STUDIA MATHEMATICA 114 (3) (1995)

On measure-preserving transformations and doubly stationary symmetric stable processes

by

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Abstract. In a 1987 paper, Cambanis, Hardin and Weron defined doubly stationary stable processes as those stable processes which have a spectral representation which is itself stationary, and they gave an example of a stationary symmetric stable process which they claimed was not doubly stationary. Here we show that their process actually had a moving average representation, and hence was doubly stationary. We also characterize doubly stationary processes in terms of measure-preserving regular set isomorphisms and the existence of σ-finite invariant measures. One consequence of the characterization is that all harmonizable symmetric stable processes are doubly stationary. Another consequence is that there exist stationary symmetric stable processes which are not doubly stationary.

1. Introduction and preliminaries. If (X_t) is a non-Gaussian symmetric stable process (definitions are given below) then, being infinitely divisible as well, it has a canonical spectral representation $(X_t) \stackrel{\mathrm{d}}{=} (\int f_t\, dN)$, where (f_t) is a process on some measure space, N is a Poisson random measure on that measure space, and the integral is appropriately defined (Maruyama, 1970). Furthermore, Maruyama (1970, Section 4) observed that if the stochastic process (X_t) is stationary, then the process (f_t) in the spectral representation is also stationary; this follows easily from the construction of (f_t), in which a Kolmogorov existence type argument was used.

But it is well known that the symmetric stable process (X_t) has another representation $(\int f_t\, dM)$ where M is a symmetric stable random measure on some measure space and (f_t) is a process of L^α functions on that space (see, for instance, Hardin, 1982). Henceforth "spectral representation" of a stable process will always refer to this representation unless stated oth-

1991 *Mathematics Subject Classification*: Primary 60G10; Secondary 60E07, 28D10, 46E30.

Key words and phrases: invariant measures, nonsingular transformations, regular set isomorphisms, double stationarity.

The first author was supported by the Lady Davis Fellowship Trust in Jerusalem, Israel and by the Hugo Steinhaus Center for Stochastic Methods in Wrocław, Poland.

The second author was partially supported by the KBN grant No. 2 1153 91 01.

erwise. It seems natural to expect, by analogy with the Poisson random measure representation described above, that the process (f_t) in this representation would also be stationary whenever (X_t) is stationary. On the other hand, the analogy is not completely valid: The canonical representation of an n-dimensional symmetric stable random vector is with respect to a stable random measure on a measure space $(E, \mathcal{E}, \mu_n)$, where μ_n is a measure concentrated on the unit sphere in $\mathbb{R}^n$. Unlike the finite-dimensional Lévy measures, these measures μ_n clearly do not satisfy the Kolmogorov consistency criteria. We therefore have intuitive reasons to suggest both the existence and the possible non-existence of a stationary spectral representation.

Cambanis, Hardin and Weron (1987) called a SαS process *doubly stationary* if it has a representation $(\int f_t\, dM)$ where the process of L^α functions (f_t) is also stationary, and they gave an example of a stationary symmetric stable process which they claimed was not doubly stationary. In this paper we show that their example was incorrect, and that it was actually a moving average process and hence doubly stationary (Example 1). We then characterize doubly stationary processes in terms of measure-preserving transformations (Theorem 6) and in terms of the existence of σ-finite invariant measures (Theorem 7), and in Example 2 we construct a stationary symmetric stable process which is not doubly stationary, using an example by Ornstein (1960) of a nonsingular transformation on $[0,1]$ which does not admit a σ-finite invariant measure equivalent to Lebesgue measure. (We have recently learned that Jan Rosinski also found the gap in the Cambanis, Hardin and Weron (1987) example, and that he observed as well that the existence of stationary, non-doubly stationary processes follows from the existence of nonsingular transformations which do not admit σ-finite invariant measures.) An additional consequence of Theorem 6 is that harmonizable processes are doubly stationary.

Now for the definitions. Let G be an arbitrary group. A stochastic process $(X_t)_{t\in G}$ is *stationary* if the distribution of $(X_{st})_{t\in G}$ is the same for all $s \in G$. A stochastic process (X_t) is said to be *symmetric α-stable* ($S\alpha S$) if every finite linear combination of the X_t's is a SαS random variable.

The symbols $\mathcal{B}_{\mathbb{R}}$ and $\mathcal{B}_{[0,1]}$ will denote the Borel σ-fields on $\mathbb{R}$ and $[0,1]$. For $p \in (0,\infty)$ and a measure space $(E, \mathcal{E}, \mu)$, we will write

$$\|f\|^p = \int |f|^p\, d\mu,$$

and let $L^p(E, \mathcal{E}, \mu)$ (or an abbreviation thereof) denote the class of all complex-valued functions f such that $\|f\|^p$ is finite. It is well known that L^p is a complete metric space with respect to the metric

$$(f,g) \mapsto \|f-g\|^{1\wedge 1/p}.$$

If (U^t) is a group of isometries on some $L^p(E, \mathcal{E}, \mu)$ and f is an L^p function, we let L_f denote the closed linear subspace generated by the functions $U^t f$; it will always be clear which isometry is implied by the notation "L_f". Given a linear subspace $F \subset L^p$, the *ratio σ-field* $\varrho(F)$ is the σ-field generated by functions of the form f/g, where f and g are functions in F (see Hardin, 1982).

It is well known (e.g. Hardin, 1982) that for every stationary SαS process $X = (X_t)_{t\in G}$ there is a linear subspace L of some $L^\alpha(E, \mathcal{E}, \mu)$, a function f in L, and a group of isometries $U = (U^t)_{t\in G}$ defined on L such that for any finite linear combination $\sum a_t X_t$ (where the a_t's are real numbers),

$$\mathbb{E}\exp\Big(i\sum a_t X_t\Big) = \exp\Big(-\Big\|\sum a_t U^t f\Big\|^\alpha\Big).$$

We say that (U, f) is a *spectral representation* for the SαS process X. Using this notation, a process is doubly stationary if it has a representation (U, f) such that $(U^t f)$ is stationary.

We will generally not write the "t" explicitly when referring to groups of isometries or groups of set transformations. For instance, we will generally write "$WU_1 = U_2 W$" instead of "$WU_1^t = U_2^t W$ for all t".

All measures in this paper will be assumed to be σ-finite. We let m denote Lebesgue measure. If two measures on the same σ-field have the same sets of measure zero, we say the measures are *equivalent*. If $(E_1, \mathcal{E}_1, \mu_1)$ and $(E_2, \mathcal{E}_2, \mu_2)$ are measure spaces, a transformation $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ is called a *regular set isomorphism* (see Lamperti (1958) or Hardin (1981)) if it preserves set complements and countable unions and intersections and if it is *nonsingular*, i.e. the measures $\mu_2\phi$ and μ_1 have the same sets of measure zero. When we say that a regular set isomorphism is invertible, we mean that its inverse is also a regular set isomorphism; a regular set isomorphism ϕ always has an inverse defined on its image $\phi\mathcal{E}_1 \subset \mathcal{E}_2$. A regular set isomorphism ϕ induces an operator, also called ϕ, on the class of measurable functions by

$$\phi 1_A = 1_{\phi A}.$$

A regular set isomorphism $\phi : \mathcal{E} \to \mathcal{E}$ is said to admit an *invariant measure* ν on $\mathcal{E}$ if $\nu\phi = \nu$. In this case we also say that ϕ is *measure-preserving* (with respect to ν). If $(E_1, \mathcal{E}_1, \mu_1)$ and $(E_2, \mathcal{E}_2, \mu_2)$ are σ-finite measure spaces and $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ is a regular set isomorphism, we write

$$\phi' = \frac{d\mu_1\phi^{-1}}{d\mu_2}.$$

2. Results. Before looking at Example (iv) of Cambanis, Hardin and Weron (1987), we state the following chain rule. We omit the proof, which is straightforward.

LEMMA 1. *If ϕ and ψ are regular set isomorphisms on σ-finite measure spaces such that the composition $\phi\psi$ is defined, then*

$$(\phi\psi)' = \psi(\phi') \cdot \psi'.$$

EXAMPLE 1. *The S1S process of Example* (iv) *of Cambanis, Hardin and Weron* (1987), *represented by* $((U^t)_{t\in\mathbb{R}}, 1)$ *on* $L^1[0,1]$, *where*

$$U^t g(x) = 2^t x^{2^t-1} g(x^{2^t}),$$

is a moving average process.

Proof. Define $\Phi_1^t : [0,1] \to [0,1]$ by

$$\Phi_1^t x = x^{2^t}, \quad t \in \mathbb{R},\ x \in [0,1].$$

Similarly, define $\Phi_2^t : \mathbb{R} \to \mathbb{R}$ by

$$\Phi_2^t x = x + t, \quad t \in \mathbb{R},\ x \in \mathbb{R}.$$

Then $(\phi_i^t)_{t\in\mathbb{R}} := ((\Phi_i^t)^{-1})_{t\in\mathbb{R}}$, $i = 1, 2$, are groups of regular set isomorphisms. Define $\Psi : \mathbb{R} \to [0,1]$ by

$$\Psi x = 2^{-2^x}, \quad t \in \mathbb{R},\ x \in \mathbb{R},$$

and define the regular set isomorphism $\psi = \Psi^{-1}$. One can check directly that

$$\Phi_1^t \Psi = \Psi \Phi_2^t, \quad t \in \mathbb{R},$$

and that therefore ϕ_1, ϕ_2 and ψ satisfy

$$\psi\phi_1 = \phi_2\psi. \tag{1}$$

Now define the positive isometries induced by the set isomorphisms by

$$U_i f = (\phi_i') \cdot (\phi_i f), \quad i = 1, 2, \quad Wf = (\psi') \cdot (\psi f).$$

(Note that $(U_1, 1)$ represents the SαS process defined in this example.) Then for all 1_A in $L^p(\mu_1)$,

$$WU_1 1_A = [\phi_1(\psi') \cdot \phi_1'] \cdot 1_{\psi\phi_1 A},$$
$$U_2 W 1_A = [\psi(\phi_2') \cdot \psi'] \cdot 1_{\phi_2\psi A}.$$

It now follows from (1) and an application of the chain rule (Lemma 1) that

$$WU_1 = U_2 W.$$

Thus $(U_2, W1)$ represents the same SαS process as $(U_1, 1)$. But U_2 is the group of shift operators, i.e.

$$U_2^t g(x) = g(x+t), \quad t \in \mathbb{R},\ g \in L^1(\mathbb{R}),$$

so the process given in the example is a moving average. ∎

Remark. The proof in Cambanis, Hardin and Weron (1987) that the process has no stationary representation is correct until the last line, where it is implicitly assumed that $\mu(E) < \infty$. Thus it was actually proven that the process has no stationary representation on a space of finite measure. This also follows from results in Gross (1994), where it is shown that an ergodic SαS process separable in probability cannot have a stationary representation on a space of finite measure; it is easy to check, using the results in Gross (1994), that this particular example is mixing.

All the results in this paper rely on the representation of a linear isometry of an L^α space into itself by a regular set isomorphism on the σ-field. The first general result was proven by Lamperti (1958); we re-state it here.

THEOREM 2 (Lamperti, 1958). *Let U be an isometry from $L^p(E,\mathcal{E},\mu)$ into $L^p(E,\mathcal{E},\mu)$, where μ is σ-finite and p is a positive real number not equal to 2. Then there exist h in $L^0(E,\mathcal{E},\mu)$ and a regular set isomorphism $\phi : \mathcal{E} \to \mathcal{E}$ defined by*

$$\phi(A) = \operatorname{supp}(U1_A), \quad \mu(A) < \infty,$$

such that

$$Uf = h \cdot (\phi f), \quad f \in L^p(\mu).$$

Necessarily then, $|h|^p = \phi'$.

Our goal now is to characterize doubly stationary processes in terms of measure-preserving regular set isomorphisms (Theorem 6). The proof, contained in the next three lemmas, is as follows: Let X be a stationary SαS process, separable in probability. Then X has a representation on a Borel subset of a separable complete metric space (Lemma 3), so by an argument due to Rosinski (1994) the regular set isomorphisms in the spectral representation are induced by point transformations (Lemma 4). But then we show that if the point transformations are measure-preserving, then we can construct a *stationary* spectral representation of the process on some product space (Lemma 5).

LEMMA 3. *Let $((V^t)_{t\in G}, g)$ be a representation of a stationary SαS process, separable in probability. Then there exists a spectral representation $((U^t)_{t\in G}, f)$ for the same process, defined on some $L^\alpha(E,\mathcal{E},\mu)$ where $E = \mathbb{R}^S \setminus \{0\}$ for some countable $S \subset G$, such that*

$$(U^t f)_{t\in G} \stackrel{\mathrm{d}}{=} (V^t g)_{t\in G}.$$

Proof. Since this proof consists of standard measure-theoretic arguments with only slight modifications, we will present an outline only.

Let $S \subset G$ be a countable subset such that $\{X_t\}_{t\in S}$ is dense in $\{X_t\}_{t\in G}$ with respect to convergence in probability. Define $E = \mathbb{R}^S \setminus \{0\}$ and let $\mathcal{E}$ be generated by the cylinder sets of E. Say the representation (V, g) is on the measure space $L^\alpha(M, \mathcal{F}, \nu)$. By the separability assumption, we can assume without loss of generality that $\mathcal{F}$ is generated by the $V^t g$'s, $t \in S$.

We will follow the usual argument in the Kolmogorov existence theorem with a minor modification to account for the fact that the measure may be infinite. As in the Kolmogorov existence theorem, we begin by defining a measure μ on the field generated by finite Cartesian products of Borel subsets of the real line. In addition, we require that at least one of the Borel subsets in the Cartesian product be a compact subset of $\mathbb{R} \setminus \{0\}$. We define μ on this field in the usual way as the measure induced by the process $(V^t g)_{t \in S}$. Because of our additional requirement on the cylinder sets, and the fact that the $V^t g$'s are L^α functions, the measure μ is finite on this field; therefore, by elementary measure-theoretic arguments, it is also regular (see, for instance, Billingsley, 1986, Theorem 12.3). Hence we can continue with the usual Kolmogorov proof, extending μ to a unique measure on the σ-field; the process $(f_t)_{t \in S}$ on $(E, \mathcal{E}, \mu)$ defined by $f_t(x) = x(t)$ thus has the same distribution as $(V^t g)_{t \in S}$.

We will now show that the process $(f_t)_{t \in G}$ has the same distribution as $(V^t g)_{t \in G}$. We will show first that convergence in measure implies a sort of "convergence in distribution"; the argument follows the argument for the case of probability measures in Billingsley (1986, Theorem 25.2), but again we need to adapt it slightly to take into account infinite measures. Let $\{t_1, \ldots, t_d\} \subset G$ be an arbitrary finite index set, and suppose that vectors $(f_{t_1,n}, \ldots, f_{t_d,n})$ of L^α functions converge in measure to $(f_{t_1}, \ldots, f_{t_d})$ as $n \to \infty$. For any nonnegative real numbers $a_1, \ldots, a_d$ with

$$\mu\{f_{t_1} = a_1, \ldots, f_{t_d} = a_d\} = 0$$

and at least one a_k strictly positive, an adaptation of the standard proof of convergence in distribution (Billingsley, 1986, Theorem 25.2) shows that

$$\mu\{f_{t_1,n} \geq a_1, \ldots, f_{t_d,n} \geq a_d\} \to \mu\{f_{t_1} \geq a_1, \ldots, f_{t_d} \geq a_d\}. \tag{2}$$

Here the requirement that some a_k be strictly positive insures that the sets have finite measure.

Note that sets of the form $\{f_{t_1} \geq a_1, \ldots, f_{t_d} \geq a_d\}$ with the a_k's as described above, together with the similar sets for nonpositive a_k's, generate $\mathcal{E}$. Thus (2), together with the similar statement for nonpositive a_k's, the equality in distribution of $(f_t)_{t \in S}$ and $(V^t g)_{t \in S}$, and separability, imply that the processes $(f_t)_{t \in G}$ and $(V^t g)_{t \in G}$ have the same distribution. ■

LEMMA 4. *If $U = (U^t)_{t \in G}$ is a continuous group of isometries on $L^\alpha(E, \mathcal{E}, \mu)$, where E is a Borel subset of a complete separable metric space and μ is a σ-finite Borel measure, then U has a representation of the form*

$$U^t g = h_t \cdot (g \circ \tau^t), \quad t \in G, \ g \in L^\alpha, \tag{3}$$

where $(\tau^t)_{t \in G}$ is a group of nonsingular point transformations and the functions h_t are $\mathcal{E}$-measurable and satisfy

$$h_{s+t} = h_s \cdot (h_t \circ \tau^s). \tag{4}$$

This statement is essentially proven by Rosinski (1994) in the proof of his Theorem 5.1. Consequently, we will omit the proof here. (Note that the continuity of $t \mapsto U^t$ is equivalent to the separability in probability of the SαS process.)

LEMMA 5. *Let (U_1, f_1) be a spectral representation for a SαS process, where $f_1 \in L^\alpha(E, \mathcal{E}, \mu)$ for an arbitrary σ-finite measure space $(E, \mathcal{E}, \mu)$. If U_1 has the representation*

$$U_1^t g = h_t \cdot (g \circ \tau^t), \quad t \in G, \ g \in L^\alpha,$$

where (τ^t) is a group of measure-preserving point transformations, then the SαS process also has a representation (U_2, f_2), where

$$U_2^t g = g \circ T^t, \quad t \in G, \ g \in L^\alpha,$$

for some group of measure-preserving point transformations (T^t).

Proof. We will construct the stationary representation using a skew product transformation on the product space of E with the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Let $\mathbb{T}$ denote the unit circle; we will always take the measure on $\mathbb{T}$ to be normalized Lebesgue measure.

Define the point transformations $T^t : E \times \mathbb{T} \to E \times \mathbb{T}$ by

$$T^t(x, y) = (\tau^t x, h_t(x) \cdot y),$$

where τ and h are induced by U_1 as described in the statement of the lemma. Define the $L^\alpha(E \times \mathbb{T})$ isometries U_2^t by $U_2^t f = f \circ T^t$.

Define the point transformation $\psi : E \times \mathbb{T} \to E$ by

$$\psi(x, y) = x,$$

and define the $L^0(E \times \mathbb{T})$ function h_ψ by

$$h_\psi(x, y) = y.$$

Now define $W : L^\alpha(E) \to L^\alpha(E \times \mathbb{T})$ by

$$W f = h_\psi \cdot (f \circ \psi).$$

Note that W is an isometry since ψ is measure-preserving and $|h_\psi| = 1$.

Direct calculation shows that $W U_1 = U_2 W$. Hence, taking $f_2 = W f_1$, we have constructed a representation (U_2, f_2) having the desired form. ■

We now combine the results of Lemmas 3–5 into a single theorem.

THEOREM 6. *A necessary and sufficient condition for a stationary SαS process, separable in probability, to be doubly stationary is that it have a* *representation $((U^t)_{t\in G}, f)$, where the group of isometries U is induced by a group of measure-preserving regular set isomorphisms ϕ, in the sense that*

$$U^t f = (\phi^t f)\cdot h_t.$$

Proof. The necessity is clear—the transformations ϕ^t are the shifts defined on the σ-field generated by the cylinder sets of $(U^t f)$ and the h_t's are 1. Sufficiency was proven in Lemmas 3–5. ∎

An immediate consequence of Theorem 6 is that harmonizable processes are doubly stationary. (A SαS process indexed by $\mathbb{R}$ is *harmonizable* if it has a representation given by isometries $U^t f(x) = f(x)e^{itx}$; harmonizable processes indexed by arbitrary locally compact abelian groups are defined similarly.)

We now apply Theorem 6 to state another characterization of double stationarity. As we observed in the introduction, Jan Rosinski also noted the connection between double stationarity and the existence of σ-finite invariant measures, independently of our work.

THEOREM 7. *Let X be a stationary SαS process, $\alpha\in(0,2)$, separable in probability, with spectral representation (U,f) on some $L^\alpha(E,\mathcal{E},\mu)$ with μ σ-finite. Suppose that U is given by $Ug = h\cdot(\phi g)$ as described in Theorem 2. If ϕ admits a σ-finite invariant measure equivalent to μ, then X is doubly stationary.*

Proof. Assume that ν is a σ-finite invariant measure for ϕ, equivalent to μ. Take $\psi : (E,\mathcal{E},\mu)\to(E,\mathcal{E},\nu)$ to be the identity, and define the isometry $W : L^\alpha(\mu)\to L^\alpha(\nu)$ by

$$Wg = (\psi')^{1/\alpha}\cdot(\psi g).$$

(Note that ψ is a regular set isomorphism because ν is equivalent to μ.) Now (WUW^{-1}, Wf) is another representation for X, and it is easy to see that for any 1_A in $L^\alpha(\nu)$, the support of $WUW^{-1}1_A$ is just ϕA. In other words, the group ϕ of regular set isomorphisms induced by the group of isometries WUW^{-1} preserves the measure ν, so by Theorem 6, X is doubly stationary. ∎

In general, nonsingular transformations on $[0,1]$ that you can write down "off the top of your head" have σ-finite invariant measures equivalent to Lebesgue measure—the question of whether there exist nonsingular transformations which do not admit such measures was open for over twenty-five years, until Ornstein's (1960) example which we use for our Example 2. Theorem 7 therefore suggests that in general, the stationary SαS processes which one is likely to deal with are doubly stationary.

EXAMPLE 2. *For every $\alpha\in(0,2)$, there exist stationary SαS processes which are not doubly stationary.*

We construct this example in the following steps: First, we will establish a relationship between the σ-field $\varrho(L_1)$ for a given isometry, and the Radon–Nikodym derivatives of the regular set isomorphism corresponding to the isometry. Then we will describe a nonsingular point transformation τ on $[0,1]$, due to Ornstein (1960), which does not admit a σ-finite, invariant, equivalent measure. Using the relationship established in the first step, we will show that this particular example can be taken so that $\varrho(L_1) = \mathcal{B}_{[0,1]}$.

We will then use Lemma 10 (stated and proven below) to show that, if the process is represented by another isometry induced by a measure-preserving transformation T on some measure space $(E,\mathcal{E},\mu)$, then there is a regular set isomorphism $\psi : \mathcal{B}_{[0,1]}\to\mathcal{E}$ such that $\psi\tau^{-1} = T\psi$. Finally, we will conclude that the SαS process represented by the isometry induced by τ together with the function $f = 1$ cannot be doubly stationary.

We will need to use a result from Hardin (1981) for our example. Recall that Lamperti's result (Theorem 2) applies only to isometric operators on a single L^p space. We will need a similar result for an isometry $W : L\to L^p(\nu)$, where L is a linear subspace of some $L^p(\mu)$. Here p is a positive number but not an even integer. Theorem 4.2 of Hardin (1981) asserts that *an isometry $W : L\to L^p(\nu)$ can be extended to an isometry $\overline{W}$ defined on the class of L^α functions of the form $r\cdot h$, where r is a $\varrho(L)$-measurable function and h is in L. Furthermore, there is an invertible regular set isomorphism $\psi : \varrho(L)\to\varrho(W(L))$ such that*

$$W(r\cdot h) = (\psi r)\cdot(Wh). \tag{5}$$

Before actually constructing our example, we establish the following fact about $\varrho(L_f)$ for the special case $f = 1$.

LEMMA 8. *Let $(E,\mathcal{E},\mu)$ be a measure space with $\mu(E)<\infty$, let $p\in(0,\infty)\setminus\{2\}$, and let $U : L^p(\mu)\to L^p(\mu)$ be an isometry with representation*

$$Uf = h\cdot(\phi f)$$

as described in Theorem 2. Then, letting "1" denote the function identically equal to 1,

$$\sigma\{\phi^n(\phi')\}_{n\in\mathbb{Z}}\subset\varrho(L_1).$$

Proof. We will show that for each integer n, $\phi^n(\phi')$ is $\varrho(L_1)$-measurable. By the chain rule, we have for $n\ge 1$,

$$(\phi^n)' = [\phi^n(\phi')]\cdot[\phi^{n-1}(\phi')]\ldots[\phi'],$$
$$(\phi^{-n})' = 1/\{[\phi^{-n}(\phi')][\phi^{-(n-1)}(\phi')]\cdots[\phi']\}.$$

Thus for $n\ge 1$,

$$\phi^n(\phi') = (\phi^n)'/(\phi^{n-1})' = |U^n1|^p/|U^{n-1}1|^p = |U^n1/U^{n-1}1|^p,$$

so $\phi^n(\phi')$ is $\varrho(L_1)$-measurable. A similar argument works for $n\le -1$. ∎

We will now describe Ornstein's (1960) example of an invertible, nonsingular point transformation $\tau : [0,1] \to [0,1]$ such that the regular set isomorphism τ^{-1} does not admit a σ-finite invariant measure equivalent to Lebesgue measure on the Borel σ-field $\mathcal{B}_{[0,1]}$.

The transformation is constructed by cutting and stacking. We will describe the procedure briefly; for a more rigorous definition, see Ornstein (1960). For the $N=1$ stage, begin with the ordered partition of [0,1) into the intervals [0,1/2) and [1/2,1), which we picture as stacked one above the other. The transformation τ is defined on each interval of the stack by mapping it linearly onto the interval above it. At each stage, τ remains undefined on the top interval. Thus at the first stage τ is only defined on [0,1/2).

If the Nth stack consists of the intervals $(I_N, \tau I_N, \ldots, \tau^{h_N-1} I_N)$ (h_N is the "height" of the Nth stack), then the $(N+1)$th stack is defined as follows: Divide the interval I_N in half, and then cut the right half into s_N consecutive subintervals of equal measure, where s_N is a parameter to be specified later. Let $I_{i,N}$, $i = 0,1,\ldots,s_N$, denote the resulting ordered partition of I_N. This defines a similar partition of any other interval $\tau^k I_N$ in the stack into the subintervals $\tau^k I_{i,N}$, $i = 0,1,\ldots,s_N$. Now create the $(N+1)$th stack by stacking each subcolumn on top of the subcolumn to its left. In other words, extend τ to part of the top interval by defining $\tau(\tau^{h_N-1} I_{i,N}) = I_{i+1,N}$, $i = 0,1,\ldots,s_N - 1$. The bottom interval I_{N+1} of the $(N+1)$th stack will now be $I_{0,N}$. By repeating this procedure, τ is defined on all of [0,1).

Ornstein (1960) showed that if s_N is chosen sufficiently large, then τ will not admit a σ-finite invariant measure equivalent to Lebesgue measure. We want the transformation to satisfy a further condition, as described in the following lemma.

LEMMA 9. *In Ornstein's example described above, the s_N's can be chosen so that*

$$\sigma\{\tau' \circ \tau^n\}_{n\in\mathbb{Z}} = \mathcal{B}_{[0,1]},$$

modulo sets of measure zero.

Proof. It is clear from the construction that the top interval in the stack is always strictly shorter than the bottom interval, for all $N > 1$, provided the s_N's are always chosen to be at least 2. Therefore, for any such choice of s_N, the subinterval $\tau^{h_N-1} I_{i,N}$ will be shorter than $I_{i+1,N}$ for all $i = 1,\ldots,s_N - 1$; this says that $\tau' > 1$ on $\tau^{h_N-1} I_{i,N}$, $i = 1,\ldots,s_N - 1$. (Here τ' coincides with the ordinary derivative of the almost everywhere differentiable function τ.)

Let β_N denote the value of τ' on $\tau^{h_N-1} I_{0,N}$, i.e.

$$\beta_N = m(I_{1,N})/m(\tau^{h_N-1} I_{0,n}).$$

(Recall that m denotes Lebesgue measure.) By choosing s_N large, we can make $m(I_{1,N})$ small and thus make β_N small. Therefore choose s_N large enough to satisfy Ornstein's criterion for nonexistence of an invariant measure, and also large enough so that $\beta_N < 1$ and $\beta_N \neq \beta_n$, $n = 1,\ldots,N-1$. This insures that

$$(\tau')^{-1}(\beta_N) = \tau^{h_N-1} I_{0,N} = \tau^{h_N-1} I_{N+1}.$$

In other words, $(\tau')^{-1}(\beta_N)$ consists of precisely one interval in the $(N+1)$th stack. Therefore every interval in the $(N+1)$th stack can be written in the form $\tau^n(\tau')^{-1}(\beta_N)$ for some integer n. Hence $\sigma\{\tau' \circ \tau^n\}_{n\in\mathbb{Z}}$ contains $\mathcal{B}_{[0,1]}$. ∎

Before continuing, we pause to note that the equality $\varrho(L_1) = \mathcal{B}_{[0,1]}$ does not hold in general for all nonsingular point transformations which do not admit σ-finite invariant measures, and hence some argument like the preceding one is necessary for the next step in our example. For an example of such a transformation where $\varrho(L_1) \neq \mathcal{B}_{[0,1]}$, consider any transformation ζ on $[0,1]$ which does not admit a σ-finite invariant measure, and define ψ on $[0,1] \times [0,1]$ with product measure by $\psi(x,y) = (\zeta x, y)$. Since the nth derivative of ψ depends only on x for each integer n, so does $U^n 1$, where U is the isometry given by $Uf(x,y) = f(\psi(x,y)) \cdot \psi'(x,y)$; thus the ratio σ-field generated by $(U^n 1)$ is strictly smaller than the Borel σ-field. But $[0,1] \times [0,1]$ is Borel isomorphic to $[0,1]$, so ψ is equivalent to some nonsingular transformation on [0,1], which also does not have an invariant measure. Thus the ratio σ-field corresponding to this transformation on $[0,1]$ is also strictly smaller than $\mathcal{B}_{[0,1]}$. (This example was provided by the referee.)

We now define the SαS sequence in terms of τ. Let τ be as in Lemma 9 above, and let

$$Ug = h \cdot (g \circ \tau),$$

where h is any function satisfying $|h|^\alpha = dm\tau/dm$. Then $(U, 1)$ will represent our stationary SαS sequence.

If the SαS sequence represented by $(U, 1)$ were doubly stationary, then by Theorem 7 there would be an $L^\alpha(E, \mathcal{E}, \mu)$, μ a σ-finite measure, and isometries $T : L^\alpha(E,\mathcal{E},\mu) \to L^\alpha(E,\mathcal{E},\mu)$ and $W : L_1 \to L^\alpha(E,\mathcal{E},\mu)$ such that T induces a measure-preserving regular set isomorphism (also called T) and on the subspace L_1,

$$WU = TW.$$

Let ψ be the regular set isomorphism defined on $\varrho(L_1)$ corresponding to W in Hardin's (1981) Theorem 4.2 as described above. It is easy to see that the transformations τ^{-1} and T also correspond to the operators U and T as described in Hardin's Theorem 4.2, so our Lemma 10 (stated below) applies and

$$\psi\tau^{-1} = T\psi$$

on $\varrho(L_1) = \mathcal{B}_{[0,1]}$. But T was assumed to be measure-preserving, so

$$\mu\psi\tau^{-1} = \mu T\psi = \mu\psi.$$

Thus $\mu\psi$ is a σ-finite invariant measure for τ, and $\mu\psi$ is equivalent to Lebesgue measure since ψ is a regular set isomorphism. But τ was chosen so as not to admit such an invariant measure. We conclude that the stationary SαS sequence represented by $(U, 1)$ is not doubly stationary. ■

We now state and prove the lemma that was used in Example 2.

LEMMA 10. *Let L be a linear subspace of some $L^p(\mu)$, p a positive real number but not an even integer, let $U_1 : L \to L$ be an isometry, let $W : L \to L^p(\nu)$ be an isometry, and define U_2 on $W(L)$ by $U = WU_1W^{-1}$. If ϕ_1, ϕ_2 and ψ are the regular set isomorphisms on $\varrho(L)$, $\varrho(W(L))$ and $\varrho(L)$ induced by U_1, U_2 and W by Hardin's* (1981) *Theorem* 4.2 *as described above, then*

$$\phi_2 = \psi\phi_1\psi^{-1}.$$

Proof. We begin with a fact about $\varrho(L)$. By Lemma 3.2 of Hardin (1981), there is a function η of "full support" in L, i.e., the support of η contains the support of every function in L. In the proof of Theorem 4.2 (1981), Hardin shows that $\varrho(L)$ is the σ-field generated by functions of the form g/η, for arbitrary g in L. One final fact: Lemma 3.4 of Hardin (1981) asserts that if V is an isometry and η has full support in L, then $V\eta$ has full support in $V(L)$.

It suffices to show that

$$\psi\phi_1(g/\eta) = \phi_2\psi(g/\eta), \quad g \in L.$$

Now taking $r = g/\eta$ and $h = \eta$ in the representation (5) of the isometry U_1, we have

$$\psi\phi_1(g/\eta) = \psi(U_1g/U_1\eta).$$

Taking $r = U_1g/U_1\eta$ and $h = U_1\eta$ in (5) for W (this is justified since $U_1\eta$ has full support in $L = U_1(L)$), we get

$$\psi(U_1g/U_1\eta) = WU_1g/WU_1\eta = U_2Wg/U_2W\eta.$$

Noting that $W\eta$ has full support in $W(L)$, then converting back from U_2 to ϕ_2 and from W to ψ, we obtain

$$\psi\phi_1(g/\eta) = \phi_2\psi(g/\eta).$$

Hence $\phi_2 = \psi\phi_1\psi^{-1}$, and the lemma is proved. ■

References

P. Billingsley (1986), *Probability and Measure*, Wiley, New York.

S. Cambanis, C. D. Hardin, Jr. and A. Weron (1987), *Ergodic properties of stationary stable processes*, Stochastic Process. Appl. 24, 1–18.

A. Gross (1994), *Some mixing conditions for stationary symmetric stable stochastic processes*, ibid. 51, 277–285.

C. D. Hardin, Jr. (1981), *Isometries on subspaces of L^p*, Indiana Univ. Math. J. 30, 449–465.

C. D. Hardin, Jr. (1982), *On the spectral representation of symmetric stable processes*, J. Multivariate Anal. 12, 385–401.

J. Lamperti (1958), *On the isometries of certain function spaces*, Pacific J. Math. 8, 459–466.

G. Maruyama (1970), *Infinitely divisible processes*, Probab. Theory Appl. 15, 3–23.

D. S. Ornstein (1960), *On invariant measures*, Bull. Amer. Math. Soc. 66, 297–300.

J. Rosinski (1994), *On uniqueness of the spectral representation of stable processes*, J. Theor. Probab. 7, 551–563.

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Received April 11, 1994
Revised version January 30, 1995 (3257)