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Sets in the ranges of nonlinear accretive operators in Banach spaces

by

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Abstract. Let X be a real Banach space and $G \subset X$ open and bounded. Assume that one of the following conditions is satisfied:

(i) X^* is uniformly convex and $T : \overline{G} \to X$ is demicontinuous and accretive;
(ii) $T : \overline{G} \to X$ is continuous and accretive;
(iii) $T : X \supset D(T) \to X$ is m-accretive and $\overline{G} \subset D(T)$.

Assume, further, that $M \subset X$ is pathwise connected and such that $M \cap TG \neq \emptyset$ and $M \cap \overline{T(\partial G)} = \emptyset$. Then $M \subset \overline{TG}$. If, moreover, Case (i) or (ii) holds and T is of type (S_1), or Case (iii) holds and T is of type (S_2), then $M \subset TG$. Various results of Morales, Reich and Torrejón, and the author are improved and/or extended.

1. Introduction and preliminaries. In what follows, the symbol X stands for a real Banach space with norm $\|\cdot\|$ and (normalized) duality mapping J. In what follows, "continuous" means "strongly continuous" and the symbol "$\to$" ("$\rightharpoonup$") means strong (weak) convergence. The symbol $\mathbb{R}$ ($\mathbb{R}_+$) stands for the set $(-\infty,\infty)$ ($[0,\infty)$) and the symbols $\partial D, \operatorname{int} D, \overline{D}$ denote the strong boundary, interior and closure of the set D, respectively. An operator $T : X \supset D(T) \to Y$, with Y another Banach space, is *bounded* if it maps bounded subsets of $D(T)$ onto bounded sets. It is *compact* if it is continuous and maps bounded subsets of $D(T)$ onto relatively compact sets. It is called *demicontinuous* if it is strong-weak continuous on $D(T)$.

An operator $T : X \supset D(T) \to 2^X$ is *accretive* if for every $x, y \in D(T)$ there exists $j \in J(x-y)$ such that

$$(*) \qquad \langle u - v, j\rangle \geq 0 \quad \text{for every } u \in Tx,\ v \in Ty.$$

An accretive operator T is *strongly accretive* if 0 in the right-hand side of $(*)$ is replaced by $\alpha\|x-y\|^2$, where $\alpha > 0$ is a fixed constant. An accretive

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operator T is called *m-accretive* if $R(T+\lambda I) = X$ for every $\lambda > 0$, where I denotes the identity operator on X.

We denote by $B_r(0)$ the open ball of X with center at zero and radius $r > 0$. For an m-accretive operator T, the *resolvents* $J_\lambda : X \to D(T)$ of T are defined by $J_\lambda = (I+\lambda T)^{-1}$ for all $\lambda \in (0,\infty)$. The *Yosida approximants* $T_\lambda : X \to X$ of T are defined by $T_\lambda = (1/\lambda)(I - J_\lambda)$.

Some of the main properties of J_λ and T_λ are given below:

1. $\|J_\lambda x - J_\lambda y\| \le \|x-y\|$ for all $x, y \in X$.
2. $\|J_\lambda x - x\| = \lambda\|T_\lambda x\| \le \lambda \inf\{\|y\| : y \in Tx\}$ for all $x \in D(T)$.
3. T_λ is m-accretive on X and $\|T_\lambda x - T_\lambda y\| \le (2/\lambda)\|x-y\|$ for all $\lambda > 0$ and $x, y \in X$.
4. $T_\lambda x \in TJ_\lambda x$ for all $x \in X$.

For facts involving accretive operators, and other related concepts, the reader is referred to Barbu [1], Browder [2], Cioranescu [3] and Lakshmikantham and Leela [19].

The main purpose of this paper is to give some results involving the existence of certain sets in the range of an accretive operator. These sets are either balls, or more general connected sets. This paper is a continuation of a series of papers on the subject by a good number of authors. A central role in this work is played by the *Leray–Schauder condition*:

$$Tx \not\ni \mu(x - x_0), \quad (\mu, x) \in (-\infty, 0) \times (\partial G \cap D(T)),$$

where G is a fixed open, bounded set and $x_0 \in G \cap D(T)$ is fixed. It turns out that this condition is sufficient, and often necessary, in various settings, for the existence of a zero of an accretive, or m-accretive, operator. The reader is referred to the papers by Gatica and Kirk [5], [6], and Kirk and Schöneberg [17] for applications of the Leray–Schauder condition in the study of pseudo-contractive mappings. We also cite the paper [16] of Kirk for such considerations involving nonexpansive mappings. The author gave in [8] a series of results involving ranges of m-accretive operators in Banach spaces X with X^* uniformly convex. These results have seen a good number of extensions by various authors. For example, Kirk and Schöneberg gave the first extensions in [18]. They were followed by Reich and Torrejón [24], Torrejón [25] and Morales [20]. Demicontinuous accretive mappings were studied by the author in [9]. These results were extended, for example, by Morales in [21] and [22].

It is our intention here to give a general result, Proposition 1, that allows a pathwise connected set M to lie in the range of an accretive operator T. We then use this result to obtain a result, Theorem 3, where a ball lies in the range of the accretive operator T. Various results in the above-mentioned papers are extended and/or improved. For a study of maximal monotone operators in connection with the Leray–Schauder condition, the reader is referred to the author's paper [14]. A survey article on compact perturbations and compact resolvents of accretive operators can be found in [12]. Applications of these results include the controllability of nonlinear evolutions with pre-assigned responses in Banach spaces ([7], [15]), as well as the construction of methods of lines for nonlinear functional evolution equations in Banach spaces [13].

2. Main results. A mapping $T : X \supset D(T) \to X$ is called *ϕ-expansive on* $M \subset X$ if

$$\|Tx - Ty\| \ge \phi(\|x-y\|), \quad x, y \in M \cap D(T),$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing function, depending on M, with $\phi(0) = 0$. T is called *ϕ-expansive* if it is ϕ-expansive on $D(T)$. T is called *locally ϕ-expansive* on an open set $Q \subset D(T)$ if every point $x_0 \in Q$ has a neighborhood $M = M(x_0) \subset Q$ such that T is ϕ-expansive on M. The function ϕ here depends on the neighborhood $M(x_0)$ of x_0.

An operator $T : X \supset D(T) \to 2^X$ is *of type* (S_1) if for every bounded sequence $\{x_n\} \subset D(T)$ with $y_n \to y$, for some sequence of terms $y_n \in Tx_n$, we have $y \in R(T)$. We say that T is *of type* (S_1) *at zero* if $y = 0$ in the previous definition. We say that T is *of type* (S_2) if for every bounded sequence $\{x_n\} \subset D(T)$ with $y_n \to y$, for some sequence of terms $y_n \in Tx_n$, there is $x \in X$ such that $x_n \to x$ and $y \in Tx$. T is *of type* (S_2) *at zero* if $y = 0$ in the previous definition.

It is easy to see that every single-valued, demicontinuous and ϕ-expansive (hence strongly accretive) operator on a closed set is of type (S_2). Every m-accretive and ϕ-expansive (hence strongly accretive) operator is also of type (S_2) because it is a closed operator (its graph is a closed subset of $X \times X$). In addition, every m-accretive operator is of type (S_1) provided that X is a (BCC) *space*, i.e., every nonempty, bounded, closed and convex subset of X has the fixed point property for nonexpansive self-mappings (cf. Reich and Torrejón [24]). A real Banach space is called a (BUC) *space* if its closed unit ball has the fixed point property for nonexpansive self-mappings.

The following lemma is Theorem 1.8 in Torrejón's paper [25].

LEMMA A. *Let X be a (BUC) space and let $T : X \supset D(T) \to 2^X$ be m-accretive. Then the following statements are equivalent*:

(i) $0 \in R(T)$;

(ii) *there exist $r > 0$ and $x_0 \in D(T)$ such that for every $x \in \partial B_r(x_0) \cap D(T)$ there exists $j \in J(x - x_0)$ such that*

$$\langle y, j\rangle \ge 0 \quad \text{for every } y \in Tx;$$

(iii) *there exist* $r > 0$ *and* $x_0 \in D(T)$ *such that*

$$Tx \not\ni \mu(x-y), \quad (\mu, x, y) \in (-\infty, 0) \times (\partial B_r(x_0) \cap D(T)) \times B_r(x_0).$$

If (ii) *or* (iii) *holds, then* $0 \in T(\overline{B_r(x_0)} \cap D(T))$.

A variant to Condition (iii) of Lemma A is Condition (jjj) below, which will be needed in Theorem 1.

(jjj) there exists $r > 0$ and $x_0 \in D(T)$ such that

$$Tx \not\ni \mu(x - x_0), \quad (\mu, x) \in (-\infty, 0) \times (\partial B_r(x_0) \cap D(T)).$$

LEMMA 1. *Let* $T : X \supset D(T) \to 2^X$ *be m-accretive and of type* (S_1) *at zero. Then*

(i) *Conditions* (i), (ii) *of Lemma A and* (jjj) *are equivalent. If, moreover, T is of type* (S_2) *at zero and* (ii) *or* (jjj) *holds, then* $0 \in T(\overline{B_r(x_0)} \cap D(T))$;

(ii) *Condition* (i) *of Lemma A and the following two conditions are equivalent*:

(iia) *there exists an open, bounded* $G \subset X$ *and* $x_0 \in G \cap D(T)$ *such that for every* $x \in \partial G \cap D(T)$ *there exists* $j \in J(x - x_0)$ *such that*

$$\langle y, j \rangle \geq 0 \quad \text{for every } y \in Tx;$$

(iib) *there exists an open, bounded* $G \subset X$ *and* $x_0 \in G \cap D(T)$ *such that*

$$Tx \not\ni \mu(x - x_0), \quad (\mu, x) \in (-\infty, 0) \times (\partial G \cap D(T)).$$

If, moreover, T is of type (S_2) *at zero and* (iia) *or* (iib) *holds, then* $0 \in T(\overline{G} \cap D(T))$.

Proof. Since (jjj) is obviously implied by (ii) by Lemma A, assume that (jjj) is true. Then $g(t) \equiv (tT + I)^{-1}x_0$ maps $[0, \infty) \to D(T)$ and is continuous on $[0, \infty)$. As such, it enters the ball $B_r(x_0)$ at $t = 0$. Assume that it leaves the ball at some point $t > 0$. Let $x_t = g(t)$. Then $\|x_t\| = r$ and, for some $y_t \in Tx_t, ty_t + x_t = x_0$. It follows that $y_t = (-1/t)(x_t - x_0)$, which is a contradiction to (jjj). It follows that $g(t) \in B_r(x_0)$ for all $t \in \mathbb{R}_+$. This says that for every $t \in \mathbb{R}_+$ there exists $x_t \in B_r(x_0)$ such that

$$tTx_t + x_t \ni x_0.$$

Let $\{t_n\} \subset \mathbb{R}_+$ with $t_n \to \infty$ as $n \to \infty$ and $t_n > 0$ for all n. Then, for $x_n \equiv x_{t_n}$ and some $y_n \in Tx_n$,

$$y_n = (-1/t_n)(x_n - x_0) \to 0 \quad \text{as } n \to \infty.$$

Since T is of type (S_1), we have $0 \in \overline{R(T)}$. If, moreover, T is of type (S_2) at zero, then $x_n \to$ (some) $x \in \overline{B_r(x_0)}$ with $0 \in Tx$. Thus, $0 \in T(\overline{B_r(x_0)} \cap D(T))$. In the second part of the lemma we easily see that (i)⇒(iia)⇒(iib). The fact that (iib)⇒(i) follows as above by replacing $B_r(x_0)$ by G. ∎

We apply Lemma A to obtain the following range theorem for m-accretive operators. For a set $A \subset X$, we set $|A| = \inf\{\|x\| : x \in A\}$.

THEOREM 1. *Assume that* $T : X \supset D(T) \to 2^X$ *is m-accretive and of type* (S_1). *Assume, further, that there exist* $q > 0$ *and* $x_0 \in D(T)$ *such that*

$$|Tx_0| < q \leq |Tx|, \quad x \in \partial B_r(x_0) \cap D(T).$$

Then $B_q(0) \subset \overline{R(T)}$. *If, moreover, T is of type* (S_2), *then* $B_q(0) \subset T(\overline{B_r(x_0)} \cap D(T))$.

Proof. There is no loss of generality if we assume that $x_0 = 0$. If this is not true, we consider instead the operator $\widetilde{T}$ defined by $\widetilde{T}x = T(x + x_0)$, $x \in D(\widetilde{T})$, where $D(\widetilde{T}) \equiv D(T) - x_0$. In this case we should also replace the ball $B_r(x_0)$ by the ball $B_r(0)$.

Now, let $T_1x \equiv Tx - v$, where v is a fixed point in $B_{(q-|T(0)|)/2}(0)$. Taking into consideration the proof of the Lemma in Morales' paper [20], we see easily that (iib) holds, with T_1 in place of T. Thus, by Lemma 1, we have $B_{(q-|T(0)|)/2}(0) \subset \overline{R(T)}$. If, in addition, T is of type (S_2), then $B_{(q-|T(0)|)/2}(0) \subset T(\overline{B_r(x_0)} \cap D(T))$. Actually,

$$B_{(q-|T(0)|)/2}(0) \subset T(B_r(x_0) \cap D(T))$$

because $y \in T(\partial B_r(0) \cap D(T))$ implies $\|y\| \geq q > (q - |T(0)|)/2$.

Now, we continue the proof of the first part of the theorem, i.e., we only assume that T is of type (S_1). We fix $u \in B_q(0)$ and consider the set

$$Q = \{t \in [0, 1] : tu \in \overline{R(T)}\}.$$

As in Kirk and Schöneberg [18, Theorem 3], we can show that $t = 1 \in Q$, i.e., that $B_q(0) \subset \overline{R(T)}$.

Now, assume that T is of type (S_2) and consider the set

$$Q = \{t \in [0, 1] : tu \in T(B_r(x_0) \cap D(T))\}.$$

We have $Q \neq \emptyset$ because $0 \in Q$. We let $t_0 = \sup Q$ and assume that $t_0 < 1$. We let $t_n \in Q$ with $t_n \to t_0$ and $u_n \in B_r(x_0) \cap D(T)$ be such that $t_n u \in Tu_n$. We define the operators T_n as follows:

$$T_n x = T(x + u_n) - t_n u, \quad x \in D(T) - u_n.$$

We have $0 \in T_n(0)$ and

$$0 = |T_n(0)| < q - t_n q < q - t_n\|u\| \leq |T_n(x)|,$$
$$x \in \partial(B_r(x_0) - u_n) \cap (D(T) - u_n).$$

Since T_n is m-accretive and of type (S_2), we may quote the above argument to obtain $B_{(q-t_n\|u\|)/2} \subset T((B_r(x_0) - u_n) \cap (D(T) - u_n))$. Since $t_0 < 1$, there exists $t \in (t_0, 1)$ and some n so that $(t - t_n)\|u\| < (q - t_n\|u\|)/2$. This is true because $t - t_n \to 0$ as $(t, n) \to (t_0^+, \infty)$. It follows that we may choose $v_n \in (B_r(x_0) - u_n) \cap (D(T) - u_n)$ such that $(t - t_n)u \in T_n(v_n)$, which implies that $tu \in T(v_n + u_n)$, i.e., $tu \in T(B_r(x_0) \cap D(T))$. Since $t > t_0$, we have the desired contradiction. The proof is complete. ■

The above theorem yields the following general invariance of domain result for m-accretive operators, which improves Theorem 3 of Morales [20] and Theorem 3 of the author in [8]. The author assumed in [8] that X^* is uniformly convex and Morales assumed in [20] that X is a (BCC) space.

THEOREM 2. *Assume that $T : X \supset D(T) \to X$ is m-accretive and $G \subset D(T)$ is open and bounded. If T is locally ϕ-expansive on G and of type (S_1), then TG is open. If T is of type (S_2), $B_r(x_0) \subset D(T)$, for some $r > 0$, and T is ϕ-expansive on $\partial B_r(x_0)$, then $B_{\phi(r)}(Tx_0) \subset T(B_r(x_0))$.*

Proof. Let $x_0 \in G$. Then T is ϕ-expansive on $\overline{B_r(x_0)} \subset G$ for some $r > 0$. As in the proof of Theorem 1, we may assume that $x_0 = 0$. We observe that the ϕ-expansiveness of T on $B_r(0)$ implies that

$$(1) \qquad 0 = \|T(0)\| < \phi(r) = \phi(\|x\|) \le \|Tx\|, \quad x \in \partial B_r(0).$$

Working as in the proof of Theorem 1 (i.e., using the proof of the Lemma in [20]), we now see that if $\widetilde{T}x = Tx - v$, where v is a fixed point in $B_{(\phi(r) - \|T(0)\|)/2}$, then (iib) holds. From Theorem 1, we thus obtain $B_{(\phi(r) - \|T(0)\|)/2} \subset R(T)$. Here we could also use Lemma 1 of the author in [8]. A careful examination of the proof of Theorem 1 now reveals that since $\widetilde{T}$ is actually ϕ-expansive on the ball $\overline{B_r(0)}$ and closed, we have $B_{(\phi(r) - \|T(0)\|)/2} \subset T(B_r(0))$. It follows that TG is open.

To show the second part of the theorem, we first observe that (1) is still true in this part. Since T is now of type (S_2), we may quote Theorem 1 in order to obtain $B_{\phi(r)}(0) \subset T(B_r(0))$. ■

A set $M \subset X$ is *pathwise connected* if for every $x, y \in M$ there exists a continuous function $s : [0, 1] \to M$ such that $s(0) = x$ and $s(1) = y$. We are now ready for a basic result involving the existence of pathwise connected sets lying in the range of an accretive operator.

PROPOSITION 1. *Let $G \subset X$ be open and bounded. Assume that one of the following conditions is satisfied:*

(i) *X^* is uniformly convex and $T : \overline{G} \to X$ is demicontinuous and accretive;*

(ii) *$T : \overline{G} \to X$ is continuous and accretive;*

(iii) *$T : X \supset D(T) \to X$ is m-accretive and $\overline{G} \subset D(T)$.*

Assume, further, that $M \subset X$ is pathwise connected and such that $M \cap TG \neq \emptyset$ and $M \cap \overline{T(\partial G)} = \emptyset$. Then $M \subset \overline{TG}$. If, moreover, Case (i) or (ii) holds and T is of type (S_1), or Case (iii) holds and T is of type (S_2), then $M \subset TG$.

Proof. Let (i) hold. We may (and do) assume that $0 \in M \cap TG, 0 \in G$ and $T(0) = 0$. In fact, if this is not already true, we fix $y_0 \in M \cap TG$ and consider instead of T the operator $\widetilde{T}x \equiv T(x + x_0) - y_0, x \in \overline{\widetilde{G}}$, where $Tx_0 = y_0$ and $\widetilde{G} = G - x_0$. We also consider instead of M the set $\widetilde{M} \equiv M - y_0$. It is easy to see that $\widetilde{M}$ is pathwise connected, $\widetilde{M} \cap \overline{\widetilde{T}(\partial\widetilde{G})} = \emptyset$ and $\widetilde{M} \cap \widetilde{T}(\widetilde{G}) \ni 0, 0 \in \widetilde{G}$ and $\widetilde{T}(0) = 0$. Moreover, $\widetilde{T}$ is demicontinuous and accretive.

We consider the operators $T_n x \equiv Tx + (1/n)x, n = 1, 2, \ldots, x \in D(T)$. We fix $y \in M$ and let $s : [0, 1] \to M$ be continuous and such that $s(0) = 0$ and $s(1) = y$.

We show first that $s(t) \notin T_n(\partial G)$ for all large n. In fact, if this is not true, then we may assume, without loss of generality, that there exists an infinite sequence $\{t_n\} \subset [0, 1]$ such that $s(t_n) \in T_n(\partial G)$ for all n. Since $[0, 1]$ is compact, we may also assume that $t_n \to t_0 \in [0, 1]$. Then $s(t_n) = Tx_n + (1/n)x_n$ for some sequence $\{x_n\} \subset \partial G$, and so $s(t_n) \to s(t_0)$. Since $\{x_n\}$ is bounded, we have $s(t_0) \in \overline{T(\partial G)}$, which contradicts our assumption.

Let $s(t) \notin T_n(\partial G)$ for $n \ge n_0$, for some $n_0 \ge 1$. From now on we consider only such values of n. Since the operator T_n is demicontinuous and strongly accretive, the invariance of domain theorem of the author [9, Theorem 1] implies that $T_n G$ is open. This says in turn that the set $Q_n \equiv \{t \in [0, 1] : s(t) \in T_n G\}$ is open in $[0, 1]$ (with $[0, 1]$ endowed with the relative topology as a subspace of $\mathbb{R}$) because $Q_n = s^{-1}(T_n G)$. Obviously, $Q_n \neq \emptyset$ because $0 \in Q_n$.

In order to show that Q_n is also closed in $[0, 1]$, we let $\{t_m\} \subset Q_n$ be such that $t_m \to t \in [0, 1]$ and we observe that $s(t_m) = T_n x_m$ for some sequence $\{x_m\} \subset G$, and $s(t_m) \to s(t)$. Since

$$s(t_m) = Tx_m + (1/n)x_m,$$

we have

$$(1/n) \limsup_{m,j\to\infty} \langle x_m - x_j, J(x_m - x_j)\rangle = (1/n) \limsup_{m,j\to\infty} \|x_m - x_j\|^2 \le 0,$$

which implies that $\{x_m\}$ is a Cauchy sequence. Letting $x_m \to \widetilde{x} \in \overline{G}$, we have $s(t) = T_n\widetilde{x}$. Since $s(t) \notin T_n(\partial G)$, we have $s(t) \in T_n G$. It follows that Q_n is closed in $[0, 1]$.

Since it is also open in $[0, 1]$, we must have $Q_n = [0, 1]$. Thus, there exists a sequence $\{x_n\} \subset G$ such that $s(1) = T_n x_n$, or

$$Tx_n + (1/n)x_n = y.$$

The boundedness of $\{x_n\}$ implies $Tx_n \to y$. Thus, $y \in \overline{TG}$. If, moreover, T is of type (S_1), then $y \in R(T) = T\overline{G}$. Since $y \in M$, we cannot have $y \in T(\partial G)$. Thus, $y \in TG$.

A very similar proof covers the case (ii). The basic difference is that we now make use of the invariance of domain result of Deimling [4, Theorem 3].

In Case (iii), T_n is m-accretive and ϕ-expansive. Thus, T_n is of type (S_1). Theorem 2 implies that T_nG is open. Letting $x_m \to \widetilde{x} \in \overline{G}$, we have $T_nx_m \to s(t)$ as $m \to \infty$. Since T_n is closed, being m-accretive, we have $s(t) = T_n\widetilde{x}$. It follows that $y \in \overline{TG}$. If T is of type (S_2), then there exists $x \in \overline{G}$ such that $Tx = y$. Since $y \notin T(\partial G)$, we must have $y \in TG$. ∎

We now give an application of Proposition 1, where it is shown that a ball lies in the range of a certain accretive operator. An operator $T : X \supset D(T) \to X$ is *locally bounded* on $G \subset D(T)$ if for every point $x \in G$ there exists an open ball $B_r(x) \subset G$ (with r depending on x) such that $T(B_r(x))$ is bounded.

THEOREM 3. *Let $G \subset X$ be open, bounded. Assume that $T : X \supset D(T) \to X$ is accretive, with $\overline{G} \subset D(T)$, and that there exists $x_0 \in G$ such that*

$$\|Tx_0\| < \|Tx\|, \quad x \in \partial G, \tag{2}$$

with $r = \inf\{\|Tx\| : x \in \partial G\} > 0$. Let T satisfy one of the conditions (i), (ii) *of Proposition* 1 *and be of type* (S_1), *or Condition* (iii) *and be of type* (S_2). *Assume that T is locally bounded on G. Then $B_r(0) \subset TG$.*

Proof. We observe first that we may take $x_0 = 0$. If this is not already true, we consider instead of T the operator $T_1x \equiv T(x + x_0)$, $x \in \overline{G}_1$, where $G_1 \equiv G - x_0$. Thus, we have $\|T(0)\| < \|Tx\|$, $x \in \partial G$.

We start the proof assuming that Case (iii) of Proposition 1 holds, i.e., we assume that T is m-accretive and of type (S_2). We introduce a perturbation to the problem by considering the operator $T_\alpha \equiv T + \alpha I$, $\alpha > 0$.

To show that T_α satisfies Condition (iib) of Lemma 1, assume that this is false and let $T_\alpha x = \mu x$ for some $(\mu, x) \in (-\infty, 0) \times \partial G$. Then, for some $j_x \in Jx$,

$$\langle Tx - T(0), j_x\rangle + \langle T(0), j_x\rangle = (\mu - \alpha)\langle x, j_x\rangle,$$

which implies

$$-\|T(0)\| \cdot \|x\| \le (\mu - \alpha)\langle x, j_x\rangle = (\mu - \alpha)\|x\|^2$$

and $\|T(0)\| \ge (\alpha - \mu)\|x\| = \|Tx\|$. This contradicts (2). It follows that (iib) holds.

Since T_α is of type (S_2) at zero, Lemma 1 says that $Tx + \alpha x = 0$ is solvable with solution $x_\alpha \in \overline{G}$ for every $\alpha > 0$. Since $Tx \ne -\alpha x$ for $x \in \partial G$, we have $x_\alpha \in G$. Since G is bounded, we let $\alpha \to 0$ to obtain $Tx_\alpha \to 0 \in \overline{TG}$.

This implies that for every $\varepsilon \in (0, r)$ we have $\overline{B_{r-\varepsilon}(0)} \cap TG \ne \emptyset$. We also observe that $\|Tx\| \ge r$, $x \in \partial G$, which implies $\overline{B_{r-\varepsilon}(0)} \cap \overline{T(\partial G)} = \emptyset$. By Proposition 1, with $M = \overline{B_{r-\varepsilon}(0)}$, we have $\overline{B_{r-\varepsilon}(0)} \subset TG$. Since $\varepsilon \in (0, r)$ is arbitrary, we have $B_r(0) \subset TG$ and the proof of this part is complete.

In case (i) of Proposition 1 holds, we need to quote Theorem 4 of Morales [21], where the Leray–Schauder condition (iib) is used in connection with the demicontinuous accretive mapping T in order to obtain

$$\inf\{\|Tx\| : x \in \overline{G}\} = 0.$$

Actually, the local boundedness assumption is redundant here because a demicontinuous accretive operator is locally bounded on the interior of its domain whenever X^* is uniformly convex (cf., for example, [12, Theorem 3.1]).

Now, assume that (ii) of Proposition 1 is true. We fix $\alpha > 0$ and consider the set

$$Q = \{t \in \mathbb{R}_+ : tTx + \alpha x = 0 \text{ for some } x \in G\}.$$

Since $0 \in Q$, $Q \ne \emptyset$. We are going to show that the set Q is open and closed in $\mathbb{R}_+$. To show that it is closed, let $\{t_n\} \subset Q$ be such that $t_n \to t \in \mathbb{R}_+$. Then, for some sequence $\{x_n\} \subset G$,

$$t_nTx_n + \alpha x_n = 0.$$

Since $0 \in Q$, we assume that $t > 0$. We may also assume that $t_n > 0$. We have, for an appropriate $j \in J(x_n - x_m)$,

$$t_n\langle Tx_n - Tx_m, j\rangle + (t_n - t_m)\langle Tx_m, j\rangle + \alpha\langle x_n - x_m, j\rangle = 0,$$

which, using the accretiveness of T, implies

$$\alpha\|x_n - x_m\|^2 \le |t_n - t_m| \cdot \|Tx_m\| \cdot \|x_n - x_m\| \to 0 \quad \text{as } m, n \to \infty,$$

because $\|Tx_m\| = (\alpha/t_m)\|x_m\|$ and the boundedness of G imply the boundedness of $\{Tx_m\}$. It follows that $x_n \to$ (some) $x \in \overline{G}$, and, by continuity, $tTx + \alpha x = 0$. Since, as in the first case considered above, $Tx \ne (-\alpha/t)x$, $x \in \partial G$, we have $x \in G$. Thus, Q is closed.

To show that Q is open, assume that it is not and let $\{t_n\} \subset \mathbb{R}_+$ be such that $t_n \to t \in Q$ and $t_n \notin Q$, $n = 1, 2, \ldots$ Obviously, $t_n > 0$ for all n. We let $g_t(x) \equiv tTx + \alpha x$. Since $t \in Q$, there is $x \in G$ such that $g_t(x) = 0$. Since G is open, there exists $r > 0$ such that $B \equiv \overline{B_r(x)} \subset G$. Since T is locally bounded on G, we may assume that TB is bounded. We have

$$y_n \equiv g_{t_n}(x) \in g_{t_n}(B)$$

and $0 \notin g_{t_n}(B)$. Thus, the line segment $[0, y_n]$, with endpoints 0 and y_n, must intersect the set $\partial g_{t_n}(B)$. Let $v_n \in [0, y_n] \cap \partial g_{t_n}(B)$. The strong accretiveness of g_{t_n} implies that $g_{t_n}(B)$ is open by Deimling's invariance of domain theorem [4, Theorem 3]. It is also easy to see that $g_{t_n}(\overline{B})$ is closed.

This implies that $\partial g_{t_n}(B) \subset g_{t_n}(\partial B)$. Thus, there exists $x_n \in \partial B$ with $g_{t_n}(x_n) = v_n$. Since $y_n = g_{t_n}(x) \to g_t(x) = 0$, we have $v_n \to 0$. From

$$t_n T x_n + \alpha x_n = v_n$$

and the boundedness of $\{Tx_n\}$, we can easily see that x_n is a Cauchy sequence. Letting $x_n \to \overline{x} \in \partial G$, we have $g_t(\overline{x}) = 0$. Since g_t is injective, we must have $\overline{x} = x$, i.e., a contradiction. It follows that $Q = \mathbb{R}_+$, which implies that the equation $Tx + (1/n)x = 0$ is solvable for every $n = 1, 2, \ldots$ with solution $x_n \in G$. Thus, $0 \in \overline{TG}$. The rest of the proof of this part follows as in the proof of the first part above. It is therefore omitted. ■

From the proof of Theorem 3 we have the following result.

THEOREM 4. *Let $T : \overline{G} \to X$ be continuous, accretive and of type (S_1) at zero, where G is open and bounded. Let $Tx \neq 0, x \in \partial G$. Then the following statements are equivalent*:

(i) $0 \in TG$;

(ii) *there exists $x_0 \in G$ such that for every $x \in \partial G$ there exists $j \in J(x - x_0)$ such that $\langle Tx, j \rangle \geq 0$;*

(iii) *there exists $x_0 \in G$ such that $Tx \neq \mu(x - x_0)$ for every $(\mu, x) \in (-\infty, 0) \times \partial G$.*

Proof. To show (i)⇒(ii), it suffices to take $x_0 \in G$ so that $Tx_0 = 0$ and use the accretiveness of T. To show that (ii)⇒(iii), assume that (ii) is true and observe that

$$Tx = \mu(x - x_0), \quad (\mu, x) \in (-\infty, 0) \times \partial G,$$

implies, for an appropriate $j \in J(x - x_0)$,

$$0 \leq \langle Tx, j \rangle = \mu \langle x - x_0, j \rangle = \mu \|x - x_0\|^2 < 0,$$

i.e., a contradiction. The implication (iii)⇒(i) follows from the proof of Theorem 3 because, as we saw there, $Tx + (1/n)x = 0$ is solvable, for every $n = 1, 2, \ldots$, with solution $x_n \in G$. Thus, $Tx_n \to 0$ implies $0 \in T\overline{G}$. Since $0 \notin T(\partial G)$, $TG \ni 0$. ■

The assumption that $Tx \neq 0, x \in \partial G$, in Theorem 4 is implied by the assumption that there exists $\overline{x} \in G$ such that $\|T\overline{x}\| < \|Tx\|, x \in \partial G$. A related result is contained in the following theorem.

THEOREM 5. *Let $T : X \supset D(T) \to X$ be accretive and $\overline{G} \subset D(T)$, where G is open and bounded. Assume, further, that T is of type (S_1), locally ϕ-expansive on G and such that $TG \cap T(\partial G) = \emptyset$ and $T\overline{G}$ is closed. Then if T satisfies one of the assumptions* (i)–(iii) *of Proposition* 1, *the following statements are equivalent*:

(i) $0 \in TG$;

(ii) *there exists $x_0 \in G$ such that $\|Tx_0\| \leq \|Tx\|, x \in \partial G$.*

Proof. We give the proof only for the case (iii) of Proposition 1. All the other cases can be treated similarly in connection with the proofs of the respective parts of Proposition 1. It is obvious that (i) implies (ii). We assume that (ii) is true. We know that T is m-accretive, locally ϕ-expansive on G and of type (S_1). By Theorem 2, we see that TG is open. Since $T\overline{G}$ is closed and $TG \cap T(\partial G) = \emptyset$, it follows that $\partial TG \subset T(\partial G)$. We are going to show that the set

$$M = \{tTx_0 : t \in [0, 1]\}$$

lies in TG. Let us first show that $M \cap \partial TG = \emptyset$. To this end, assume that $y \in M \cap \partial TG$. Then $y \in T(\partial G)$ and so $y = T\overline{x}$ for some $\overline{x} \in \partial G$. If $Tx_0 = 0$, we are done. Thus, we may assume that $Tx_0 \neq 0$. If $y = 0$, then $\|Tx_0\| \leq \|T\overline{x}\| = \|y\| = 0$, i.e., a contradiction. Hence, $y \neq 0$ and there exists $t \in (0, 1]$ such that $y = T\overline{x} = tTx_0$. If $t = 1$, then $y \in TG \cap T(\partial G)$, i.e., a contradiction. Consequently, $t \in (0, 1)$, which implies $t\|Tx_0\| < \|Tx_0\| \leq \|T\overline{x}\| = t\|Tx_0\|$, i.e., a contradiction again. We conclude that $M \cap \partial TG = \emptyset$. Since M is connected and $M \cap TG \neq \emptyset$ ($Tx_0 \in TG$), we must have $M \subset TG$. It follows that $0 \in TG$ and the proof is complete. ■

3. Compact perturbations. In this section we give an improvement of Theorem 3 of the author in [11], where it was assumed that C is also uniformly continuous on bounded sets. Namely, we give the following result.

THEOREM 6. *Let $T : X \supset D(T) \to 2^X$ be m-accretive and $C : \overline{D(T)} \to X$ compact. Let $p \in X$ and assume that there exists a positive constant b such that $x \in D(T)$ and $\|x\| \geq b$ imply that there exists $j \in Jx$ such that*

$$(*) \qquad \langle u + Cx - p, j \rangle \geq 0$$

for all $u \in Tx$. Then $p \in \overline{(T + C)(B_b(0) \cap D(T))}$.

Before we give the proof of this theorem, we quote a result from Nagumo's paper [23, Theorem 7].

THEOREM A. *Let $T : [0, 1] \times \overline{G} \to X$ be continuous, with G an open subset of X, and such that $T([0, 1], \overline{G}) \subset K$, where K is a compact set. Assume that $s : [0, 1] \to X$ is continuous and such that $s(t) \notin (I + T(t, \cdot))(\partial G)$ for every $t \in [0, 1]$. Then $d(I + T(t, \cdot), G, s(t)) =$ const.*

Proof of Theorem 6. In the proof of Theorem 3 of [11] we have shown that there exists a ball $B_q(0)$ such that $x + U(t, x) = 0$ has no solution $x_t \in \partial B_q(0)$ for any $t \in [0, 1)$, where

$$U(t, u) \equiv t[C(ntT + I)^{-1}(nu) - p].$$

If such a solution x_t exists for $t = 1$, we are done. Thus, we may assume that no such solution exists for $t \in [0,1]$. Let

$$y_t(u) \equiv (ntT + I)^{-1}(nu).$$

We have, for $t > 0$, $t_0 > 0$ and $u \in \overline{B_q(0)}$,

$$\|y_t(u) - y_{t_0}(u_0)\| \le \|y_t(u) - y_{t_0}(u)\| + \|y_{t_0}(u) - y_{t_0}(u_0)\|$$
$$\le \frac{2|t - t_0|}{t_0}\|nu - x_0\| + n\|u - u_0\| \to 0 \quad \text{as } (t,u) \to (t_0, u_0),$$

where we have used estimates from the proof of Theorem 3 in [11]. Thus, $Cy_t(u) \to Cy_{t_0}(u_0)$ as $(t,u) \to (t_0,u_0)$ with $t_0 > 0$. Also, $(t,u) \to tCy_t(u)$ is continuous at (t_0,u_0) provided that $t_0 > 0$ and $u_0 \in \overline{B_q(0)}$. It follows that the operator $U(t,u)$ is continuous at every (t_0,u_0) with $t_0 > 0$.

From the above inequality for $\|y_t(u) - y_{t_0}(u_0)\|$, we see that the set $M \equiv \{y_t(u) : (t,u) \in (0,1] \times \overline{B_q(0)}\}$ is bounded. This says that the set $\overline{CM - p}$ is compact and implies that $\|U(t,u)\| = t\|Cy_t(u) - p\| \to 0$ as $t \to 0^+$ uniformly w.r.t. u. It follows that the operator $U(t,u)$ is continuous at any point $(t_0,u_0) \in [0,1] \times \overline{B_q(0)}$. The set $U([0,1],\overline{B_q(0)}) = \{t[Cy_t(u) - p] : t \in [0,1], u \in \overline{B_q(0)}\}$ lies inside the compact set $K = [0,1] \cdot \overline{CM - p}$ because multiplication $((t,x) \to tx)$ is continuous and the set $[0,1] \times \overline{CM - p}$ is compact.

Applying Nagumo's theorem above, for $T(t,u) \equiv U(t,u)$, $G = B_q(0)$ and $s(t) \equiv 0$, we obtain the solvability of the equation $x + U(1,x) = 0$ for every n and we are done. In fact,

$$Tx_n + Cx_n + (1/n)x_n \ni p,$$

for some sequence $\{x_n\} \subset B_b(0) \cap D(T)$. Since $\{x_n\}$ is bounded we have our conclusion. ■

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