Contents of Volume 115, Number 1

Q. LAI, Two-weight mixed Φ-inequalities for the one-sided maximal function . 1–22

P. BISTRÖM, J. A. JARAMILLO and M. LINDSTRÖM, Algebras of real analytic functions: Homomorphisms and bounding sets 23–37

T. V. PEDERSEN, Some results about Beurling algebras with applications to operator theory 39–52

M. FUHRMAN, Analyticity of transition semigroups and closability of bilinear forms in Hilbert spaces 53–71

R. G. FABER, A lifting theorem for locally convex subspaces of L_0 73–85

B. A. BARNES, Convergence in the generalized sense relative to Banach algebras of operators and in LMC-algebras 87–103

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Two-weight mixed Φ-inequalities for the one-sided maximal function

by

QINSHENG LAI (Leeds)

Abstract. Suppose u, v, w, and t are weight functions on an appropriate measure space (X,μ), and Φ_1, Φ_2 are Young functions satisfying a certain relationship. Let T denote an operator to be specified below. The main purpose of this paper is to characterize

(i) the strong type mixed Φ-inequality

$$\Phi_2^{-1}\Big(\int_X \Phi_2(T(fv))w\,d\mu\Big) \le \Phi_1^{-1}\Big(\int_X \Phi_1(Cf)v\,d\mu\Big),$$

(ii) the weak type mixed Φ-inequality

$$\Phi_2^{-1}\Big(\int_{\{|Tf|>\lambda\}} \Phi_2(\lambda w)t\,d\mu\Big) \le \Phi_1^{-1}\Big(\int_X \Phi_1(Cfu)v\,d\mu\Big)$$

and

(iii) the extra-weak type mixed Φ-inequality

$$|\{x\in X: |Tf(x)|>\lambda\}|_{w\,d\mu} \le \Phi_2\Phi_1^{-1}\Big(\int_X \Phi_1\Big(\frac{Cfu}{\lambda}\Big)v\,d\mu\Big),$$

when T is the one-sided maximal function M_g^+; as well to characterize (iii) for the Fefferman–Stein type fractional maximal operator and the Hardy-type operator.

1. Introduction. Let g be a locally integrable and positive function on the real line $\mathbb{R}$. The one-sided maximal function M_g^+ is defined by

$$M_g^+ f(x) = \sup_{h>0} \frac{1}{g(x,x+h)} \int_x^{x+h} |f(y)|g(y)\,dy, \tag{1}$$

where $f\in L_{\rm loc}$ and $g(x,x+h)=\int_x^{x+h} g(y)\,dy$. Symmetrically, we can define

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$$M_g^- f(x) = \sup_{h>0} \frac{1}{g(x-h,x)} \int_{x-h}^{x} |f(y)|g(y)\,dy. \tag{2}$$

Recently, much work has been done in order to characterize the weight norm inequalities for these two operators (see [14]–[17], [20]). Their main results are the two-weight strong type (p,p) inequalities ($1 < p < \infty$) and weak type (p,p) inequalities ($1 \le p < \infty$), which are parallel to those for the two-sided maximal function M_g defined by

$$M_g f(x) = \sup_{x\in(a,b)} \frac{1}{g(a,b)} \int_a^b |f(y)|g(y)\,dy. \tag{3}$$

In particular, the following characterizations have been proved.

THEOREM A ([15], [19]). *Suppose $w(x)$, $v(x)$ are weight functions, i.e. nonnegative measurable functions on $\mathbb{R}$, and $1 < p < \infty$. Then the strong type inequality*

$$\int_{-\infty}^{\infty} (M_g^+ f(x))^p w(x)\,dx \le C \int_{-\infty}^{\infty} |f(x)|^p v(x)\,dx$$

holds for all measurable f if and only if there exists a constant $C > 0$ such that for every interval $I = (a,b)$ with $\int_{(-\infty,a)} w > 0$, the inequality

$$\int_a^b (M_g^+(\chi_I g^{1/(p-1)}\sigma))^p w\,dx \le C \int_a^b g^{p'}\sigma\,dx < \infty \tag{4}$$

holds, where $\sigma = v^{-1/(p-1)}$, $1/p + 1/p' = 1$ and χ_E is the characteristic function of the measurable set E.

THEOREM B ([15], [20]). *Suppose $1 \le p < \infty$. Let (w,v) be a pair of weight functions. Then the weak type (p,p) inequality*

$$w(\{x : M_g^+ f(x) > \lambda\}) \le \left(\frac{C(\int_{-\infty}^{\infty} |f(x)|^p v(x)\,dx)^{1/p}}{\lambda}\right)^p \tag{5}$$

holds for all measurable f and $\lambda > 0$ if and only if there exists a constant $C > 0$ such that for all $a < b < c$,

$$\int_a^b w\Big(\int_b^c g^{p'}\sigma\Big)^{p-1} \le C\Big(\int_a^c g\Big)^p \tag{6}$$

holds, where σ is the same as above.

In comparison with the results for the two-sided maximal function M_g, there is an obvious shortcoming, that is, both Theorem A and Theorem B deal with just the single index case. In this paper we will characterize the two-weight strong type (p,q) and weak type (p,q) inequalities for the one-sided maximal function M_g^+ in the cases of $1 < p \le q < \infty$. These characterizations are the analogues of those for the two-sided maximal function. All results in this paper concerning M_g^+ have their counterparts for the M_g^-, however, we shall omit presenting and proving those. Moreover, we will establish our theorems in general Orlicz classes in the form in which the statements appear in the abstract. The weighted (p,q) inequalities in the cases $q < p$ are discussed in our papers [11] and [12].

In Section 2, we shall list some preliminaries on Young functions and Orlicz spaces and state our theorems for M_g^+. The proofs will be given in Section 3. The method used in Section 3 lets us characterize extra-weak type mixed Φ-inequalities for the Fefferman–Stein type maximal operator (see [5], [19]) and the Hardy-type operator (see [1]). We shall present those in Section 4.

2. Notations and results concerning M_g^+. First, we list some necessary notations and properties of Young functions and Orlicz spaces. We refer to [6] for the details.

On the real line $\mathbb{R}$, a *Young function* $\Phi(t)$ (i.e. an N-function in [6]) is given by the representation

$$\Phi(t) = \int_0^{|t|} \phi(x)\,dx,$$

where $\phi(t)$ is a nondecreasing function, positive for $t > 0$ and continuous from the right for $t \ge 0$, satisfying the conditions $\phi(0) = 0$ and $\phi(t) \to \infty$ as $t \to \infty$.

Further, let

$$\phi^{-1}(s) = \sup\{t : \phi(t) \le s\}$$

be the right inverse of $\phi(t)$. The function

$$\Psi(t) = \int_0^{|t|} \phi^{-1}(s)\,ds$$

is called the *complementary function* to $\Phi(t)$. It is also a Young function.

Let μ be a measure on $\mathbb{R}$ and Φ a Young function. The *Orlicz space* $L_\Phi(\mu)$ consists of all μ-measurable functions f on $\mathbb{R}$ for which the *Luxemburg norm*

$$\|f\|_{\Phi(\mu)} = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}} \Phi(|f(x)|/\lambda)\,d\mu(x) \le 1\Big\} \tag{7}$$

is finite.

The following inequalities are well known and important for our argument.

(i) *Hölder inequality*:

$$\int_{\mathbb{R}} |f(x)g(x)|\,d\mu \le 2\|f\|_{\Phi(\mu)}\|g\|_{\Psi(\mu)}; \tag{8}$$

(ii) *Young inequality*:

$$st \le \Phi(s) + \Psi(t) \quad \text{for every } s,t \in [0,\infty); \tag{9}$$

(iii) the inequality

$$\Phi(D\Psi(t)) \le \Psi(t) \quad \text{for all } t>0, \tag{10}$$

where $D\Psi(t) = \Psi(t)/t$. Usually, the last inequality appears in the form $t \le \Phi^{-1}(t)\Psi^{-1}(t)$.

As in [10], for a given pair of Young functions Φ_1 and Φ_2, we write $\Phi_1 \ll \Phi_2$ if the inequality

$$\sum \Phi_2\Phi_1^{-1}(a_i) \le \Phi_2\Phi_1^{-1}\Big(\sum a_i\Big) \tag{11}$$

holds for every sequence $\{a_i\}$ with $a_i \ge 0$.

The condition (11) is fulfilled if $\Phi_2\Phi_1^{-1}$ is convex, in particular, if $\Phi_2(t) = |t|^q$ and $\Phi_1(t) = |t|^p$ with $1 < p \le q < \infty$.

Throughout this paper, Ψ represents the complementary function to Φ. For a weight function w and measurable set E, $w(E) = |E|_w = \int_E w(x)\,dx$. In particular, $w((a,b)) = w(a,b)$, and $|E|$ is the Lebesgue measure of E. Unspecified letters $A,B,C,\ldots$ etc. will be absolute constants not necessarily the same at each occurrence. However, when the theorems concern the estimates for the best constants, they will be the same in a certain theorem. We shall keep the usual conventions $t\cdot\infty = \infty$ ($t \in (0,\infty]$), $0\cdot\infty = 0$, $1/\infty = 0$ and $1/0 = \infty$.

Now we state our main results concerning M_g^+.

THEOREM 1. *Suppose Φ_1 and Φ_2 are Young functions satisfying $\Phi_1 \ll \Phi_2$, and there exists a constant $A>0$ such that*

$$\int_{\mathbb{R}} \Phi_1(M_{gv}f)gv\,dx \le \int_{\mathbb{R}} \Phi_1(Af)gv\,dx \tag{12}$$

holds for all $f \ge 0$. Let w and v be weight functions. Then there exists a constant $C>0$ such that

$$\Phi_2^{-1}\Big(\int_{\mathbb{R}} \Phi_2(M_g^+(fv))w\Big) \le \Phi_1^{-1}\Big(\int_{\mathbb{R}} \Phi_1(Cf)gv\Big) \tag{13}$$

holds for all measurable f if and only if

$$\Phi_2^{-1}\Big(\int_a^b \Phi_2(M_g^+(\varepsilon\chi_{(a,b)}v))w\Big) \le \Phi_1^{-1}\Big(\Phi_1(B\varepsilon)\int_a^b gv\Big) < \infty \tag{14}$$

holds for all $\varepsilon>0$ and intervals (a,b) with $\int_{-\infty}^a w > 0$.

Furthermore, for the best constants C and B in (13) *and* (14) *respectively, we have*

$$B \le C \le 8AB.$$

Remark 1. The condition (12) is rather artificial. However, it follows from $\Psi_1 \in \Delta_2$ (see [21] or [3]), i.e. from the existence of a constant $K>0$ such that

$$\Psi_1(2t) \le K\Psi_1(t) \quad \text{for all } t>0. \tag{12$'$}$$

Conversely, for a given Young function Φ, we have

PROPOSITION 1. *Suppose w is a weight on $\mathbb{R}$, positive and integrable on an interval (a,b). If*

$$\int_{\mathbb{R}} \Phi(M_w f)w\,dx \le \int_{\mathbb{R}} \Phi(Af)w\,dx$$

holds for all $f \ge 0$, then $\Psi \in \Delta_2$.

Proof. Put $I = (a,a+h)$ and $f = \varepsilon\chi_I$. Using an obvious inequality

$$\frac{t}{2}\phi\Big(\frac{t}{2}\Big) \le \Phi(t) \le t\phi(t),$$

we have

$$\begin{aligned} A\varepsilon\phi(A\varepsilon)w(I) \ge \Phi(A\varepsilon)w(I) &\ge \int_{\mathbb{R}} \Phi(M_w f)w\,dx \\ &\ge \int_{a+h}^b \Phi\Big(\frac{\varepsilon\int_I w}{\int_a^x w}\Big)w(x)\,dx \ge \int_{a+h}^b \frac{(\varepsilon/2)\int_I w}{\int_a^x w}\phi\Big(\frac{(\varepsilon/2)\int_I w}{\int_a^x w}\Big)w(x)\,dx \\ &= \frac{\varepsilon}{2}w(I)\int_{\delta(I)}^{\varepsilon/2}\frac{\phi(y)}{y}\,dy \end{aligned}$$

with the change of variable $y = ((\varepsilon/2)w(I))/\int_a^x w$, where $\delta(I) = ((\varepsilon/2)w(I))/\int_a^b w$. Hence

$$\int_{\delta(I)}^{\varepsilon/2}\frac{\phi(y)}{y}\,dy \le 2A\phi(A\varepsilon).$$

Letting $h \to 0$, we get the Dini condition

$$\int_0^{\varepsilon/2}\frac{\phi(y)}{y}\,dy \le 2A\phi(A\varepsilon),$$

which is equivalent to $\Psi \in \Delta_2$ (see [3]). Proposition 1 is proved.

Furthermore, Theorem 7 of [3] can be rewritten as

PROPOSITION 2. *Let Φ be a Young function, and t, u, v, w be weight functions with $0 < t(x), u(x), v(x), w(x) < \infty$ a.e. Then, in order for*

$$\int \Phi(w(x)M_g^+ f(x))t(x)\,dx \le \int \Phi(Cu(x)f(x))v(x)\,dx$$

to hold for all $f \ge 0$, we must have $\Psi \in \Delta_2$.

Proof. The proof is nearly the same as that of Theorem 7 in [3], so we just give an outline.

There exists a constant $K > 0$ such that the set

$$E = \{x : K^{-1} \le t(x), u(x), v(x), w(x), g(x) \le K\}$$

has positive measure. Choose x to be a point of density of E and a Lebesgue point of g. Let $r_0 > 0$ such that

$$|(x-r,x)\cap E| \ge \frac{1}{2}r \quad \text{and} \quad \frac{g(x-r,x)}{r} \le 2g(x) \le 2K$$

for all $0 < r \le r_0$. Write $B_m = (x - 2^{-m}r_0, x)$ $(m = 0, 1, \ldots)$, and let $f_m = \chi_{E\cap B_m}$. One has

$$\int\limits_{(B_0\cap E)\setminus B_m} \frac{dy}{x-y} \ge Cm$$

and

$$M_g^+ f_m(y) \ge CK^{-2}|B_m\cap E|(x-y)^{-1}$$

when $y \in B_0\setminus B_m$. Then Bloom–Kerman's argument shows that

$$\Psi(2y) \le C2^m K^2 \Psi(y)$$

for all $y > 0$ if we choose $m \ge 2CK^6$. Proposition 2 is proved.

Using Proposition 2, one can withdraw the condition (12) from the assumptions of Theorem 1 when $\Phi_1 = \Phi_2$. We shall omit the presentation to avoid the dull repetitions.

We would like to thank the referee for informing us of the equivalence between (12) and (12′) and suggesting the previous propositions.

The condition (12) is always satisfied if $\Phi_1(t) = |t|^p$ with $1 < p < \infty$. Therefore, using Theorem 1 and a well-known and obvious change, we obtain the strong (p,q) inequality with $p \le q$ for M_g.

COROLLARY 1. *Suppose $1 < p \le q < \infty$. Let w, v be a pair of weight functions. Then the strong type inequality*

$$\Big(\int\limits_{\mathbb{R}} (M_g^+ f)^q w\Big)^{1/q} \le C\Big(\int\limits_{\mathbb{R}} |f|^p v\Big)^{1/p}$$

holds for all locally integrable f if and only if

$$\Big(\int\limits_a^b (M_g^+(\chi_{(a,b)}g^{1/(p-1)}\sigma))^q w\Big)^{1/q} \le B\Big(\int\limits_a^b g^{p'}\sigma\Big)^{1/p} < \infty$$

holds for all intervals (a,b) with $\int_{-\infty}^a w > 0$, where $\sigma = v^{-1/(p-1)}$.

When $p = q$, this is Theorem 2 of [15] which generalizes Theorem 2 of [20].

THEOREM 2. *Suppose Φ_1 and Φ_2 are Young functions with $\Phi_1 \ll \Phi_2$, ϱ is a Borel measure on $\mathbb{R}$ and w, u, v are weight functions. Then the following statements are equivalent:*

(i) *there exists a constant $C > 0$ such that the weak type inequality*

$$\Phi_2^{-1}\Big(\int\limits_{\{M_g^+ f>\lambda\}} \Phi_2(\lambda w(x))\,d\varrho(x)\Big) \le \Phi_1^{-1}\Big(\int\limits_{\mathbb{R}} \Phi_1(Cf(x)u(x))v(x)\,dx\Big) \tag{15}$$

holds for all measurable f and $\lambda > 0$;

(ii) *there exists a constant B such that*

$$\int\limits_b^c \Psi_1\Big(\frac{J(\varepsilon,(a,b))g(x)}{B\varepsilon g(a,c)u(x)v(x)}\Big)v(x)\,dx \le J(\varepsilon,(a,b)) < \infty \tag{16}$$

holds for all $a < b < c$ and $\varepsilon > 0$, where

$$J(\varepsilon,(a,b)) = \Phi_1\Phi_2^{-1}\Big(\int\limits_a^b \Phi_2(\lambda w(x))\,d\varrho(x)\Big); \tag{17}$$

(iii) *there exists a constant D such that*

$$\Big\|\frac{g(\cdot)\chi_{(b,c)}(\cdot)}{\varepsilon u(\cdot)v(\cdot)}\Big\|_{\Psi_1(\varepsilon v)} \le Dg(a,c)\eta \tag{18}$$

holds for all $\varepsilon > 0$ and every $a < b < c$, where $\eta = \eta(\varepsilon,(a,b)) = \eta(\varepsilon)$ is defined by

$$\eta = \sup\{\theta \ge 0 : J(\theta,(a,b)) \le 1/\varepsilon\}. \tag{19}$$

Furthermore, for the best constants in (15), (16) *and* (18), *we have*

$$D \le B \le C \le 8D.$$

THEOREM 3. *Suppose Φ_1 and Φ_2 are Young functions with $\Phi_1 \ll \Phi_2$, and w, u, v are weight functions. Then the following statements are equivalent:*

(i) *there exists a constant $C > 0$ such that the extra-weak type inequality*

$$w(\{M_g^+ f > \lambda\}) \le \Phi_2\Phi_1^{-1}\Big(\int\limits_{\mathbb{R}} \Phi_1\Big(\frac{C|f|u}{\lambda}\Big)v\Big) \tag{20}$$

holds for all measurable f and all $\lambda > 0$;

(ii) *there exists a constant* $B > 0$ *such that*

$$\int_b^c \Psi_1\left(\frac{\Phi_1\Phi_2^{-1}(w(a,b))g(x)}{Bg(a,c)v(x)u(x)}\right)v(x)\,dx \le \Phi_1\Phi_2^{-1}(w(a,b)) < \infty \tag{21}$$

for every $a < b < c$;

(iii) *there exists a constant* $D > 0$ *such that*

$$\left\|\frac{g(\cdot)\chi_{(b,c)}(\cdot)}{u(\cdot)v(\cdot)}\right\|_{\Psi_1(v/\Phi_1\Phi_2^{-1}(w(a,b)))} \le \frac{Dg(a,c)}{\Phi_1\Phi_2^{-1}(w(a,b))} \tag{22}$$

holds for all $a < b < c$ *with* $\Phi_1\Phi_2^{-1}(w(a,b)) > 0$, *and* $\Phi_1\Phi_2^{-1}(w(a,b)) < \infty$ *for all bounded intervals* (a,b).

Furthermore, for the best constants C, B *and* D *in* (20), (21) *and* (22), *we have*

$$D \le B \le C \le 8D.$$

Theorem 2 or Theorem 3 directly imply

COROLLARY 2. *Suppose* w, v *are weight functions and* $1 < p \le q < \infty$. *Then the following statements are equivalent*:

(i) *there exists* $C > 0$ *such that the weak type inequality*

$$w(\{x : M_g^+ f(x) > \lambda\}) \le \left(\frac{C(\int_{\mathbb{R}} |f|^p v)^{1/p}}{\lambda}\right)^q$$

holds for all measurable f *and* $\lambda > 0$;

(ii) *there exists* $B > 0$ *such that*

$$\frac{(w(a,b))^{1/q}(\sigma(b,c))^{1/p'}}{g(a,c)} \le B$$

holds for all $a < b < c$, *where* $\sigma = (g/v)^{p'}v$.

Moreover, for the best constants C *and* B *above, we have*

$$A_1 B \le C \le A_2 B,$$

where A_1, A_2 *are absolute constants depending only on* p *and* q.

The special case of $p = q$ gives Theorem 1 ($p > 1$) of [15].

Remark 2. It is obvious that both the weak type mixed Φ-inequality like (15) and the extra-weak type mixed Φ-inequality like (20) are extensions of the usual weak type (p,q) inequality to Orlicz classes. The terminology "extra-weak type" appears in [18] in the case of $\Phi_1 = \Phi_2$. It stems from the fact that the weak type inequality implies the extra-weak type inequality for all homogeneous operators T.

Remark 3. When $\Phi_1 = \Phi_2 = \Phi$, the weak type and extra-weak type inequalities ((15) and (20)) have been discussed in [17] and [16]. In this

particular case, our condition (21) becomes the $A_e^+(\Phi, g)$ condition in Theorem 2 of [17]. However, our condition (16) is apparently different from the corresponding $A_w^+(\Phi, g)$ condition in Theorem 1 of [17] and its equivalent $A_\Phi^+(g)$ condition in Theorem 1 of [16]. Their condition is that

$$\sup_{\varepsilon>0}\sup_{a<b<c}\left(\frac{\varepsilon w(a,b)}{g(a,c)}\right)R_\Phi\left(\frac{1}{Bg(ac)}\int_b^c S_\Phi\left(\frac{g(x)}{\varepsilon v(x)}\right)g(x)\,dx\right) \le K,$$

where $R_\Phi(t) = \Phi(t)/t$ and $S_\Phi(t) = \Psi(t)/t$.

3. Proofs of Theorems 1–3

Proof of Theorem 1. (13)⇒(14) follows from a standard argument (see e.g. [19], [20]) that involves testing (13) with $f = \varepsilon\chi_{(a,b)}$. Conversely, suppose (14) holds. Without loss of generality, we assume f is nonnegative, bounded and supported by an interval bounded from above.

Let N be a positive integer. For every integer k, let

$$\Omega_k = \{x : M_g^+(fv)(x) > 2^k\} \cap (-N, \infty).$$

Each Ω_k is an open set, therefore there exists a sequence $\{(a_{j,k}, b_{j,k})\}$ such that the $\{(a_{j,k}, b_{j,k})\}_j$ are pairwise disjoint and $\Omega_k = \bigcup_j (a_{j,k}, b_{j,k})$. Furthermore, we have (see [20])

$$\int_x^{b_{j,k}} fgv \ge 2^k \int_x^{b_{j,k}} g \quad \text{for every } x \in (a_{j,k}, b_{j,k}). \tag{23}$$

For every $(a_{j,k}, b_{j,k})$, let

$$d_{j,k} = \inf\left\{x \in (a_{j,k}, b_{j,k}) : \int_x^{b_{j,k}} gv < \infty\right\}.$$

It is clear that $\int_x^{b_{j,k}} gv < \infty$ if $x > d_{j,k}$, and $w = 0$ for a.e. $x \in (a_{j,k}, d_{j,k})$ by (14). With $E_{j,k} = (a_{j,k}, b_{j,k}) \setminus \Omega_{k+1}$ and $F_{j,k} = (d_{j,k}, b_{j,k}) \setminus \Omega_{k+1}$, we have

$$\begin{aligned} \int_{-N}^{\infty} \Phi_2(M_g^+(fv))w &= \sum_{k,j}\int_{E_{j,k}} \Phi_2(M_g^+(fv))w \\ &= \sum_{k,j}\int_{F_{j,k}} \Phi_2(M_g^+(fv))w \le \sum_{k,j}\int_{F_{j,k}} \Phi_2(2^{k+1})w \\ &\le \sum_{k,j}\int_{F_{j,k}} \Phi_2\left(2\frac{\int_x^{b_{j,k}} fgv}{\int_x^{b_{j,k}} g}\right)w \quad \text{(by (23)).} \end{aligned} \tag{24}$$

For every integer m, let

$$A_{j,k,m} = F_{j,k} \cap \left\{x : 2^m < \frac{\int_x^{b_{j,k}} fgv}{\int_x^{b_{j,k}} gv} \le 2^{m+1}\right\}$$
$$= \left\{x \in (d_{j,k}, b_{j,k}) : 2^m < \frac{\int_x^{b_{j,k}} fgv}{\int_x^{b_{j,k}} gv} \le 2^{m+1}\right\} \cap F_{j,k}$$
$$= H^m_{j,k} \cap F_{j,k}.$$

Fix m, set $e_{j,k} = \inf H^m_{j,k}$ and $J^m_{j,k} = (e_{j,k}, b_{j,k})$. Then

$$A_{j,k,m} \subset J^m_{j,k} \cap F_{j,k}.$$

The sets $\{A_{j,k,m}\}_{j,k}$ are pairwise disjoint for fixed m. It is obvious that the inequality

$$2^m \le \int_{J^m_{j,k}} fgv \Big/ \int_{J^m_{j,k}} gv \le 2^{m+1}$$

holds for all j, k and m. Therefore

$$\bigcup_{j,k} J^m_{j,k} \subset \{x : M_{gv}f(x) \ge 2^m\}. \tag{25}$$

Let $I_{j,k} = (d_{j,k}, b_{j,k})$. Then $J^m_{j,k} \subset I_{j,k}$. Moreover, the collection $\{I_{j,k}\}$ is nested, that is, $k < s$ implies that either $I_{i,s} \subset I_{j,k}$ or $I_{i,s} \cap I_{j,k} = \emptyset$ for all i, j. Since all the lengths of $I_{j,k}$ are uniformly bounded above, we can select maximal elements from the $\{I_{j,k}\}$. Denote this subfamily by $\{I_i\}$. Let $G(i) = \{(m,j,k) : J^m_{j,k} \subset I_i\}$. Noting that $\bigcup_{(m,j,k)\in G(i)} J^m_{j,k}$ is an open set, we can write

$$\bigcup_{j,k:\,(m,j,k)\in G(i)} J^m_{j,k} = \bigcup_s O^{i,m}_s,$$

where $\{O^{i,m}_s\}$ is a sequence of pairwise disjoint open intervals with $O^{i,m}_s \subset I_i$ for all s and m. Fix m, and rewrite the $\{O^{i,m}_s\}_{i,s}$ as a sequence $\{O_{n,m}\}_n$ which has the following properties:

(i) $\{O_{n,m}\}_n$ are pairwise disjoint, since $\{I_i\}$ are pairwise disjoint;
(ii) every $J^m_{j,k}$ belongs to a certain $O_{n,m}$;
(iii) $\bigcup_n O_{n,m} = \bigcup_{j,k} J^m_{j,k} \subset \{x : M_{gv}f(x) \ge 2^m\}$ by (25).

Now, we estimate the right side of (24). Noting that, for $x \in A_{j,k,m}$,

$$\frac{\int_x^{b_{j,k}} fgv}{\int_x^{b_{j,k}} g} = \frac{\int_x^{b_{j,k}} fgv \int_x^{b_{j,k}} gv}{\int_x^{b_{j,k}} gv \int_x^{b_{j,k}} g} \le 2^{m+1} M^+_g(\chi_{J^m_{j,k}} v)(x), \tag{26}$$

we have

$$\sum_{k,j} \int_{F_{j,k}} \Phi_2\left(2\frac{\int_x^{b_{j,k}} fgv}{\int_x^{b_{j,k}} g}\right) w \tag{27}$$
$$\le \sum_{m=-\infty}^{\infty} \sum_{(n,m)} \sum_{\{(k,j):J^m_{j,k}\subset O_{n,m}\}} \int_{A_{j,k,m}} \Phi_2\left(2\frac{\int_x^{b_{j,k}} fgv}{\int_x^{b_{j,k}} g}\right) w$$
$$\le \sum_m \sum_{(n,m)} \sum_{\{(k,j):J^m_{j,k}\subset O_{n,m}\}} \int_{A_{j,k,m}} \Phi_2(M^+_g(2^{m+2}\chi_{J^m_{j,k}} v))w \quad \text{(by (26))}$$
$$\le \sum_m \sum_{(n,m)} \int_{O_{n,m}} \Phi_2(M^+_g(2^{m+2}\chi_{O_{n,m}} v))w$$
$$\le \sum_m \sum_{(n,m)} \Phi_2\Phi_1^{-1}\Big(\Phi_1(8B2^{m-1}) \int_{O_{n,m}} gv\Big) \quad \text{(by (14))}.$$

Let $\Phi_3(t) = \Phi_1(8Bt)$. Then (12) is still available with the same constant A. Therefore the right side of (27) is bounded by

$$\Phi_2\Phi_1^{-1}\Big(\sum_m \sum_{(n,m)} |O_{n,m}|_{gv}\Phi_3(2^{m-1})\Big) \quad (\text{since } \Phi_1 \ll \Phi_2) \tag{28}$$
$$\le \Phi_2\Phi_1^{-1}\Big(\sum_m |\{x : M_{gv}f(x) \ge 2^m\}|_{gv}\Phi_3(2^{m-1})\Big)$$
$$\le \Phi_2\Phi_1^{-1}\Big(\int_{\mathbb{R}} \Phi_3(M_{gv}f)gv\Big) \le \Phi_2\Phi_1^{-1}\Big(\int_{\mathbb{R}} \Phi_3(Af)gv\Big) \quad \text{(by (12))}$$
$$= \Phi_2\Phi_1^{-1}\Big(\int_{\mathbb{R}} \Phi_1(8ABf)gv\Big).$$

Combining (24), (27), (28) and letting $N \to \infty$, we complete the proof of Theorem 1.

Proof of Theorem 2. To avoid trivial cases, we may assume that $u(x)v(x)$ is not identical to infinity on any interval (a,∞). Otherwise, we can present and prove our theorem on $(-\infty,a)$ instead of $(-\infty,\infty)$.

(15)⇒(16). The proof of the right inequality depends on the fact that

$$(a,b) \subset \{x : M^+_g(f\chi_{(b,c)})(x) > m^+_g(f)\} \tag{29}$$

holds for every $a < b < c$ and $f \ge 0$, where

$$m^+_g(f) = \frac{1}{g(a,c)} \int_b^c fg.$$

Let $E = \{x \in \mathbb{R} : \text{there exists } \delta > 0 \text{ such that } J(\lambda, (x-\delta, x+\delta)) < \infty$ for all $\lambda > 0\}$ and $\theta = \inf\{\mathbb{R} \setminus E\}$. Then $J(\lambda, (a,b)) < \infty$ for all $\lambda > 0$ and all $(a,b) \subset (-\infty, \theta)$. Moreover, we claim that θ must be ∞. Indeed, suppose $\theta < \infty$. Given $\delta > 0$ we consider the set $S_M = \{x \in (\theta+\delta, \infty) : u(x)v(x) < M\}$ for arbitrary $M > 0$. Noting the definitions of θ and E, we can choose $a < b = \theta + \delta$ such that $(a+b)/2 \in [\theta, \theta+\delta)$ and that there exists $\lambda > 0$ satisfying $J(\lambda, (a,b)) = \infty$.

Suppose $|S_M| > 0$. First, we assume $|\{x \in S_M : u(x) < \infty\}| > 0$. Then we can choose $c > b$ and a set $S \subset (b,c) \cap S_M$ such that $|S| > 0$ and $u(x) \le N$ on S for some $N > 0$ large enough. On the other hand, on setting $f(x) = \chi_S(x)$, it follows from (29) and (15) that

$$J(\lambda, (a,b)) \le \Phi_1 \Phi_2^{-1}\Big(\int\limits_{\{M_g^+(g(a,c)\lambda\chi_S/g(S))>\lambda\}} \Phi_2(\lambda w(x))\, d\varrho(x)\Big) \tag{30}$$

$$\le \int\limits_S \Phi_1\left(\frac{g(a,c)C\lambda u(x)}{g(S)}\right) v(x)\, dx$$

$$\le \Phi_1\left(\frac{g(a,c)C\lambda N}{g(S)}\right)|S| + \Phi_1\left(\frac{g(a,c)C\lambda M}{g(S)}\right)|S| < \infty.$$

(In the last inequality above we use the fact that $s\Phi_1(t) \le \Phi_1(st)$ for all $s \ge 1$ and $t \ge 0$). This is a contradiction.

Secondly, assume $|\{x \in S_M : u(x) < \infty\}| = 0$. Then we have $v(x) = 0$ a.e. on S_M. The same argument as that used above also produces a contradiction because in this case the right side of (30) is 0 due to the convention $\infty \cdot 0 = 0$. These contradictions imply $|S_M| = 0$. Therefore, we conclude that $u(x)v(x) = \infty$ a.e. on (θ, ∞), a trivial case we have excluded.

For the left inequality in (16), given $a < b < c$ we may assume $J(\varepsilon, (a,b)) > 0$, otherwise (16) holds trivially. It follows that $u(x)v(x) > 0$ a.e. on (b,c). Indeed, if $|F| = |\{x \in (b,c) : u(x)v(x) = 0\}| > 0$, then taking $f = \chi_F$, we obtain, from (29) and (15),

$$J(\lambda, (a,b)) \le \int\limits_F \Phi_1\left(\frac{g(a,c)C\lambda u(x)}{g(S)}\right) v(x)\, dx = 0.$$

This is a contradiction. Noting that the left side of (16) is zero if $u(x)$ or $v(x) = \infty$, we may assume both $u(x)$ and $v(x)$ are finite a.e. on (b,c); otherwise, we can consider $(b,c) \setminus \{x : u(x) \text{ or } v(x) = \infty\}$ instead of (b,c).

For arbitrary $\eta > 1$, let

$$h(x) = \frac{J(\varepsilon, (a,b))g(x)}{\eta C\varepsilon g(a,c)u(x)v(x)}\chi_{E_n}(x),$$

where $E_n = \{x \in (b,c) : 1/n \le g(x), u(x), v(x) \le n\}$, and

$$f(x) = \Psi_1(h(x)) \frac{\eta C\varepsilon g(a,c)v(x)}{J(\varepsilon, (a,b))g(x)}.$$

Then

$$T = \int\limits_b^c \Psi_1(h(x))v(x)\, dx = \frac{J(\varepsilon, (a,b))}{\eta C\varepsilon} m_g^+(f),$$

where $m_g^+(f)$ is defined in (29).

We claim that $m_g^+(f) \le C\varepsilon$, hence

$$T \le \frac{J(\varepsilon, (a,b))}{\eta},$$

and (16) follows by taking $n \to \infty$ and $\eta \to 1$. Furthermore, $B \le C$.

Indeed, suppose $m_g^+(f) > C\varepsilon$. It follows that

$$T \le \frac{m_g^+(f)}{\eta C\varepsilon} \int\limits_{E_n} \Phi_1\left(\frac{C\varepsilon f(x)u(x)}{m_g^+(f)}\right) v(x)\, dx$$

$$\text{(by (29) and (15), also cf. (30))}$$

$$\le \frac{1}{\eta} \int\limits_{E_n} \Phi_1(f(x)u(x))v(x)\, dx = \frac{1}{\eta} \int\limits_{E_n} \Phi_1(D\Psi_1(h(x)))v(x)\, dx$$

$$\text{(since } C\varepsilon/m_g^+(f) < 1 \text{ and } \Phi_1 \text{ is convex with } \Phi_1(0) = 0)$$

$$\le \frac{1}{\eta} \int\limits_{E_n} \Psi_1(h(x))v(x)\, dx = \frac{T}{\eta} \quad \text{(by (10))}.$$

This contradiction completes the proof of (15)⇒(16).

(16)⇒(18). Given $a < b < c$, suppose $J(\theta, (a,b)) > 0$ for all $\theta > 0$. Then it follows from (16) and (7), the definition of the Luxemburg norm, that

$$\left\| \frac{g(\cdot)\chi_{(b,c)}(\cdot)J(\theta, (a,b))}{B\theta g(a,c)u(\cdot)v(\cdot)} \right\|_{\Psi_1(v/J(\theta,(a,b)))} \le 1 \tag{31}$$

for all $\theta > 0$. Since $J(\theta, (a,b))$ is a continuous increasing function of θ for fixed (a,b), and takes values from 0 to infinity, we can choose θ such that $1/J(\theta, (a,b)) = \varepsilon$ for given $\varepsilon > 0$. Then (18) follows from (31) immediately.

When $J(\theta_0, (a,b)) = 0$ for some $\theta_0 > 0$, we have $w(x) = 0$ ϱ-a.e. on (a,b). Then $J(\theta, (a,b)) = 0$ for all $\theta > 0$. In this case, $\eta = \eta(\varepsilon, (a,b)) = \infty$ for all $\varepsilon > 0$, and (18) holds trivially.

Now we prove (18)⇒(15). Without loss of generality, we can assume $f \ge 0$. For given $\lambda > 0$, as in the proof of Theorem 1, we can write $\{x :$

$M_g^+ f(x) > \lambda\} = \bigcup (a_j, b_j)$ with

$$\lambda \le \frac{1}{g(x, b_j)} \int_x^{b_j} fg \quad \text{for all } j \text{ and } x \in (a_j, b_j). \tag{32}$$

Fix j and let $(a,b) = (a_j, b_j)$. Following the "cutting method" of F. J. Martín-Reyes ([14]), we put $x_0 = a$ and let x_k be such that

$$\int_{x_k}^b fg = 2 \int_{x_{k+1}}^b fg, \quad k = 0, 1, \ldots$$

Then $\{x_k\}$ is an increasing sequence with limit b, and we have

$$\lambda \le \frac{1}{g(x_{k-1}, b)} \int_{x_{k-1}}^b fg \le \frac{4}{g(x_{k-1}, x_{k+1})} \int_{x_k}^{x_{k+1}} fg. \tag{33}$$

Let $f_k = f\chi_{(x_k, x_{k+1}]}$. It follows from the Hölder inequality (8) that

$$\int_{x_k}^{x_{k+1}} fg \le 2\|f_k u\|_{\Phi_1(\varepsilon_k v)} \left\| \frac{g\chi_{(x_k, x_{k+1})}}{\varepsilon_k uv} \right\|_{\Psi_1(\varepsilon_k v)} \tag{34}$$

for any $\varepsilon_k > 0$. For $\delta > 1$ arbitrary, choose ε_k such that

$$\int \Phi_1(8\delta D f_k(x) u(x)) \varepsilon_k v(x)\, dx = 1.$$

Then

$$\|f_k\|_{\Phi_1(\varepsilon_k v)} \le \frac{1}{8\delta D}. \tag{35}$$

Combining (32)–(35), we have

$$\lambda \le \frac{1}{D\delta g(x_{k-1}, x_{k+1})} \left\| \frac{g\chi_{(x_k, x_{k+1})}}{\varepsilon_k uv} \right\|_{\Psi_1(\varepsilon_k v)} \le \frac{\eta}{\delta} \quad \text{(by (18))}. \tag{36}$$

Recalling the definition of η (see (19)) and observing that $J(\theta, (a,b))$ is increasing in θ for fixed (a,b), we obtain

$$J(\lambda, (x_{k-1}, x_k)) \le 1/\varepsilon_k.$$

Summing over k and j, we obtain

$$\int_{\{M_g^+ f > \lambda\}} \Phi_2(\lambda w(x))\, d\varrho(x) \le \sum_j \sum_k \Phi_2 \Phi_1^{-1}(1/\varepsilon_k)$$
$$\le \Phi_2 \Phi_1^{-1}\Big(\sum_j \sum_k \int \Phi_1(8\delta D f_k) v\Big) \le \Phi_2 \Phi_1^{-1}\Big(\int \Phi_1(8\delta D f) v\Big).$$

The inequality (15) with $C \le 8D$ follows, since $\delta > 1$ is arbitrary. The proof of Theorem 2 is complete.

Proof of Theorem 3. The proof of (20)⇒(21) is similar to that of (15)⇒(16), we shall only give an outline. The details ignored can be found above.

For $a < b < c$ given, we may assume that $0 < w(a,b) < \infty$ and $0 < u(x), v(x) < \infty$ a.e. on (b,c). Let

$$E_n = \{x \in (b,c) : 1/n < u(x), v(x), g(x) < n\},$$

where n is a positive integer. For arbitrary $\eta > 1$, put

$$h(x) = \frac{\Phi_1 \Phi_2^{-1}(w(a,b)) g(x)}{\eta C g(a,c) v(x) u(x)} \chi_{E_n},$$

and $f(x) = D\Psi_1(h(x))/u(x)$. Let

$$T = \int_{E_n} \Psi_1(h(x)) v(x)\, dx.$$

Then

$$m_g^+(f) = \frac{\eta C}{\Phi_1 \Phi_2^{-1}(w(a,b))} T. \tag{37}$$

We claim that $m_g^+(f) \le C$, hence

$$T \le \frac{\Phi_1 \Phi_2^{-1}(w(a,b))}{\eta},$$

and (21) follows by taking $n \to \infty$ and $\eta \to 1$. Furthermore, $B \le C$.

Indeed, if $m_g^+(f) > C$, it follows that

$$\begin{aligned} T &= \frac{m_g^+(f)}{\eta C} \Phi_1 \Phi_2^{-1}(w(a,b)) \quad \text{(by (37))} \\ &\le \frac{m_g^+(f)}{\eta C} \int_{E_n} \Phi_1\Big(\frac{CD\Psi_1(h(x))}{m_g^+(f)}\Big) v(x)\, dx \quad \text{(by (29) and (20))} \\ &\le \frac{1}{\eta} \int_{E_n} \Phi_1(D\Psi_1(h(x))) v(x)\, dx \quad \text{(since } C/m_g^+(f) < 1) \\ &\le \frac{1}{\eta} \int_{E_n} \Psi_1(h(x)) v(x)\, dx = \frac{T}{\eta} \quad \text{(by (10))}. \end{aligned}$$

This is absurd.

The implication (21)⇒(22) follows from the definition of the Luxemburg norm (see (7)) directly, and we shall omit it.

(22)⇒(20). Using the same notations as in the proofs of (18)⇒(15), we write $\{M_g^+ f > \lambda\} = \bigcup (a_j, b_j) : (a_j, b_j) = (a,b) = \bigcup (x_k, x_{k+1}]$ for any fixed

j, where both $\{(a_j, b_j)\}$ and $\{(x_k, x_{k+1}]\}$ are pairwise disjoint. Moreover,

$$\lambda \le \frac{4}{g(x_{k-1}, x_{k+1})} \int_{x_k}^{x_{k+1}} fg. \tag{38}$$

Given $\delta > 1$ arbitrary, let $\lambda = 8\delta D$. We claim that

$$w(x_{k-1}, x_k) \le \Phi_2\Phi_1^{-1}\Big(\int_{x_k}^{x_{k+1}} \Phi_1(fu)v\Big) \tag{39}$$

holds for all (x_{k-1}, x_k) with $w(x_{k-1}, x_k) > 0$. Then it follows that

$$w(\{x : M_g^+ f > 8\delta D\}) \le \Phi_2\Phi_1^{-1}\Big(\int_{\mathbb{R}} \Phi_1(fu)v\Big),$$

and this is (20) with $C \le 8\delta D$.

Now we prove (39). In order to reach a contradiction, we assume that

$$w(x_{k-1}, x_k) > \Phi_2\Phi_1^{-1}\Big(\int_{x_k}^{x_{k+1}} \Phi_1(fu)v\Big) \tag{40}$$

for some k. Writing $x_{k-1} = a$, $x_k = b$ and $x_{k+1} = c$, we have

$$\begin{aligned} 8\delta D &\le \frac{4}{g(a,c)} \int_b^c fg \quad \text{(by (38))} \\ &\le \frac{8\Phi_1\Phi_2^{-1}(w(a,b))}{g(a,c)} \|f\chi_{(b,c)}u\|_{\Phi_1(\mu)} \Big\|\frac{g\chi_{(b,c)}}{uv}\Big\|_{\Psi_1(\mu)} \\ &\qquad \text{(by Hölder's inequality (8))}, \end{aligned} \tag{41}$$

where $d\mu = v(x)\,dx/\Phi_1\Phi_2^{-1}(w(a,b))$. However, the inequality (40) shows

$$\|f\chi_{(b,c)}u\|_{\Phi_1(\mu)} \le 1, \tag{42}$$

and the condition (22) yields

$$\Big\|\frac{g\chi_{(b,c)}}{uv}\Big\|_{\Psi_1(\mu)} \le \frac{Dg(a,c)}{\Phi_1\Phi_2^{-1}(w(a,b))}. \tag{43}$$

Substituting (42) and (43) into (41), we get

$$8\delta D \le 8D.$$

This is absurd. The proof of Theorem 3 is complete.

4. Some extra-weak type mixed Φ-inequalities. In this section, we shall establish some extra-weak type mixed Φ-inequalities. The corresponding weak type Φ-inequalities are given in [10] (also cf. [8] and [9]). However, in previous articles the results are established under the assumption that all weight functions are positive and finite almost everywhere. For the Hardy–Littlewood maximal operator, this restriction implies that there is no solution for the weak type (p,q) inequality when $p < q$ (see [13]), and we are able to avoid it by using some ideas of the proof of Theorem 2. The theorems in this section not only present some new results, but also show how to omit this restriction in previous work.

The first result concerns the Fefferman–Stein type fractional maximal operator on homogeneous-type spaces, which includes the well known Hardy–Littlewood maximal function and fractional maximal operator on $\mathbb{R}^n$.

In what follows (X, d, μ) will denote a homogeneous-type space in Coifman–Weiss' sense. This is a set X with a quasimetric d and a measure μ. The constant associated with the quasimetric d will be denoted by K. It is assumed that every ball $B = B(x,r) = \{y \in X : d(x,y) < r\}$ is measurable, and μ satisfies the doubling condition

$$\mu(2B) \le C\mu(B)$$

for all balls $B = B(x,r)$, where $2B = B(x, 2r)$. Generally, for $\delta > 0$, $\delta B = \delta B(x,r) = B(x, \delta r)$. We will use A to denote the constant such that

$$\mu(K(2K+1)B) \le A\mu(B)$$

for all balls B. The details about homogeneous-type spaces can be found in [4].

Let $X_+ = \{(x,t) : x \in X,\ t \ge 0\}$, $\widehat{B} = \widehat{B}(x,r) = \{(y,t) : y \in B(x,r),\ 0 \le t < r\}$ and for $\delta > 0$, $\delta\widehat{B} = \delta\widehat{B}(x,r) = \widehat{B}(x, \delta r)$.

Given $\alpha \in [0,1)$, the Fefferman–Stein type fractional maximal operator M_α is defined by

$$M_\alpha f(x,t) = \sup_{t \le r} \frac{1}{\mu(B(x,r))^{1-\alpha}} \int_{B(x,r)} |f(y)|\,d\mu(y),$$

which maps locally integrable functions on X into functions on X_+. In the formula above, if $r = 0$, let

$$\frac{1}{\mu(B(x,r))^{1-\alpha}} \int_{B(x,r)} |f(y)|\,d\mu(y) = 0.$$

THEOREM 4. *Suppose (X, d, μ) is a homogeneous-type space, and ϱ is a measure on X_+. Let u and v be weight functions on X. Suppose Φ_1, Φ_2 are Young functions with $\Phi_1 \ll \Phi_2$. The following statements are equivalent:*

(i) *there exists a constant $C > 0$ such that the extra-weak type inequality*

$$\varrho(\{(x,t) : M_\alpha f(x,t) > \lambda\}) \le \Phi_2\Phi_1^{-1}\Big(\int_X \Phi_1\Big(\frac{Cfu}{\lambda}\Big)v\,d\mu\Big) \tag{44}$$

holds for all measurable $f \ge 0$ and all $\lambda > 0$;

(ii) *there exists a constant* $N > 0$ *such that*

$$\varrho(\widehat{B}) \le \Phi_2\Phi_1^{-1}\Big(\int_B \Phi_1\Big(\frac{Nfu}{m_B^\alpha(f)}\Big)v\,d\mu\Big) \tag{45}$$

holds for all locally integrable $f \ge 0$ *and balls* B, *where*

$$m_B^\alpha(f) = \frac{1}{\mu(B)^{1-\alpha}}\int_B f(y)\,d\mu(y);$$

(iii) *there exists a constant* $D > 0$ *such that*

$$\int_B \Psi\Big(\frac{\Phi_1\Phi_2^{-1}(\varrho(\widehat{B}))}{D\mu(B)^{1-\alpha}u(x)v(x)}\Big)v(x)\,d\mu(x) \le \Phi_1\Phi_2^{-1}(\varrho(\widehat{B})) < \infty \tag{46}$$

holds for all balls B.

Moreover, for the best constants C, N *and* D, *we have*

$$N \le AC, \quad D \le N \quad \text{and} \quad C \le 2MD,$$

where M *is a positive constant such that*

$$\mu(5K^2B)^{1-\alpha} \le M\mu(B)^{1-\alpha}$$

holds for all balls B.

Proof. Since $B(x,r) \subset B(y,2Kr) \subset B(x,K(2K+1)r)$ for every $y \in B(x,r)$, we have

$$\widehat{B} \subset \Big\{(x,t) : M_\alpha f > \frac{1}{A}m_B^\alpha(f)\Big\}$$

for all balls B and $f \ge 0$. Then it is obvious that (45) follows from (44) and $N \le AC$.

(45)⇒(46). The process is similar to that used in Theorem 2 (cf. the proof of (15)⇒(16)), and we just give the sketch.

Suppose $\Phi_1\Phi_2^{-1}(\varrho(B_0)) = \infty$ for a ball B_0. For every $B \supset B_0$ and given $H > 0$, suppose $\mu(S_H) = \mu(\{x \in B : u(x)v(x) < H\}) > 0$. On the one hand, if $\mu(\{x \in S_H : u(x) < \infty\}) > 0$, then we can choose a subset $S \subset S_H$ such that $\mu(S) > 0$ and $u(x) \le L$ on S for some L large enough. On setting $f = \chi_S$, it follows from (45) that

$$\begin{aligned}\infty \le \Phi_1\Phi_2^{-1}(\varrho(\widehat{B})) &\le \int_S \Phi_1\Big(\frac{N\mu(B)^{1-\alpha}u(x)}{\mu(S)}\Big)v(x)\,d\mu \\ &\le \Phi_1\Big(\frac{N\mu(B)^{1-\alpha}H}{\mu(S)}\Big)\mu(S) + \Phi_1\Big(\frac{N\mu(B)^{1-\alpha}L}{\mu(S)}\Big)\mu(S) \quad \text{(cf. (30))}.\end{aligned}$$

This is a contradiction. On the other hand, if $\mu(\{x \in S_H : u(x) < \infty\}) = 0$, then $v(x) = 0$ on S_H. By using the same method as previously, we obtain a contradiction. Then we conclude that $\mu(\{x \in B : u(x)v(x) < H\}) = 0$ for every $B \supset B_0$ and $H > 0$. Therefore we have $u(x)v(x) = \infty$ μ-a.e. on X, and this is a trivial case we should exclude. The right inequality in (46) is proved.

For a given ball B, write

$$B_n = \{x \in B : 1/n \le u(x), v(x) \le n\} \quad \text{for every positive integer } n;$$

$$h(x) = \frac{\Phi_1\Phi_2^{-1}(\varrho(\widehat{B}))}{\eta N\mu(B)^{1-\alpha}u(x)v(x)}\chi_{B_n}(x) \quad \text{for } \eta > 1;$$

$$f(x) = D\Psi_1(h(x))/u(x)$$

and

$$I = \int_B \Psi_1(h(x))v(x)\,d\mu(x) = \frac{\Phi_1\Phi_2^{-1}(\varrho(\widehat{B}))}{\eta N}m_B^\alpha(f).$$

We can prove $m_B^\alpha(f) \le N$ by the same argument as in Theorem 2 (proof of $m_g^+(f) \le C\varepsilon$ in the implication (15)⇒(16)). Then we obtain

$$I \le \frac{\Phi_1\Phi_2^{-1}(\varrho(\widehat{B}))}{\eta},$$

and the left inequality in (46) with $D \le N$ follows.

(46)⇒(44). For arbitrary $R > 0$, let

$$M_\alpha^R f(x,t) = \sup_{t \le r \le R}\frac{1}{\mu(B(x,r))^{1-\alpha}}\int_{B(x,r)}|f(y)|\,d\mu(y).$$

It is sufficient to prove (44) for M_α^R with constant $C \le 2MD$ which is independent of R.

Fix a $x_0 \in X$, let $B_m = B(x_0,m)$, $E_\lambda = \{(x,t) : M_\alpha^R f(x,t) > \lambda\}$ for given $f \ge 0$ and $\lambda > 0$ and $E_\lambda^m = E_\lambda \cap B_m$. It follows from a well-known decomposition lemma in X_+ (see [7] or [10]) that there exists a sequence of balls $\{B_j\}$ which are pairwise disjoint and satisfy

$$E_\lambda^m \subset \bigcup 5K^2\widehat{B}_j, \tag{47}$$

$$m_{B_j}^\alpha(f) > \lambda. \tag{48}$$

It follows from (48), Young inequality (9) and (46) that

$$\begin{aligned}\lambda &\le \frac{1}{\mu(B_j)^{1-\alpha}}\int_{B_j} f\,d\mu \\ &\le \frac{D}{\Phi_1\Phi_2^{-1}(\varrho(5K^2\widehat{B}_j))}\Big(\int_{B_j}\Phi_1(Mfu)v\,d\mu + \int_{B_j}\Psi_1\Big(\frac{\Phi_1\Phi_2^{-1}(\varrho(5K^2\widehat{B}))}{DM\mu(B_j)^{1-\alpha}uv}\Big)v\,d\mu\Big) \\ &\le D\Big(\frac{\int_{B_j}\Phi_1(Mfu)v\,d\mu}{\Phi_1\Phi_2^{-1}(\varrho(5K^2\widehat{B}_j))} + 1\Big).\end{aligned}$$

Put $\lambda = 2D$. We have

$$\varrho(5K^2\widehat{B}_j) \le \Phi_2\Phi^{-1}\Big(\int_{B_j} \Phi_1(Mf(x)u(x))v(x)\,d\mu\Big) \quad \text{for every } j.$$

From (47) and $\Phi_1 \ll \Phi_2$, we obtain

$$\varrho(E_{2D}^m) \le \Phi_2\Phi_1^{-1}\Big(\sum \int_{B_j} \Phi_1(Mfu)v\,d\mu\Big) \le \Phi_2\Phi_1^{-1}\Big(\int_X \Phi_1(Mfu)v\,d\mu\Big).$$

Letting $m \to \infty$, we finish the proof of Theorem 4.

Finally, we consider the extra-weak type mixed Φ-inequality for the Hardy-type operator defined by

$$Tf(x) = \int_0^x K(x,y)f(y)\,d\mu(y), \qquad x \ge 0, \tag{49}$$

where μ is a regular and nonatomic positive measure on $(0,\infty)$, and the kernel $K(x,y)$ on $\mathbb{R}^+ \times \mathbb{R}^+$ satisfies

(i) $K(x,y) > 0$ if $x > y$;
(ii) $K(x,y)$ is nondecreasing in x and nonincreasing in y (see [1]).

THEOREM 5. *Suppose T is defined in* (49), *and Φ_1, Φ_2 are Young functions. Let w, u and v be weight functions. Then the following statements are equivalent*:

(i) *there exists a constant $C > 0$ such that the extra-weak type inequality*

$$|\{x : |Tf(x)| > \lambda\}|_{w\,d\mu} \le \Phi_2\Phi_1^{-1}\Big(\int_0^\infty \Phi_1\Big(\frac{C|f|u}{\lambda}\Big)v\,d\mu\Big) \tag{50}$$

holds for all measurable f and $\lambda > 0$;

(ii) *there exists a constant $D > 0$ such that*

$$\int_0^r \Psi_1\Big(\frac{K(r,y)\Phi_1\Phi_2^{-1}(|(r,\infty)|_{w\,d\mu})}{Dv(y)u(y)}\Big)v(y)\,d\mu(y) \le \Phi_1\Phi_2^{-1}(|(r,\infty)|_{w\,d\mu}) < \infty \tag{51}$$

holds for all $r > 0$.

Moreover, for the best constants C, D, we have

$$D \le C \le 2D.$$

The corresponding four-weight weak-type inequality, like (15) in Theorem 2, has been characterized in [2] and [10] independently. The necessary modifications of the proof of the weak-type inequality in order to obtain the extra weak-type inequality are demonstrated in the proof of Theorem 3 (also by comparing Theorem 4 with Theorem 3 in [10]). Therefore we shall omit the proof of Theorem 5.

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Algebras of real analytic functions: Homomorphisms and bounding sets

by

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Abstract. This article deals with bounding sets in real Banach spaces E with respect to the functions in $\mathcal{A}(E)$, the algebra of real analytic functions on E, as well as to various subalgebras of $\mathcal{A}(E)$. These bounding sets are shown to be relatively weakly compact and the question whether they are always relatively compact in the norm topology is reduced to the study of the action on the set of unit vectors in l_∞ of the corresponding functions in $\mathcal{A}(l_\infty)$. These results are achieved by studying the homomorphisms on the function algebras in question, an idea that is also reversed in order to obtain new results for the set of homomorphisms on these algebras.

In this paper we are interested in subsets of a real Banach space on which different classes of functions are bounded. In [3] it is shown that if a subset B of a real Banach space E has the property that each C^∞-function on E is bounded on B, then B is relatively compact. As the continuous polynomials $\mathcal{P}(E)$ on E are bounded on bounded sets the focus of interest in this article is on the algebras between $\mathcal{P}(E)$ and $\mathcal{A}(E)$, the algebra of real analytic functions on E. Let $A(E)$ denote such an algebra. Then we say that a set in E is A-bounding if all functions in $A(E)$ are bounded on it.

A main theme in this paper is the close interplay between the homomorphisms on $A(E)$ and the A-bounding sets. Using appropriate properties of the homomorphisms on the algebra $\mathcal{R}(E)$ of rational forms of elements in $\mathcal{P}(E)$, we deduce that the $\mathcal{R}$-bounding sets are always relatively compact in the weak topology of the Banach space E. With this result at hand, we show that the problem of the $\mathcal{A}$-bounding sets being relatively compact in arbitrary Banach spaces E is reduced to the study of the behaviour of the real analytic functions on l_∞ acting on the set of unit vectors in l_∞. Further we show that the $\mathcal{R}$-bounding and the relatively norm compact

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