence of polynomials on a Banach space E with uniformly bounded degrees. Then there is a point $a\in E$ such that $\phi(p_n)=p_n(a)$ for all n.*

Proof. Choose a sequence (α_n) of positive reals such that the sums

$$f(x)=\sum_{n=1}^{\infty}\alpha_n(p_n(x)-\phi(p_n))^2\quad\text{and}\quad g(x)=\sum_{n=1}^{\infty}\frac{\alpha_n}{n}(p_n(x)-\phi(p_n))^2$$

are pointwise convergent with respect to each series of k-homogeneous terms for fixed k separately. Then f and g are continuous polynomials on E by the Banach–Steinhaus theorem. The rest of the proof can now be carried out in the same way as in the proof of Theorem 1.

Theorem 11. *Let $A(E)$ be a single-set evaluating algebra containing $\mathcal{AE}(E)$. Let $\phi\in\operatorname{Hom}A(E)$, let $f\in A(E)$ and let (f_n) be a sequence in $\mathcal{AE}(E)$. Then there is a point $a\in E$ such that $\phi(f)=f(a)$, $\phi(f_n)=f_n(a)$ for all n. In particular, the restriction $\phi|_{\mathcal{AE}(E)}$ is sequentially $\mathcal{AE}_s(E)$-continuous.*

Proof. Let (p_n) be a sequence in $\bigcup_{k=1}^{\infty}\mathcal{P}(^kE)$. Denote by $d(p_n)$ the degree of the polynomial p_n. Take a sequence (k_n) of odd natural numbers with $k_1=1$ and $k_{n+1}>2k_nd(p_n)$ for $n=1,2,\ldots$ Then

$$|p_n^{k_n}(x)|\le\|p_n\|^{k_n}\|x\|^{k_nd(p_n)}\quad\text{for every } x\in E.$$

Set

$$g(x)=\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\cdot\frac{1}{2^n}\cdot\frac{1}{n^{2k_nd(p_n)}}(p_n^{k_n}(x)-\phi(p_n^{k_n}))^2,$$

where (α_n) is a sequence of reals with

$$\alpha_n>\|p_n\|^{2k_n}+2|\phi(p_n^{k_n})|\|p_n\|^{k_n}+\phi(p_n^{2k_n})\quad\text{for all } n.$$

Then

$$\begin{aligned}g(x)&\le\sum_{n=1}^{\infty}\frac{1}{2^n}\cdot\frac{1}{n^{2k_nd(p_n)}}(\|x\|^{2k_nd(p_n)n}+\|x\|^{k_nd(p_n)}+1)\\&\le\sum_{n=1}^{\infty}\frac{1}{2^n}\bigg(\Big\|\frac{x}{n}\Big\|^{2k_nd(p_n)}+\Big\|\frac{x}{n}\Big\|^{k_nd(p_n)}+1\bigg)<\infty\quad\text{for all } x\in E.\end{aligned}$$

Since g is pointwise convergent, it is by construction a function in $\mathcal{AE}(E)$. By using the same technique as in the proof of Proposition 10, we find that there is some $x\in E$ with $\phi(f)=f(x)$ and $\phi(p_n^{k_n})=p_n^{k_n}(x)$ for all n. As each k_n is odd, it follows that $\phi(p_n)=p_n(x)$ for all n. Since each f_k is a polynomial sum $\sum p_{n,k}$, there is a point $x\in E$ with $\phi(f_k)=f_k(x)$ and $\phi(p_{n,k})=p_{n,k}(x)$ for all n, hence $\phi(f_k)=\sum\phi(p_{n,k})$. Let $a\in E$ with $\phi(f)=f(a)$ and $\phi(p_{n,k})=p_{n,k}(a)$ for all n and k. Then

$$\phi(f_k)=\sum_n\phi(p_{n,k})=\sum_n p_{n,k}(a)=f_k(a)\quad\text{for all } k,$$

and the statement is proved.

Corollary 12. *For every Banach space E the inverse-closed algebra $\mathcal{RAE}(E)$ is sequentially evaluating.*

For (E_n) a sequence of Banach spaces, consider the Banach spaces $(\bigoplus_n E_n)_{l_p}$ $(1\le p\le\infty)$ and $(\bigoplus_n E_n)_{c_0}$ [16, p. 374].

Proposition 13. *Let (E_n) be a sequence of Banach spaces such that each E_n admits a continuous linear injection into $l_q(\Gamma)$ for some $q<\infty$ and some set Γ of non-measurable cardinality. If E equals $(\bigoplus_n E_n)_{l_p}$ or $(\bigoplus_n E_n)_{c_0}$, then $E=\operatorname{Hom}\mathcal{RAE}(E)$.*

Proof. Since each E_n injects into $l_q(\Gamma)$, $E_n=\operatorname{Hom}\mathcal{RAE}(E_n)$ by Theorem 2.4 of [13]. In view of the injection into $l_q(\Gamma)$ for each $x_n\in E_n$ there is a continuous polynomial p_{x_n} on E_n separating x_n from any other point in E_n. According to Theorem 9 of [19], it follows that $\prod_{n=1}^{\infty}E_n=\operatorname{Hom}\mathcal{RAE}(\prod_{n=1}^{\infty}E_n)$. Take $\phi\in\operatorname{Hom}\mathcal{RAE}(E)$. Let I denote the natural injection from E into $\prod_{n=1}^{\infty}E_n$. Since the map $\psi:\prod_{n=1}^{\infty}E_n\to\mathbb{R}$, where $\psi(g)=\phi(g\circ I)$, is a homomorphism, there is a unique point $a=(a_n)$ in $\prod_{n=1}^{\infty}E_n$ with $\psi(g)=g(a)$ for each $g\in\mathcal{RAE}(\prod_{n=1}^{\infty}E_n)$. If $\operatorname{pr}_n:\prod_{n=1}^{\infty}E_n\to E_n$ are the natural projections, the sequence $(p_{a_n}\circ\operatorname{pr}_n)$ in $\mathcal{RAE}(\prod_{n=1}^{\infty}E_n)$ separates a from any other point in $\prod_{n=1}^{\infty}E_n$. Take $f\in\mathcal{RAE}(E)$. By Corollary 12, there is a point $x\in E$ such that $\phi(f)=f(x)$ and $\phi(p_{a_n}\circ\operatorname{pr}_n\circ I)=p_{a_n}(\operatorname{pr}_n(x))$ for all n. Hence $p_{a_n}(\operatorname{pr}_n(x))=p_{a_n}(\operatorname{pr}_nn(a))$ for each n, which gives $x=a$ and the proof is complete.

Remark. Examples of Banach spaces admitting a continuous linear injection into some $l_q(\Gamma)$ include all super-reflexive spaces [17] as well as those with weak*-separable duals.

According to Theorem 7.1 of [6], the restriction of any $p \in \mathcal{P}(E)$ to a weakly compact set is weakly continuous if E has the Dunford–Pettis property and, consequently, p is sequentially weakly continuous. By Theorems 4.4.7 and 4.5.9 of [20], p is then weakly uniformly continuous on bounded sets if E, in addition, does not contain a copy of l_1. As a consequence of Theorem 4.3.7 of [20], we therefore get the following:

LEMMA 14. *Let E be a Banach space with the Dunford–Pettis property that does not contain a copy of l_1. Then $\mathcal{P}_f(E)$ is dense in $\mathcal{P}(E)$ with respect to the topology of uniform convergence on bounded sets in E.*

THEOREM 15. *Let E be a weakly Lindelöf Banach space not containing a copy of l_1 with the Dunford–Pettis property. Then*

$$E = \operatorname{Hom}\mathcal{R}(E) = \operatorname{Hom}\mathcal{RAE}(E).$$

Proof. Let $\phi : \mathcal{R}A \to \mathbb{R}$ be a homomorphism, where A is either $\mathcal{P}(E)$ or $\mathcal{AE}(E)$. Since E is weakly Lindelöf and every continuous finite type polynomial on E is weakly continuous, there is, according to Proposition 10 and Theorem 11, a point $a \in E$ such that $\phi(p) = p(a)$ for all $p \in \mathcal{P}_f(E)$. Take an arbitrary $f \in A$. By Lemma 14, there is a sequence (p_n) in $\mathcal{P}_f(E)$ converging pointwise to f. Again, by Proposition 10 and Theorem 11, there is some $x \in E$ such that $\phi(f) = f(x)$ and $\phi(p_n) = p_n(x)$ for all n. Since $\lim_n p_n(x) = f(x)$ and $\lim_n p_n(a) = f(a)$, obviously $\phi(f) = f(a)$.

Examples of spaces E satisfying the premises in Theorem 15 are $c_0(\Gamma)$, $C(K)$ for any scattered Corson compact space K [21] and the Ciesielski–Pol space X_0 of $C(K)$-type that is not even injectable into $c_0(\Gamma)$ [8]. The space $c_0(\Gamma)$ is special in the sense that $c_0(\Gamma) = \operatorname{Hom}\mathcal{A}(c_0(\Gamma))$ [13].

LEMMA 16. *Let $A(E)$ be a single-set evaluating algebra. Then every homomorphism on $A(E)$ extends to $\mathcal{R}A(E)$.*

Proof. Let $\phi : A(E) \to \mathbb{R}$ be a homomorphism and take $g \in \mathcal{R}A(E)$. The function g is of the form f_1/f_2, where $f_1, f_2 \in A(E)$. Then the map $\psi : g \mapsto \phi(f_1)/\phi(f_2)$ is the obvious extension of ϕ to $\mathcal{R}A(E)$. Since $A(E)$ is single-set evaluating, there is a point $x \in E$ such that $\phi(f_i) = f_i(x)$ for $i = 1, 2$, and hence the map ψ is well-defined.

PROPOSITION 17. *The algebra $\mathcal{AE}(l_\infty)$ is not single-set evaluating.*

Proof. Suppose that $\mathcal{AE}(l_\infty)$ is a single-set evaluating algebra. Then $\operatorname{Hom}\mathcal{AE}(l_\infty) = \operatorname{Hom}\mathcal{RAE}(l_\infty)$ by Lemma 16, and therefore $\mathcal{AE}(l_\infty)$ is sequentially evaluating according to Corollary 12. Since $(l_\infty)'$ admits a point separating sequence, $l_\infty = \operatorname{Hom}\mathcal{AE}(l_\infty)$. Hence every $\mathcal{AE}$-bounding subset of l_∞ is relatively compact in $l_{\infty\mathcal{AE}(l_\infty)}$, in particular relatively compact in the $\sigma(l_\infty, (l_\infty)')$-topology. However, by Proposition 8, the unit ball $B_{c_0} \subset l_\infty$ is $\mathcal{AE}$-bounding. The topologies $\sigma(c_0, l_1)$ and $\sigma(l_\infty, (l_\infty)')$ coincide on c_0, and consequently, B_{c_0} is $\sigma(c_0, l_1)$-compact, an obvious contradiction.

If E is finite-dimensional, then $\mathcal{P}_f(E) = \mathcal{P}(E)$, and hence $\mathcal{P}(E)$ is sequentially evaluating. Since E' contains a point separating sequence, every homomorphism on $\mathcal{P}(E)$ is a point evaluation. If E is infinite-dimensional the situation is entirely different.

PROPOSITION 18. *Let E be an infinite-dimensional Banach space. Then $\mathcal{P}(E)$ is not a single-set evaluating algebra.*

Proof. Suppose that $\phi(p) \in p(E)$ for all $p \in \mathcal{P}(E)$. As in Proposition 10, it can be shown that $\phi|_{P_f(E)}$ is sequentially evaluating for each $\phi \in \operatorname{Hom} P(E)$. Using the interchangeable double limit property technique as in Theorem 1, one sees that each bounded set is relatively weakly compact. Hence E has to be reflexive and therefore weakly Lindelf. Since $\phi|_{P_f(E)}$ is sequentially evaluating, it therefore follows that $\phi|_{P_f(E)}$ is a point evaluation on E for each $\phi \in \operatorname{Hom} P(E)$. On the other hand, by [12], given $\omega \in E'^*$ there exists $\phi \in \operatorname{Hom} P(E)$ with $\omega(l) = \phi(l)$ for every $l \in E'$, which gives a contradiction since $E'^* \setminus E \neq \emptyset$.

PROPOSITION 19. *Let E be a non-reflexive Banach space. Then neither $\mathcal{P}_f(E)$ nor the inverse-closed algebra $\mathcal{RP}_f(E)$ are sequentially evaluating.*

Proof. By linear algebra, a linear functional l is a linear combination of linear functionals $l_1, \ldots, l_n$ if and only if $\operatorname{Ker}(l) \supset \bigcap_{i=1}^n \operatorname{Ker}(l_i)$, and therefore $\mathcal{P}_f(E)$ is single-set evaluating. Assume, contrary to the statement, that $\mathcal{P}_f(E)$ is sequentially evaluating. Let B_E be the unit ball in E. Take sequences (x_n) in B_E and (l_m) in the polar B_E°. As in the proof of Theorem 1, B_E is relatively compact in $\operatorname{Hom}\mathcal{P}_f(E)$. Let ϕ and l_0 be cluster points of the sequences (x_n) and (l_m) respectively. By assumption, there is a point $a \in E$ with $\phi(l_m) = l_m(a)$ for $m = 0, 1, \ldots$ If all limits involved exist, then

$$\begin{aligned}\lim_m \lim_n l_m(x_n) &= \lim_m \phi(l_m) = \lim_m l_m(a) = l_0(a) = \phi(l_0)\\ &= \lim_n l_0(x_n) = \lim_n \lim_m l_m(x_n).\end{aligned}$$

Thus B_E has the interchangeable double limit property with B_E°. By the Eberlein–Grothendieck theorem [11, p. 15], the set B_E is weakly compact. As this is a contradiction, the algebra $\mathcal{P}_f(E)$ cannot be sequentially evaluating.

Then, according to Lemma 16, the algebra $\mathcal{RP}_f(E)$, although inverse-closed, is not sequentially evaluating.

Acknowledgements. The second named author wishes to thank Javier Gómez for valuable discussions.

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Received June 30, 1993 (3124)
Revised version October 28, 1994