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On weakly A-harmonic tensors

by

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Abstract. We study very weak solutions of an A-harmonic equation to show that they are in fact the usual solutions.

0. Introduction. Let $\Lambda^l = \Lambda^l(\mathbb{R}^n)$ denote the linear space of l-covectors in $\mathbb{R}^n$, $l = 1, \ldots, n$. This is an inner product space of dimension $\binom{n}{l}$. A differential form of degree l on $\Omega \subset \mathbb{R}^n$ is simply a function or Schwarz distribution on Ω with values in Λ^l. We shall consider a nonlinear mapping

$$A : \Omega \times \Lambda^l(\mathbb{R}^n) \to \Lambda^l(\mathbb{R}^n)$$

such that

$$\text{(H1)} \quad |A(x,\xi) - A(x,\zeta)| \le b|\xi - \zeta|(|\xi| + |\zeta|)^{p-2},$$
$$\text{(H2)} \quad \langle A(x,\xi) - A(x,\zeta) \mid \xi - \zeta\rangle \ge a|\xi - \zeta|^2(|\xi| + |\zeta|)^{p-2},$$
$$\text{(H3)} \quad A(x,\lambda\zeta) = |\lambda|^{p-2}\lambda A(x,\zeta),$$

for almost every $x \in \Omega$, $\lambda \in \mathbb{R}$, and all $\xi, \zeta \in \Lambda^l(\mathbb{R}^n)$. The exponent $p > 1$ will determine the natural Sobolev class, denoted by $W^p_d(\Omega, \Lambda^{l-1})$, in which to consider the A-harmonic equation

$$d^*A(x, du) = 0 \tag{0.1}$$

(see §1 for the definition of $W^p_d(\Omega, \Lambda^{l-1})$ and other Sobolev classes of differential forms). This equation means that

$$\int_\Omega \langle A(x, du) \mid d\phi\rangle = 0 \tag{0.2}$$

for every $\phi \in W^p_d(\Omega, \Lambda^{l-1})$ with compact support. Such differential forms u will be referred to as A-harmonic tensors.

Definition 0.1. A differential form $u \in W^s_{d,\mathrm{loc}}(\Omega, \Lambda^{l-1})$, $s \ge \max\{1, p-1\}$, is called a *weakly A-harmonic tensor* if it satisfies equation (0.1)

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in the distributional sense, that is, the integral identity (0.2) holds for all $\phi \in W_d^{s/(s-p+1)}(\Omega, \Lambda^{l-1})$ with compact support.

Our main result is the following regularity theorem:

THEOREM 1. *There exist exponents $p_1 = p_1(n,p) < p$ and $p_2 = p_2(n,p) > p$ such that if $u \in W^{p_1}_{d,\mathrm{loc}}(\Omega, \Lambda^{l-1})$ is weakly A-harmonic, then $u \in W^{p_2}_{d,\mathrm{loc}}(\Omega, \Lambda^{l-1})$. In particular, $u \in W^{p}_{d,\mathrm{loc}}(\Omega, \Lambda^{l-1})$ is A-harmonic in the usual sense.*

Partial differential equations for differential forms are of growing interest due to the recent developments in nonlinear elasticity theory and quasiconformal mappings. Many elasticity results involving determinants are better understood if they are formulated in terms of differential forms. The key tools are the Hodge decomposition and a Poincaré-type inequality as developed in [IL]. The p-harmonic equation $d^*(|du|^{p-2}du) = 0$ plays an important role in this study. For example, to every conformal mapping $f = (f^1, \ldots, f^n) : \Omega \to \mathbb{R}^n$ there corresponds a p-harmonic tensor u defined by

$$du = df^1 \wedge \ldots \wedge df^l, \qquad p = \frac{n}{l}.$$

A systematic study of nonlinear equations for differential forms has originated from the work by L. M. Sibner and R. B. Sibner [SS] and K. Uhlenbeck [U]. Just to mention that, using the well known technique of Nash–De Giorgi–Moser, K. Uhlenbeck derived the C^α-regularity for p-harmonic tensors.

Our study has arisen from the following general question: what is the minimal degree of integrability of the gradient of a very weak solution of an elliptic equation which still guarantees that the solution is in fact the usual one? Theorem 1 can be regarded as dual to the higher integrability result of F. W. Gehring [G] and Elcrat–Meyers [EM]. In the scalar case, this problem has been previously solved by T. Iwaniec and C. Sbordone [IS] and later by J. Lewis [L].

Our result for differential forms is particularly applicable to quasiconformal mappings (see [I2], [I3], [IM1], [IM2] and [IMNS]).

1. Notation and preliminary results. Throughout we use the notation of [IL]. For the sake of completeness we list basic notions of exterior calculus. Denote by $\Lambda^l = \Lambda^l(\mathbb{R}^n)$ the space of l-covectors in $\mathbb{R}^n$. For $l = 0$ we put $\Lambda^0(\mathbb{R}^n) = \mathbb{R}^n$. Also, $\Lambda^l = \Lambda^l(\mathbb{R}^n) = 0$ if $l < 0$ or $l > n$. Then the direct sum $\Lambda(\mathbb{R}^n) = \bigoplus_{l=0}^n \Lambda^l(\mathbb{R}^n)$ is an exterior algebra with respect to the wedge product $\wedge$. We denote the Hodge star operator by $*$,

$$* : \Lambda^l(\mathbb{R}^n) \to \Lambda^{n-l}(\mathbb{R}^n).$$

By definition, $*1$ equals the usual volume element in $\mathbb{R}^n$. It induces, in a standard way, a scalar product between forms of the same degree, namely $*1\langle\alpha \mid \beta\rangle = \alpha \wedge *\beta$. Then $*$ becomes an isometry between Λ^l and Λ^{n-l}.

Let $\Omega \subset \mathbb{R}^n$ be an open subset with smooth boundary. A *differential l-form* ω on Ω is a locally integrable function or a Schwarz distribution on Ω with values in $\Lambda(\mathbb{R}^n)$. If $x_1, \ldots, x_n$ denote the coordinate functions in $\mathbb{R}^n$, then the natural generators for the algebra of differential forms are the differentials $dx_1, \ldots, dx_n$. Thus each $\alpha : \Omega \to \Lambda^l(\mathbb{R}^n)$ can be written as

$$\alpha(x) = \sum_{1 \le i_1 < \ldots < i_l \le n} a_{(i_1,\ldots,i_l)} dx_{i_1} \wedge \ldots \wedge dx_{i_l}$$

where $a_{(i_1,\ldots,i_l)}$ are either functions or distributions.

There is a linear operator $d : \mathcal{D}'(\Omega, \Lambda^l) \to \mathcal{D}'(\Omega, \Lambda^{l+1})$ (called the *exterior derivative*) uniquely determined by the following conditions:

(i) for $p = 0$, df is the differential of f,

(ii) for $\alpha \in \mathcal{D}'(\Omega, \Lambda^l)$ and $\beta \in \mathcal{D}'(\Omega, \Lambda^k)$, we have

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^l \alpha \wedge d\beta,$$

(iii) $d(d\alpha) = 0$.

The elements of the kernel of $d : \mathcal{D}'(\Omega, \Lambda^l) \to \mathcal{D}'(\Omega, \Lambda^{l+1})$ are called *closed l-forms* and those in the image of $d : \mathcal{D}'(\Omega, \Lambda^{l-1}) \to \mathcal{D}'(\Omega, \Lambda^l)$ are the *exact l-forms*.

The formal adjoint operator, also known as the *Hodge codifferential*, is given by

$$d^* = (-I)^{nl+1} * d * : \mathcal{D}'(\Omega, \Lambda^{l+1}) \to \mathcal{D}'(\Omega, \Lambda^l).$$

The forms in the image of $d^* : \mathcal{D}'(\Omega, \Lambda^{l+1}) \to \mathcal{D}'(\Omega, \Lambda^l)$ are called *coexact l-forms*.

We denote by $C^\infty(\overline{\Omega}, \Lambda^l)$ the class of infinitely differentiable l-forms on $\overline{\Omega} \subset \mathbb{R}^n$. Since Ω is a smooth domain, near a boundary point of Ω we can consider a local coordinate system such that $x_n = 0$ on $\partial\Omega$ and the x_n-curve is orthogonal to the remaining x_i-curves. In these coordinates every differential form $\omega \in C^\infty(\overline{\Omega}, \Lambda^l)$ can be expressed as $\omega(x) = \omega_{\mathrm{T}}(x) + \omega_{\mathrm{N}}(x)$, where

$$\omega_{\mathrm{T}}(x) = \sum_{1 \le i_1 < \ldots < i_l < n} \omega_{i_1,\ldots,i_l}(x) dx_{i_1} \wedge \ldots \wedge dx_{i_l},$$

$$\omega_{\mathrm{N}}(x) = \sum_{1 \le i_1 < \ldots < i_l = n} \omega_{i_1,\ldots,i_l}(x) dx_{i_1} \wedge \ldots \wedge dx_{i_l},$$

These forms are called, respectively, the *tangential part* and the *normal part* of ω. Now, the duality between d and d^* is established by the following identity:

$$\int_\Omega \langle v \mid du\rangle = \int_\Omega \langle d^*v \mid u\rangle \quad \forall u\in C^\infty(\overline{\Omega},\Lambda^l),\ v\in C^\infty(\overline{\Omega},\Lambda^{l+1})$$

provided $u_{\mathrm T}=0$ or $v_{\mathrm N}=0$.

We make a brief list of spaces of differential forms:

- $L^p(\Omega,\Lambda^l)$ — the space of differential forms with coefficients in $L^p(\Omega)$;
- $L^p_1(\Omega,\Lambda^l)$ — the space of differential forms ω such that $\nabla\omega$ is a regular distribution of class $L^p(\Omega,\Lambda^l)$;
- $W^{1,p}(\Omega,\Lambda^l)$ — the Sobolev space of l-forms defined by $L^p(\Omega,\Lambda^l)\cap L^p_1(\Omega,\Lambda^l)$;
- $W^p_d(\Omega,\Lambda^l)$ — the space of l-forms ω such that $d\omega\in L^p(\Omega,\Lambda^{l+1})$;
- $W^p_{d^*}(\Omega,\Lambda^l)$ — the space of l-forms ω such that $d^*\omega\in L^p(\Omega,\Lambda^{l-1})$;
- $W^p_{d,0}(\Omega,\Lambda^l)$ — the completion of $C^\infty(\overline{\Omega},\Lambda^l)$ in W^p_d with respect to the norm

$$\|\omega\|_{p,\Omega}+\|d\omega\|_{p,\Omega}.$$

The l-forms $u\in W^p_{d,0}(\Omega,\Lambda^l)$ are said to have *vanishing tangential component* at the boundary of Ω in the distributional sense. Analogously, we define $W^p_{d^*,0}(\Omega,\Lambda^l)$ to be the class of l-forms with vanishing normal component on the boundary of Ω.

Now, we can extend the duality between d and d^*. For p, q Hölder conjugate and $u\in W^p_{d,0}(\Omega,\Lambda^l)$, $v\in W^q_{d^*,0}(\Omega,\Lambda^l)$, by an approximation argument, we have

$$\int_\Omega \langle v\mid du\rangle=\int_\Omega\langle d^*v\mid u\rangle.$$

Let D be a convex bounded domain in $\mathbb{R}^n$. Suppose $\omega\in L^p_{\mathrm{loc}}(D,\Lambda^l)$ is such that $d\omega\in L^p_{\mathrm{loc}}(D,\Lambda^{l+1})$, $1\le p<\infty$, $l=1,\dots,n$. Then it is possible to construct an integral operator T such that $d(T\omega)$ is a regular distribution of class $L^p_{\mathrm{loc}}(D,\Lambda^l)$ and $\omega=T(d\omega)+d(T\omega)$. For more details concerning the homotopy operators we refer to [IL]. It turns out that $T:L^p(D,\Lambda^l)\to W^{1,p}(D,\Lambda^{l-1})$ is a bounded operator whose norm is estimated by

$$\|T\omega\|_{W^{1,p}(D)}\le A_p(n)\mu(D)\|\omega\|_{p,D}, \tag{1.1}$$

where the constant $\mu(D)$ measures the flatness of the domain D. In particular,

$$\mu(\lambda D)=\mu(D)\quad\text{for all positive }\lambda.$$

In order to formulate an analogue of the Poincaré–Sobolev inequality for differential forms we define an l-form $\omega_D\in\mathcal{D}'(D,\Lambda^l)$ by

$$\omega_D=\begin{cases}|D|^{-1}\int_D\omega(y)\,dy & \text{if } l=0,\\ d(T\omega) & \text{if } l=1,\dots,n.\end{cases}$$

Clearly ω_D is a closed form and for $l>0$, ω_D is an exact form. Furthermore, as a consequence of (1.1), we have:

PROPOSITION 1. *Let $\omega\in\mathcal{D}'(D,\Lambda^l)$ be such that $d\omega\in L^p(D,\Lambda^{l+1})$, $1<p<\infty$. Then $\omega-\omega_D$ is in $W^{1,p}(D,\Lambda^l)$ and*

$$\|\omega-\omega_D\|_{W^{1,p}(D)}\le A_p(n)\|d\omega\|_{p,D} \tag{1.2}$$

for D a cube or a ball in $\mathbb{R}^n$.

Now, a Poincaré-type inequality for differential forms reads as follows:

COROLLARY 1 ([IL]). *Suppose that $\omega\in\mathcal{D}'(D,\Lambda^l)$ and $d\omega\in L^p(D,\Lambda^{l+1})$, $l=0,1,\dots,n$ and $1<p<n$. Then $\omega-\omega_D$ is in $L^{np/(n-p)}(D,\Lambda^l)$ and we have the following uniform estimate:*

$$\Big(\int_D|\omega-\omega_D|^{np/(n-p)}\Big)^{(n-p)/(np)}\le c_p(n)\Big(\int_D|d\omega|^p\Big)^{1/p} \tag{1.3}$$

for D a cube or a ball in $\mathbb{R}^n$.

COROLLARY 2. *Let $\omega\in\mathcal{D}'(D,\Lambda^l)$ be such that $d^*\omega\in L^p(D,\Lambda^{l-1})$, $l=0,1,\dots,n$. Then there exists a coclosed form $\omega^*_D\in L^p(D,\Lambda^{l-1})$ such that*

$$\Big(\int_D|\omega-\omega^*_D|^{np/(n-p)}\Big)^{(n-p)/(np)}\le c_p(n)\Big(\int_D|d^*\omega|^p\Big)^{1/p},\quad 1<p<n. \tag{1.4}$$

Note that ω^*_D is in fact a coexact form if $l<n$.

Proof. We use the fact that the Hodge star operator is an isometry. In particular, we have

$$\|\omega-\omega^*_D\|_{np/(n-p)}=\|(*\omega)-*(\omega^*_D)\|_{np/(n-p)}.$$

Here, the l-form ω^*_D is defined by $*(\omega^*_D)=(*\omega)_D$. Accordingly,

$$d(*(\omega^*_D))=d((*\omega)_D)=0.$$

Hence, $d^*(\omega^*_D)=(-1)^{nl+1}*d*\omega^*_D=0$.

Finally, applying Corollary 1 with $*\omega$ in place of ω yields

$$\|(*\omega)-(*\omega)_D\|_{np/(n-p)}\le C(n,p)\|d(*\omega)\|_p=C(n)\|d^*\omega\|_p.$$

COROLLARY 3. *Let D be a cube or a ball, and $u\in L^s(D,\Lambda^l)$ with $du\in L^s(D,\Lambda^{l+1})$. Then*

$$\frac{1}{\operatorname{diam}D}\Big(\fint_D|u-u_D|^s\Big)^{1/s}\le c(n,s)\Big(\fint_D|du|^{ns/(n+s-1)}\Big)^{(n+s-1)/(ns)}. \tag{1.5}$$

(We denote by $\fint_D$ the integral mean over D.)

Proof. We consider two cases:

Case 1: Suppose $1 < s \le n/(n-1)$. We notice that

$$\frac{ns}{n-1} > \max\left\{s, \frac{n}{n-1}\right\},$$

so we can apply Hölder's inequality and then Corollary 1 to obtain

$$\Big(\fint_D |u-u_D|^s\Big)^{1/s} \le \Big(\fint_D |u-u_D|^{ns/(n-1)}\Big)^{(n-1)/(ns)} \le c(\operatorname{diam} D)\Big(\fint_D |du|^{ns/(n+s-1)}\Big)^{(n+s-1)/(ns)}.$$

Case 2: Suppose $s > n/(n-1)$. Then, by Corollary 1,

$$\Big(\fint_D |u-u_D|^s\Big)^{1/s} \le c\Big(\fint_D |du|^{ns/(n+s)}\Big)^{(n+s)/(ns)} = c(\operatorname{diam} D)^{(n+s)/s}\Big(\fint_D |du|^{ns/(n+s)}\Big)^{(n+s)/(ns)} \le c(\operatorname{diam} D)^{(n+s)/s}\Big(\fint_D |du|^{ns/(n+s-1)}\Big)^{(n+s-1)/(ns)}.$$

The assertion is now immediate.

COROLLARY 4. *Let D be a cube or a ball, and $u \in L^s(D,\Lambda^l)$ with $d^*u \in L^s(D,\Lambda^{l-1})$. Then*

$$\frac{1}{\operatorname{diam} D}\Big(\fint_D |u-u_D^*|^s\Big)^{1/s} \le c(n,s)\Big(\fint_D |d^*u|^{ns/(n+s-1)}\Big)^{(n+s-1)/(ns)}.$$

From now on, Ω stands for a cube or a ball which is a regular domain (see [IS] for the definition and properties).

Given $\omega \in L^p(\Omega,\Lambda^l(\mathbb{R}^n))$, there exist differential forms $\alpha \in \operatorname{Ker} d^* \cap L_1^p$ and $\beta \in \operatorname{Ker} d \cap L_1^p$ such that

$$\omega = d\alpha + d^*\beta, \quad \alpha_T = 0. \tag{1.6}$$

The forms α and β are unique and satisfy the uniform estimate

$$\|\alpha\|_{L_1^p(\Omega)} + \|\beta\|_{L_1^p(\Omega)} \le c_p(n)\|\omega\|_p \tag{1.7}$$

(see [I1–3], [IM1–2] and references there). This is what we call the *Hodge decomposition* of ω. In what follows we will be concerned with the question of stability in the Hodge decomposition.

Given $v \in W_{d,0}^s(\Omega,\Lambda^{l-1})$, consider a nonlinear perturbation of the exact form $dv \in L^s(\Omega,\Lambda^l)$, say $\omega = |dv|^\varepsilon dv$. With the aid of the Hodge decomposition we split ω as $\omega = d\phi + d^*\psi$, where $\phi_T = 0$. We wish to establish a precise relation between dv, $d\phi$ and $d^*\psi$ as ε tends to zero.

An estimate for $d^*\psi$ in terms of dv was presented in [I2], using maximal inequalities and properties of harmonic functions. More recently, T. Iwaniec and C. Sbordone [IS] have found a new approach to this problem by using complex interpolation technique. They have obtained the following result.

Let (X,μ) be a measure space and let E be a separable complex Hilbert space. Consider a bounded linear operator $T : L^r(X,E) \to L^r(X,E)$ for all $r \in [r_1, r_2]$, where $1 \le r_1 < r_2 \le \infty$. Denote its norm by $\|T\|_r$.

THEOREM 2. *Suppose that $r/r_2 \le 1+\varepsilon \le r/r_1$. Then*

$$\|T(|f|^\varepsilon f)\|_{r/(1+\varepsilon)} \le K|\varepsilon|\,\|f\|_r^{1+\varepsilon}$$

for each $f \in L^r(X,E) \cap \operatorname{Ker} T$, where

$$K = \frac{2r(r_2-r_1)}{(r-r_1)(r-r_2)}(\|T\|_{r_1} + \|T\|_{r_2}).$$

This result can be viewed as a part of a general theory of interpolation and nonlinear commutators due to R. Rochberg and G. Weiss [RW] (see also [M1–2], [MS], and [CLMS], [JRW], [R]).

2. The nonhomogeneous A-harmonic equation. Let Ω be a ball or a cube in $\mathbb{R}^n$. Given $g \in L^s(\Omega,\Lambda^l)$ and $f \in L^{s/(p-1)}(\Omega,\Lambda^l)$ where $s \ge \max\{1, p-1\}$, we consider the nonhomogeneous equation

$$d^*(A(x, g+dv)) = d^*f \quad \text{for } v \in W_{d,0}^s(\Omega,\Lambda^{l-1}). \tag{2.1}$$

This equation, referred to as the *A-harmonic equation*, has to be understood in the distributional sense, that is,

$$\int_\Omega \langle A(x,g+dv) \mid d\phi\rangle = \int_\Omega \langle f \mid d\phi\rangle \quad \text{for each } \phi \in W_{d,0}^{s/(s-p+1)}(\Omega,\Lambda^{l-1}).$$

We emphasize that the "natural" space in which to consider the A-harmonic equation is $W_{d,0}^p$, where the Sobolev exponent p is the one appearing in (H1)–(H3). In this case $s = p$ and one could apply Browder's methods of monotone operators. However, we are interested in weakly A-harmonic tensors of class $W_{d,0}^s$, with s below the natural exponent, $s < p$. The trouble with weakly A-harmonic tensors is that the standard choice of test functions ϕ fails. This technique, used by Stampacchia [S] in the linear case, has also been extended to nonlinear equations [BG1–2], but only with the natural exponent (see also [BMS], [St] for the nonisotropic case).

In our approach the key tools are the Hodge decomposition (applied to a nonlinear perturbation of an exact form) and the Poincaré–Sobolev type inequality. We start with a lemma:

LEMMA 1. *Given the A-harmonic equation* (2.1) *there exists* $\varepsilon = \varepsilon(n,p,a,b) \in (0,p-1)$ *such that*

$$\int_\Omega |dv|^s \le c \int_\Omega (|g|^s + |f|^{s/(p-1)}) \tag{2.2}$$

for $p-\varepsilon \le s \le p+\varepsilon$.

Proof. Let us begin with the Hodge decomposition

$$|dv|^{s-p}dv = d\phi + h.$$

We apply Theorem 2 to the measure space (Ω, dx) and $E = \Lambda^l(\mathbb{R}^n)$. The operator T is then defined by $T\omega = d^*\beta$, where $\omega = d\alpha + d^*\beta$ (see the Hodge decomposition (1.6)). In view of (1.7), $T : L^r(\Omega,\Lambda) \to L^r(\Omega,\Lambda)$ is a bounded linear operator for all $1 < r < \infty$.

It follows from the uniqueness of the Hodge decomposition that the kernel of T consists of the exact forms from $dW^r_{d,0}(\Omega,\Lambda^{l-1})$, while the range of T consists of the coexact forms. In view of Theorem 2, we have

$$\|h\|_{s/(s-p+1)} = \|T(|dv|^{s-p}dv)\|_{s/(s-p+1)} \le K|s-p|\|dv\|_s^{s-p+1}. \tag{2.3}$$

We can use $d\phi = h - |dv|^{s-p}dv$ as a test form for equation (2.1) to get

$$\int_\Omega \langle A(x,g+dv) \mid |dv|^{s-p}dv - h\rangle = \int_\Omega \langle f \mid |dv|^{s-p}dv - h\rangle,$$

$$\int_\Omega \langle A(x,dv) \mid |dv|^{s-p}dv\rangle = \int_\Omega \langle A(x,dv) - A(x,g+dv) \mid |dv|^{s-p}dv\rangle$$
$$+ \int_\Omega \langle A(x,g+dv) \mid h\rangle + \int_\Omega \langle f \mid d\phi\rangle.$$

Using the monotonicity inequality (H2) and the Lipschitz type inequality (H1), we obtain

$$a\int_\Omega |dv|^s$$
$$\le b\int_\Omega |g|(|dv|+|g+dv|)^{p-2}|dv|^{s-p+1} + b\int_\Omega |g+dv|^{p-1}|h| + \int_\Omega |f||d\phi|$$
$$\le b\int_\Omega |g|(|dv|+|g+dv|)^{s-1}$$
$$+ b\Big(\int_\Omega |g+dv|^s\Big)^{(p-1)/s}\|h\|_{s/(s-p+1)} + \|f\|_{s/(p-1)}\|d\phi\|_{s/(s-p+1)}.$$

Now, inequality (2.3) yields

$$a\int_\Omega |dv|^s \le b\Big(\int_\Omega |g|^s\Big)^{1/s}\Big[\int_\Omega (|dv|+|g+dv|)^s\Big]^{(s-1)/s}$$
$$+ bK|s-p|\|dv\|_s^{s-p+1}\Big(\int_\Omega |g+dv|^s\Big)^{(p-1)/s}$$
$$+ b\|f\|_{s/(p-1)}(1+K|s-p|)\|dv\|_s^{s-p+1}.$$

Hence, we arrive at the estimate

$$\int_\Omega |dv|^s \le C_1\|g\|_s^s + C_2\|g\|_s\|dv\|_s^{s-1} + C_3|s-p|\|g\|_s^{p-1}\|dv\|_s^{s-p+1}$$
$$+ C_4|s-p|\|dv\|_s^s + C_5\|f\|_{s/(p-1)}\|dv\|_s^{s-p+1}.$$

We now determine ε small enough to ensure that $C_4|s-p| \le 1/2$ for $p-\varepsilon < s < p+\varepsilon$. This, together with a routine use of Young's inequality, yields

$$\int_\Omega |dv|^s \le C_1\|g\|_s^s + C_2\tau\|dv\|_s^s + C_2(\tau)\|g\|_s^s$$
$$+ C_3|s-p|\eta\|dv\|_s^s + C_3(\eta)|s-p|\|g\|_s^s$$
$$+ C_5\gamma\|dv\|_s^s + C_5(\gamma)\|f\|_{s/(p-1)}^{s/(p-1)},$$

where τ, η, γ are arbitrary positive numbers. It is clear that one can choose τ,η and γ to satisfy the inequality

$$C_2\tau + 2C_3\eta + C_5\gamma \le \frac{1}{2},$$

which implies the desired estimate.

LEMMA 2. *Let* $\varepsilon = \varepsilon_p(n) > 0$ *be as in Lemma* 1. *Suppose that* $u \in W^s_{d,\mathrm{loc}}(\Omega,\Lambda^{l-1})$ *is weakly A-harmonic for some* $s \in (p-\varepsilon,p)$. *Then for any concentric cubes* $Q \subset 2Q \subset \Omega$ *we have*

$$\Big(\int_Q |du|^s\Big)^{1/s} \le C(n,p)\Big(\int_{2Q} |du|^r\Big)^{1/r}, \tag{2.4}$$

where

$$r = \max\left\{\frac{ns}{n+s-1}, \frac{ns}{np-n+s-p+1}\right\}. \tag{2.5}$$

Here the constant $C(n,p)$ *does not depend on* s *and* r *and* $r < s$.

Proof. Under the hypotheses above, we have

$$\int_\Omega \langle A(x,du) \mid d\phi\rangle = 0$$

for all $\phi \in L_1^{s/(s-p+1)}(\Omega, \Lambda)$ with compact support in Ω. In view of the homogeneity of $A(x,\xi)$ (see condition (H3) in the Introduction), we can write $d^*A(x, \eta^{p/(p-1)}du) = d^*f$, where $f = \eta^p A(x, du)$, for every nonnegative cut-off function $\eta \in C_0^\infty(2Q)$ such that $\eta = 1$ on a cube $Q \subset 2Q \subset \Omega$ and $|\nabla \eta| \le c(n)/\operatorname{diam} Q$.

Put $v = \eta^q(u - u_{2Q})$, where $q = p/(p-1)$, $u_{2Q} = d(Tu)$ and T is the homotopy operator for $D = 2Q$ introduced in §1. Clearly, $dv = \eta^q du + d(\eta^q) \wedge (u - u_{2Q})$ and $\eta^q du = dv + g$ with $g = -d(\eta^q) \wedge (u - u_{2Q})$. In this way, we arrive at a nonhomogeneous A-harmonic equation

$$d^*A(x, g + dv) = d^*f = d^*(f - \mu).$$

Here μ can be any coexact form, that is, $d^*\mu = 0$.

Applying Lemma 1 to $\Omega = 2Q$, we obtain

$$\int_{2Q} |dv|^s \le c\Big(\int_{2Q} |g|^s + |f - \mu|^{s/(p-1)}\Big),$$

$$\int_{Q} |du|^s \le c \int_{2Q} |dv|^s \le \frac{c}{\varrho^s} \int_{2Q} |u - u_{2Q}|^s + c \int_{2Q} |f - \mu|^{s/(p-1)},$$

where $\varrho = \operatorname{diam} Q$. Hence

$$\Big(\fint_{Q} |du|^s\Big)^{1/s} \le \frac{c}{\varrho}\Big(\fint_{2Q} |u - u_{2Q}|^s\Big)^{1/s} + c\Big(\fint_{2Q} |f - \mu|^{s/(p-1)}\Big)^{1/s} = I + II.$$

To estimate the first integral, we apply Corollary 3:

$$I \le c(n,s)\Big(\fint_{2Q} |du|^{ns/(n+s-1)}\Big)^{(n+s-1)/(ns)}.$$

To estimate the second integral, we apply Corollary 4 with $\omega = f$ and $\omega_D^* = \mu$. This yields

$$II = \frac{1}{\varrho^{n/s}}\Big\{\Big[\int_{2Q} |f - \mu|^{s/(p-1)}\Big]^{(p-1)/s}\Big\}^{1/(p-1)}$$

$$\le \frac{1}{\varrho^{n/s}}\Big(\int_{2Q} |d(*f)|^{ns/(np-n+s-p+1)}\Big)^{(np-n+s-p+1)/(ns)}.$$

Recall that $d^*f = d^*A(x, \eta^q du)$. By the homogeneity property of $A(x,\xi)$, we obtain

$$d^*f = d^*(\eta^p A(x, du)) = \eta^p d^*A(x, du) + (-1)^{n(l+1)} * (d\eta^p \wedge *A(x, du)).$$

This implies a pointwise inequality $|d^*f| \le b|\nabla(\eta^p)||du|^{p-1}$. Hence, we have the following estimate:

$$II \le \frac{c}{\varrho^{n/s+1/(p-1)}}\Big(\int_{2Q} |du|^{ns/(np-n+s-p+1)}\Big)^{(np-n+s-p+1)/(ns)}$$

$$= c\Big(\int_{2Q} |du|^{ns/(np-n+s-p+1)}\Big)^{(np-n+s-p+1)/(ns)}.$$

In conclusion,

$$\Big(\int_{Q} |du|^s\Big)^{1/s} \le c\Big(\int_{2Q} |du|^{ns/(s+n-1)}\Big)^{(s+n-1)/(ns)} + c\Big(\int_{2Q} |du|^{ns/(np-n+s-p+1)}\Big)^{(np-n+s-p+1)/(ns)}.$$

This is a reverse Hölder inequality for du. For r as defined in (2.5), we obtain (2.4).

THEOREM 1. *There exist exponents* $1 < p_1 = p_1(n,p) < p < p_2 = p_2(n,p) < \infty$ *such that if* $u \in W^{p_1}_{d,\mathrm{loc}}(\Omega, \Lambda^{l-1})$ *is a weakly A-harmonic tensor, then* $u \in W^{p_2}_{d,\mathrm{loc}}(\Omega, \Lambda^{l-1})$. *Thus u is an A-harmonic tensor in the usual sense.*

Proof. Let $\varepsilon = \varepsilon_p(n,a,b) > 0$ be as in Lemmas 1 and 2. We define $p_1 = p - \varepsilon_p(n)$ and $p_2 = p + \varepsilon_p(n)$. Then we find that

$$\Big(\int_{Q} |du|^{p_1}\Big)^{1/p_1} \le c\Big(\fint_{2Q} |du|^r\Big)^{1/r}$$

with some $r < p_1$ and $c = c(n,p,a,b)$. The exponent $r < p_1$ is defined by (2.5) where $s = p_1$.

We are now in a position to use Gehring's lemma (see [BI], [Gi], [GM], [Me]) to improve the degree of integrability of du. Accordingly, there exists $r_2 > r_1 = p_1$ such that $u \in W^{r_2}_{d,\mathrm{loc}}(\Omega, \Lambda^l)$.

Repeating the above arguments leads to another reverse Hölder inequality:

$$\Big(\int_{Q} |du|^{r_2}\Big)^{1/r_2} \le c(n,p,a,b)\Big(\fint_{2Q} |du|^r\Big)^{1/r},$$

where c is the same constant as before. This, in turn, allows us to increase the exponent r_2 even further, say $r_3 > r_2$, etc. The point is that the constant $c = c(n,p,a,b)$ will not change as long as $r_1 < r_2 < r_3 < \ldots$ stay in the interval (p_1, p_2), completing the proof of Theorem 1.

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