constant function on each X_k with $1 \leq k \leq n$, and thus by (10) we have the same inequality (13) for this f. It follows that $\liminf_n \|M_n\|_{pq} < \infty$.

But (11) gives, for $f = 1_E$ with $E = \{(k,1)\}$,

$$\frac{1}{\|f\|_{pq}} \left\| \frac{1}{b_k} \sum_{i=0}^{b_k-1} T^i f \right\|_{pq} = \frac{1}{b_k} (\mu(\{(k,i) : 1 \leq i \leq b_k\}))^{1/p} > k,$$

and hence we have $\sup_n \|M_n(T)\|_{pq} = \infty$.

Remark. A slight modification of the above example shows that if $(p,q) \neq (\infty,\infty)$, then we may have $\lim_n \|M_n(T)\|_{pq} = \infty$ for T which satisfies the pointwise ergodic theorem from $L(p,q)$ to itself. It follows that the converse of Theorem 2 does not hold.

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Received April 19, 1993
Revised version December 10, 1993 (3097)

STUDIA MATHEMATICA 114 (3) (1995)

Automatic extensions of functional calculi

by

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Abstract. Given a Banach algebra $\mathcal{F}$ of complex-valued functions and a closed, linear (possibly unbounded) densely defined operator A, on a Banach space, with an $\mathcal{F}$ functional calculus we present two ways of extending this functional calculus to a much larger class of functions with little or no growth conditions. We apply this to spectral operators of scalar type, generators of bounded strongly continuous groups and operators whose resolvent set contains a half-line. For f in this larger class, one construction measures how far $f(A)$ is from generating a strongly continuous semigroup, while the other construction measures how far $f(A)$ is from being bounded. We apply our constructions to evolution equations.

I. Introduction and preliminaries. Suppose $\mathcal{F}$ is a Banach algebra of complex-valued functions on a subset of the complex plane. If A is in $B(X)$, the space of bounded linear operators from the Banach space X into itself, and $\mathcal{F}$ contains both $f_0(z) \equiv 1$ and $f_1(z) \equiv z$, then an $\mathcal{F}$ functional calculus for A is a continuous algebra homomorphism, $f \mapsto f(A)$, from $\mathcal{F}$ into $B(X)$, such that $f_0(A) = I$, the identity operator, and $f_1(A) = A$.

When A is unbounded, then we cannot have $f_1 \in \mathcal{F}$. Something more indirect is required to involve A in its functional calculus. We will essentially use the definition of a functional calculus given in [8], except that we will also consider Banach algebras $\mathcal{F}$ that may not contain f_0; thus in (3) of Definition 1.2 we stipulate that functions $z \mapsto (\lambda - z)^{-m}$ are mapped where one would expect.

It is convenient to introduce terminology and important concepts before proceeding further.

TERMINOLOGY AND HYPOTHESES 1.1. All operators considered are linear. Throughout, we will assume that A is a closed, densely defined operator on a Banach space X. We will write $\mathcal{D}(A)$ for the domain of A, $\varrho(A)$ for the resolvent set of A, $B(X)$ for the Banach space of bounded operators from X to itself. We will write $\mathrm{Im}(B)$ for the image of an operator B. The space $\mathcal{F}$ will always be a Banach algebra of complex-valued functions on a subset

1991 *Mathematics Subject Classification*: Primary 47A60; Secondary 47B40, 47D05.

of the complex plane. We will write $f_0(z) \equiv 1$, $f_1(z) \equiv z$, $g_\lambda(z) \equiv (\lambda - z)^{-1}$, for complex λ.

Throughout this paper, we will use the following sort of informal terminology for our functional calculi: $(e^{tz})(A)$ will be a shorthand for $f_t(A)$, $f_t(z) \equiv e^{tz}$. In general, we will use z for the dependent variable for analytic functions, s for the dependent variable for functions defined only on the real line.

DEFINITION 1.2. An $\mathcal{F}$ *functional calculus for* A is a continuous algebra homomorphism, $f \mapsto f(A)$, from $\mathcal{F}$ into $B(X)$, such that

(1) whenever both f and ff_1 are in $\mathcal{F}$, then $\mathrm{Im}(f(A)) \subseteq \mathcal{D}(A)$, with

$$f(A)A \subseteq Af(A) = (ff_1)(A);$$

(2) whenever $f_0 \in \mathcal{F}$, then $f_0(A) = I$; and

(3) whenever $g_\lambda^m \in \mathcal{F}$, for some $m \in \mathbb{N}$, complex λ, then $\lambda - A$ is injective and $g_\lambda^m(A) = (\lambda - A)^{-m}$.

Note that, when $g_\lambda^k \in \mathcal{F}$, for $0 \le k \le m$, then (3) follows automatically from (1) and (2), and, conversely, (2) follows automatically from (1) and (3).

The following, introduced in [2], will be fundamental to our constructions. A general reference for regularized semigroups is [5].

DEFINITION 1.3. The strongly continuous family of bounded operators $\{W(t)\}_{t\ge 0}$ is a C*-regularized semigroup* if $W(0) = C$ is injective, and $W(t)W(s) = W(t+s)C$, for all $s, t \ge 0$. The *generator* is defined by

$$Bx = C^{-1}\left(\lim_{t\to 0} \frac{1}{t}(W(t)x - Cx)\right),$$

with maximal domain, that is, the domain of B equals the set of all x for which the limit exists and is in the image of C.

The generator is automatically closed, and, although it may not be densely defined, the image of C is contained in the closure of the domain of B.

In this paper, we show that, when A has an $\mathcal{F}$ functional calculus, then there exist much larger classes of functions, which we call $\mathrm{EXT}_j(\mathcal{F})$ ($j = 1, 2$), such that A has an unbounded $\mathrm{EXT}_j(\mathcal{F})$ functional calculus, that is, a map from $\mathrm{EXT}_j(\mathcal{F})$ into the set of all closed operators on X, that extends the $\mathcal{F}$ functional calculus for A, and is an algebra homomorphism, in some sense that must be specified; since $f(A)$ will be unbounded in general, it is not always clear what analogue of the multiplicative property $f(A)g(A) = (fg)(A)$ is reasonable to expect. In many cases, all growth restrictions on the functions in $\mathcal{F}$ may be removed in passing to $\mathrm{EXT}_j(\mathcal{F})$. For example, when $\mathcal{F}$ equals the space of bounded Borel measurable functions on the real line, $\mathrm{EXT}_j(\mathcal{F})$ equals the space of all Borel measurable functions on the real line.

"EXT" stands for "extension".

In Construction One, for $f \in \mathrm{EXT}_1(\mathcal{F})$, we will define $f(A)$ indirectly, by using the $\mathcal{F}$ functional calculus to define $\{(e^{tf}h)(A)\}_{t\ge 0}$, a regularized semigroup, and then defining $f(A)$ to be the generator. Essential to this construction is the following.

LEMMA 1.4 ([5, Propositions 3.10 and 3.11]). *Suppose* $\{W(t)\}_{t\ge 0}$ *is a* C*-regularized semigroup generated by* B.

(a) *If* C_1 *is an injective operator that commutes with* $W(t)$, *for all* $t \ge 0$, *then* $\{C_1W(t)\}_{t\ge 0}$ *is a* CC_1*-regularized semigroup generated by* B.

(b) $B = C^{-1}BC$.

Lemma 1.4(a) guarantees that no matter how much regularizing we do, the generator (which will be $f(A)$) will be the same.

There are two types of "good" behavior an operator can have: it can be bounded or it can generate a strongly continuous semigroup of bounded operators. From the point of view of applications, the latter is more important. Although the generator is usually unbounded, generating a strongly continuous semigroup of bounded operators corresponds to the many physical problems that may be modeled as an *abstract Cauchy problem*

$$\frac{d}{dt}u(t,x) = Bu(t,x) \quad (t \ge 0), \qquad u(0,x) = x, \tag{1.5}$$

being well-posed.

By a *mild solution* of (1.5) we mean that $t \mapsto u(t,x) \in C([0,\infty), X)$ and $\int_0^t u(s,x)\,ds \in \mathcal{D}(B)$, for all $t \ge 0$, with

$$u(t,x) = B\Big(\int_0^t u(s,x)\,ds\Big) + x \quad (t \ge 0).$$

By a *strong solution* we mean that $t \mapsto u(t,x) \in C^1([0,\infty), X) \cap C([0,\infty), \mathcal{D}(B))$ and u satisfies (1.5).

When an operator, B, is not bounded, one can measure its "unboundedness" by finding C such that BC is bounded. Similarly, generating a C-regularized semigroup measures its "ill-posedness", that is, how far from well-posedness the abstract Cauchy problem corresponding to that operator is, as is indicated by the following.

LEMMA 1.6 ([5, Theorems 3.5, 3.13, 4.13 and 5.16]). *Suppose* B *generates a* C*-regularized semigroup* $\{W(t)\}_{t\ge 0}$. *Then*

(1) (1.5) *has a unique mild solution, for all* $x \in \mathrm{Im}(C)$;

(2) (1.5) *has a unique strong solution, for all* $x \in C(\mathcal{D}(B))$;

(3) (1.5) *has a mild solution if and only if* $W(t)x \in \mathrm{Im}(C)$, *for all* $t \geq 0$, *and* $t \mapsto C^{-1}W(t)x \in C([0,\infty), X)$; *and*

(4) *there exists a Fréchet space* Z *such that*

$$[\mathrm{Im}(C)] \hookrightarrow Z \hookrightarrow X$$

and $B|_Z$ *generates a strongly continuous semigroup. If* $\{W(t)\}_{t\geq 0}$ *is exponentially bounded, then* Z *may be chosen to be a Banach space.*

Thus Construction One addresses the second sort of "good" behavior, since $f(A)$ is defined as the generator of a regularized semigroup. Construction Two addresses the first kind of "good" behavior, in the following way. Given f in $\mathrm{EXT}_2(\mathcal{F})$, we choose $h \in \mathcal{F}$ so that $h(A)$ is injective and $fh \in \mathcal{F}$; $f(A)$ is then defined to be $(h(A))^{-1}(fh)(A)$. Thus, although $f(A)$ may be unbounded, $f(A)h(A)$ is bounded.

We apply these general constructions to specific classes of operators, with specific choices of $\mathcal{F}$, in Section IV. For spectral operators of scalar type, we may choose $\mathcal{F}$ to be the space of bounded Borel measurable functions on the real line, so that, as mentioned earlier, the closed operator $f(A)$ is defined for any Borel measurable function f. For generators of bounded strongly continuous groups, $\mathcal{F}$ may be chosen to include continuously differentiable functions that decay sufficiently rapidly at infinity, so that $\mathrm{EXT}_j(\mathcal{F})$ includes all continuously differentiable functions, that is, the closed operator $f(A)$ is defined whenever f' exists and is continuous, on the real line. In both these examples, $f(A)$ will be densely defined, for all $f \in \mathrm{EXT}_j(\mathcal{F})$.

For operators whose resolvent set contains a half-line, $\mathcal{F}$ may be chosen to be functions holomorphic on a neighborhood of the spectrum of A that decay sufficiently rapidly at infinity, and $\mathrm{EXT}_j(\mathcal{F})$ then includes all functions holomorphic on a neighborhood of the spectrum of A, whose image omits a half-line. There are no conditions involving the behavior of the function at infinity. In general, these functions may not even be polynomially bounded, or meromorphic at infinity.

For both constructions, we will need injective operators, $h(A)$, for appropriate $h \in \mathcal{F}$. One way to obtain such operators is the following.

LEMMA 1.7. *Suppose* $\{W(t)\}_{t\geq 0}$ *is a regularized semigroup analytic in a neighborhood of* $(0,b]$. *Then* $W(b)$ *is injective. If, in addition,* $\mathrm{Im}(W(0))$ *is dense, then so is* $\mathrm{Im}(W(b))$.

Proof. Let B be the generator of $\{W(t)\}_{t\geq 0}$. As with strongly continuous semigroups, $(d/dt)^n W(t)|_{t=b} = B^n W(b) \in B(X)$, for any $n \in \mathbb{N}$.

If $W(b)x = 0$, then for any $n \in \mathbb{N}$, $(d/dt)^n W(t)x|_{t=b} = B^n W(b)x = 0$. Since $t \mapsto W(t)x$ is analytic in a neighborhood of $(0,b]$, and continuous on $[0,b]$, it follows that $W(0)x = 0$. Since $W(0)$ is injective, this implies that $x = 0$. Thus $W(b)$ is injective.

Now suppose $\mathrm{Im}(W(0))$ is dense. Suppose $x^* \in \mathrm{Im}(W(b))^{\perp}$. Then $\langle W(b)^*x^*, x\rangle = 0$, for all $x \in X$, so, as in the previous paragraph, we have, for all $\varepsilon > 0$,

$$\left(\frac{d}{dt}\right)^n \langle W(t)^*x^*, W(\varepsilon)x\rangle|_{t=b} = \langle (B^*)^n W(b)^*x^*, W(\varepsilon)x\rangle = 0,$$

so that by analyticity,

$$\langle x^*, W(0)^2 x\rangle = \lim_{\varepsilon\to 0}\langle W^*(0)x^*, W(\varepsilon)x\rangle = 0,$$

for all $x \in X$. Since $\mathrm{Im}(W(0))$, hence $\mathrm{Im}(W(0)^2)$, is dense, this implies that $x^* = 0$, as desired. ∎

In Section V, we apply our constructions to evolution equations, that is, to the abstract Cauchy problem (1.5). In particular, we show how one can make literal sense out of the formal solution $u(t,x) = e^{tB}x$, even when B does not generate a strongly continuous semigroup.

It is interesting that, in Construction Two, we gain information about the original $\mathcal{F}$ functional calculus by our extension. We show (Theorem 3.7(7)) that, when both f and $1/f$ are in $\mathrm{EXT}_2(\mathcal{F})$, then $f(A)$ is injective. In particular, when $f \in \mathcal{F}$ but $1/f \notin \mathcal{F}$, our construction is useful. This is applied to all our examples in Section IV.

II. Construction One.

In this section, we will ask that $\mathcal{F}$ satisfy the following hypotheses.

HYPOTHESES 2.1. For any $f \in \mathcal{F}$, there exists $h \in \mathcal{F}$ such that $h_t(z) \equiv e^{tf(z)}h(z) \in \mathcal{F}$, for any $t \geq 0$, with

$$t \mapsto h_t \in C([0,\infty), \mathcal{F}),$$

and $h(A)$ is injective whenever A is an operator with an $\mathcal{F}$ functional calculus.

LEMMA 2.2. *Suppose that* $s \mapsto k_s \in C([0,\infty), \mathcal{F})$ *and* A *has an* $\mathcal{F}$ *functional calculus. Then for any* $t \geq 0$,

$$\int_0^t k_s(A)\,ds = \Big(\int_0^t k_s\,ds\Big)(A).$$

Proof. Fix $t \geq 0$. Note that $s \mapsto k_s(A) \in C([0,\infty), B(X))$, thus the first integral is defined as a limit of Riemann sums converging in $B(X)$. The second integral is the limit of Riemann sums converging in $\mathcal{F}$, thus $(\int_0^t k_s\,ds) \in \mathcal{F}$, so that $(\int_0^t k_s\,ds)(A) \in B(X)$ is defined by the $\mathcal{F}$ functional calculus for A.

For any $n \in \mathbb{N}$, define the Riemann sum

$$R_n \equiv \frac{t}{n} \sum_{j=1}^{n} k_{jt/n}.$$

Then, by the linearity and continuity of the map $f \mapsto f(A)$, and the continuity of the integrands,

$$\int_0^t k_s(A)\,ds = \lim_{n\to\infty} \frac{t}{n} \sum_{j=1}^{n} k_{jt/n}(A)$$
$$= \lim_{n\to\infty} R_n(A) = (\lim_{n\to\infty} R_n)(A) = \Big(\int_0^t k_s\,ds\Big)(A). \blacksquare$$

COROLLARY 2.3. *Suppose A has an $\mathcal{F}$ functional calculus and h, f are as in Hypotheses* 2.1. *Then $\{h_t(A)\}_{t\ge0}$ is a regularized semigroup generated by $f(A)$.*

Proof. Since $f \mapsto f(A)$ is an algebra homomorphism and $h(A)$ is injective, $\{h_t(A)\}_{t\ge0}$ is a regularized semigroup. By Lemma 2.2, for $t \ge 0$,

$$\int_0^t h_s(A)f(A)\,ds = \int_0^t (h_s f)(A)\,ds = \Big(\int_0^t h_s f\,ds\Big)(A)$$
$$= (h_t - h)(A) = h_t(A) - h(A).$$

Differentiating at $t = 0$ gives us the result. $\blacksquare$

DEFINITION 2.4. Let $\mathrm{EXT}_1(\mathcal{F})$ be the set of all complex-valued functions g for which there exists $h \in \mathcal{F}$ such that $h_t(z) \equiv e^{tg(z)}h(z) \in \mathcal{F}$, for $t \ge 0$, with

$$t \mapsto h_t \in C([0,\infty), \mathcal{F}),$$

and $h(A)$ is injective whenever A has an $\mathcal{F}$ functional calculus.

EXAMPLE 2.5. (a) Suppose $m \in \mathbb{N}$, V is an open set in the complex plane, whose boundary is a finite union of smooth (possibly unbounded) curves, with a half-line contained in the complement of V. For $\lambda \notin V$, define

$$H_m^\infty(V) \equiv \{f \in H^\infty(V) \mid w \mapsto (\lambda - w)^m f(w) \in H^\infty(V)\},$$

with

$$\|f\|_{H_m^\infty(V)} \equiv \sup\{|(\lambda - w)^m f(w)| \mid w \in V\}.$$

By choosing $h(z) \equiv (\lambda - z)^{-m}$, it is clear that $\mathrm{EXT}_1(H_m^\infty(V))$ includes all functions holomorphic on V whose real part is bounded above.

Note that $\mathrm{EXT}_1(H_m^\infty(V))$ may include functions that are not polynomially bounded. For example, if $V \equiv H_\varepsilon$, the horizontal strip $\{z \mid |\mathrm{Im}(z)| < \varepsilon\}$, then exponentials $z \mapsto -e^{\omega z}$, $0 \le \omega \le \pi/(2\varepsilon)$, are in $\mathrm{EXT}_1(H_m^\infty(H_\varepsilon))$.

(b) If $\mathcal{F}$ is chosen to be $C_0(\mathbb{R})$, then $\mathrm{EXT}_1(C_0(\mathbb{R}))$ includes $C(\mathbb{R})$. For $f \in C(\mathbb{R})$, this may be seen by choosing, for Definition 2.4, $h(s) \equiv e^{-s^2}e^{-|f(s)|^2}$. The operator $h(A)$ is injective by Lemma 1.7, since

$$\{(e^{-s^2}e^{-z|f(s)|^2})(A)(i - A)^{-1}\}_{\mathrm{Re}(z)>0} = \{(e^{-s^2}e^{-z|f(s)|^2}g_i)(A)\}_{\mathrm{Re}(z)>0}$$

is a holomorphic regularized semigroup.

(c) If $B(\mathbb{R})$ is defined to be the set of all bounded Borel measurable functions on the real line, with the supremum norm, then the same argument as in (b) shows that $\mathrm{EXT}_1(B(\mathbb{R}))$ includes all Borel measurable functions on the real line.

(d) Let $\mathcal{F}$ be the set of Fourier transforms of L^1 functions on the real line, that is, functions f of the form

$$f(s) = \int_{\mathbb{R}} e^{-ist}F(t)\,dt,$$

where $F \in L^1(\mathbb{R})$, with

$$\|f\|_{\mathcal{F}} \equiv \|F\|_1.$$

Let $C^1(\mathbb{R})$ be the set of all continuously differentiable functions on the real line.

We claim that $C^1(\mathbb{R}) \subseteq \mathrm{EXT}_1(\mathcal{F})$.

It is well known that there exists a constant M, independent of f, such that (see [10])

$$\|F\|_1 \le M(\|f\|_2 + \|f'\|_2).$$

This implies that $\mathcal{F}$ contains all functions g that are continuously differentiable, with $(1+|s|)g(s)$ and $(1+|s|)g'(s)$ bounded on the real line, and there exists a constant K so that

(d1) $$\|g\|_{\mathcal{F}} \le K(\sup_{s\in\mathbb{R}}((1+|s|)|g(s)|) + \sup_{s\in\mathbb{R}}((1+|s|)|g'(s)|)),$$

for all $g \in C^1(\mathbb{R})$.

Fix $g \in C^1(\mathbb{R})$. Define

$$H(s) \equiv \sup_{0\le|x|\le|s|} (|g(x)|^2 + |g'(x)|^2),$$

then define

$$h(s) \equiv \begin{cases} \int_s^\infty (y - s)e^{-y^2}e^{-H(y)}\,dy & (s \ge 0), \\ \int_{-\infty}^s (s - y)e^{-y^2}e^{-H(y)}\,dy & (s \le 0). \end{cases}$$

Then, since H is nondecreasing on $[0,\infty)$ and nonincreasing on $(-\infty,0]$, there exists a constant K_1 so that

$$h(s) \le K_1 e^{-H(s)} \le K_1 e^{-|g'(s)|^2}, \quad \forall s \in \mathbb{R},$$

whence

$$t \mapsto (1+|s|)h(s)\frac{d}{ds}e^{tg(s)} \in C([0,\infty), C_0(\mathbb{R})). \tag{d2}$$

Similarly, there exists a constant K_2 so that

$$|h(s)| + \left|\frac{d}{ds}h(s)\right| \le K_2 e^{-|g(s)|^2};$$

thus

$$t \mapsto (1+|s|)\frac{d}{ds}(h(s)e^{tg(s)}) \in C([0,\infty), C_0(\mathbb{R})), \tag{d3}$$

and

$$t \mapsto (1+|s|)e^{tg(s)}h(s) \in C([0,\infty), C_0(\mathbb{R})). \tag{d4}$$

By (d1)–(d4), $t \mapsto e^{tg}h \in C([0,\infty), \mathcal{F})$.

All that remains is to show that $h(A)$ is injective, whenever A has an $\mathcal{F}$ functional calculus. This follows from the fact that h and h' are real, positive and bounded, and h' converges monotonically to zero at $\pm\infty$, since we may then write $h = e^{\log h}$, and use (d1) to show that

$$\{(e^{z\log h(s)}(1+is)^{-1})(A)\}_{z\in\mathrm{RHP}}$$

defines a holomorphic strongly continuous $(1+iA)^{-1}$-regularized semigroup. By Lemma 1.7, $h(A)(1+iA)^{-1} = (e^{\log h(s)}(1+is)^{-1})(A)$, hence $h(A)$, is injective.

Before we define $f(A)$, for $f \in \mathrm{EXT}_1(\mathcal{F})$, we will worry about possible ambiguities in our definition.

LEMMA 2.6. *Suppose A has an $\mathcal{F}$ functional calculus, $g \in \mathrm{EXT}_1(\mathcal{F})$ and, for $j = 1,2$, h_j is as in Definition 2.4, that is, $h_{j,t}(z) \equiv e^{tg(z)}h_j(z) \in \mathcal{F}$, for $t \ge 0$, $h_j(A)$ is injective, and*

$$t \mapsto h_{j,t} \in C([0,\infty), \mathcal{F}) \quad (j = 1,2).$$

Then $\{h_{j,t}(A)\}_{t\ge0}$ is an $h_j(A)$-regularized semigroup ($j = 1,2$), and the generator of $\{h_{1,t}(A)\}_{t\ge0}$ equals the generator of $\{h_{2,t}(A)\}_{t\ge0}$.

Proof. Since A has an $\mathcal{F}$ functional calculus, $\{h_{j,t}(A)\}_{t\ge0}$ has the algebraic properties of an $h_j(A)$-regularized semigroup. The continuity follows from Definition 2.4. Let us write $g(j, A)$ for the generator of $\{h_{j,t}(A)\}_{t\ge0}$. By Lemma 1.4(a), for $j = 1,2$, $g(j,A)$ is the generator of the $h_1(A)h_2(A)$-regularized semigroup $\{h_1(A)h_{2,t}(A)\}_{t\ge0} = \{h_2(A)h_{1,t}(A)\}_{t\ge0}$. Thus $g(1,A) = g(2,A)$, as desired. ■

DEFINITION 2.7. Suppose A has an $\mathcal{F}$ functional calculus. For $g \in \mathrm{EXT}_1(\mathcal{F})$, let h and h_t be as in Definition 2.4. Then $g(A)$ is defined to be the generator of the $h(A)$-regularized semigroup $\{h_t(A)\}_{t\ge0}$.

By Lemma 2.6, this definition is independent of which h is chosen.

By [5, Theorem 3.4], $\mathcal{D}(g(A))$ is dense whenever $\mathrm{Im}(h(A))$ is dense.

THEOREM 2.8. *Suppose A has an $\mathcal{F}$ functional calculus. Then the map $f \mapsto f(A)$ from Definition 2.7 is a map from $\mathrm{EXT}_1(\mathcal{F})$ into the space of all closed operators on X, extending the $\mathcal{F}$ functional calculus, such that*

(1) *if $f_0 \in \mathrm{EXT}_1(\mathcal{F})$, then $f_0(A) = I$;*

(2) *if $g_\lambda \in \mathrm{EXT}_1(\mathcal{F})$, then $\lambda - A$ is injective and $g_\lambda(A)$ is an injective extension of $(\lambda - A)^{-1}$, with*

$$g_\lambda(A) = (h(A))^{-1}(\lambda - A)^{-1}h(A),$$

where h is as in Definition 2.4 for g_λ;

(3) *if $f \in \mathcal{F}$ and $g \in \mathrm{EXT}_1(\mathcal{F})$, then*

$$f(A)g(A) \subseteq g(A)f(A);$$

(4) *if $f, gf \in \mathcal{F}$ and $g \in \mathrm{EXT}_1(\mathcal{F})$, then*

$$g(A)f(A) = (gf)(A);$$

and

(5) *if $f, f_0 \in \mathcal{F}$ and $1/f \in \mathrm{EXT}_1(\mathcal{F})$, then $f(A)$ is injective.*

Proof. By Corollary 2.3, Definition 2.7 extends the $\mathcal{F}$ functional calculus for A.

(1) Let h and h_t be as in Definition 2.4, for $g \equiv f_0$. That is, $h \in \mathcal{F}$, and $f_0(A)$ is the generator of $\{e^t h(A)\}_{t\ge0}$. For any $x \in X$, $(d/dt)|_{t=0}h_t(A)x = h(A)x$, thus $x \in \mathcal{D}(f_0(A))$, with $f_0(A)x = x$.

(2) Let $W(t) \equiv (e^{tg_\lambda}h)(A)$ $(t \ge 0)$, $C \equiv W(0)$. For $t \ge 0$, by Lemma 2.2, $(\lambda - f_1)(e^{tg_\lambda} - 1)h \in \mathcal{F}$, with

$$\int_0^t W(s)\,ds = \Big(\int_0^t e^{sg_\lambda}h\,ds\Big)(A) = ((\lambda - f_1)(e^{tg_\lambda} - 1)h)(A);$$

this implies (see Definition 1.2(1)) that, for any $x \in X$, $(W(t) - C)x = ((e^{tg_\lambda} - 1)h)(A)x \in \mathcal{D}(A)$, with

$$(\lambda - A)(W(t) - C)x = \int_0^t W(s)x\,ds; \tag{2.9}$$

and for $x \in \mathcal{D}(A)$,

$$\int_0^t W(s)x\,ds = (W(t) - C)(\lambda - A)x. \tag{2.10}$$

Thus, differentiating both sides of (2.10) at $t = 0$ tells us that $(\lambda - A)x \in \mathcal{D}(g_\lambda(A))$, with

$$x = g_\lambda(A)(\lambda - A)x \quad (x \in \mathcal{D}(A)).$$

This implies that $\lambda - A$ is injective, and $g_\lambda(A)$ is an extension of $(\lambda - A)^{-1}$. By Lemma 1.4(b), $g_\lambda(A) = C^{-1}g_\lambda(A)C$. Thus $g_\lambda(A)$ is an extension of $C^{-1}(\lambda - A)^{-1}C$.

To show equality, suppose $x \in \mathcal{D}(g_\lambda(A))$. Since A is closed, we may differentiate both sides of (2.9) at $t = 0$, to obtain

$$(\lambda - A)Cg_\lambda(A)x = Cx, \tag{2.11}$$

so that $x \in \mathcal{D}(C^{-1}(\lambda - A)^{-1}C)$, with

$$g_\lambda(A)x = C^{-1}(\lambda - A)^{-1}Cx,$$

as desired. By (2.11), $g_\lambda(A)$ is injective.

For the proof of (3) and (4), let h be as in Definition 2.4 for g, and

$$W(t) \equiv (e^{tg}h)(A) \quad (t \geq 0), \qquad C \equiv W(0).$$

(3) Suppose $x \in \mathcal{D}(g(A))$. Then for $t > 0$,

$$\frac{1}{t}(W(t) - C)f(A)x = f(A)\left(\frac{1}{t}(W(t)x - Cx)\right),$$

thus since $f(A)$ is bounded, we may take the limit as $t \to 0$, as follows:

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}(W(t) - C)f(A)x &= f(A)\left(\lim_{t\to 0}\frac{1}{t}(W(t)x - Cx)\right)\\ &= f(A)Cg(A)x = Cf(A)g(A)x.\end{aligned}$$

This implies that $f(A)x \in \mathcal{D}(g(A))$, with $g(A)f(A)x = f(A)g(A)x$, as desired.

(4) Suppose $x \in X$. For any $t \geq 0$, we may use Lemma 2.2 as follows:

$$\begin{aligned}\int_0^t W(s)(gf)(A)x\,ds &= \int_0^t (e^{sg}hgf)(A)x\,ds = \Big(\int_0^t e^{sg}hgf\,ds\Big)(A)x\\ &= (e^{tg}hf)(A)x - (hf)(A)x = W(t)f(A)x - Cf(A)x.\end{aligned}$$

This implies that $f(A)x \in \mathcal{D}(g(A))$, with $g(A)f(A)x = (gf)(A)x$, as desired.

(5) follows from (4) and (1). ∎

III. Construction Two. In Construction One, when $t \mapsto e^{tg}h$ is in $C^1([0,\infty),\mathcal{F})$, then it is not hard to show that $gh \in \mathcal{F}$ and $g(A) = (h(A))^{-1}(gh)(A)$. In general, if $gh \in \mathcal{F}$ and $h(A)$ is injective, this definition is simpler.

HYPOTHESIS 3.1. We will assume throughout this section that there exists $h \in \mathcal{F}$ such that $h(A)$ is injective whenever A has an $\mathcal{F}$ functional calculus.

DEFINITION 3.2. Let $\mathrm{EXT}_2(\mathcal{F})$ be the set of all g for which there exists $h \in \mathcal{F}$ such that $gh \in \mathcal{F}$, and $h(A)$ is injective whenever A has an $\mathcal{F}$ functional calculus.

Note that $\mathrm{EXT}_2(\mathcal{F})$ is an algebra.

EXAMPLE 3.3. (a) Let V and $H_m^\infty(V)$ be as in Example 2.5(a).

Let $\mathcal{H}(V)$ be the set of all functions holomorphic on V whose image, $f(V)$, omits a half-line.

We claim that $\mathcal{H}(V) \subseteq \mathrm{EXT}_2(H_m^\infty(V))$. To show this, we will use the results of Section II.

Let $\mathcal{H}(V)_1$ be the set of all $f \in \mathcal{H}(V)$ such that $\{\mathrm{Re}(f(z)) \mid z \in V\}$ is bounded above. Then it is not hard to see, by choosing $h(z) \equiv (\lambda - z)^{-m}$ $(\lambda \notin V)$, that $\mathcal{H}(V)_1 \subseteq \mathrm{EXT}_1(\mathcal{F})$.

Suppose now that $g \in \mathcal{H}(V)$. There exist complex α, β such that the image of $\alpha g + \beta$, $(\alpha g + \beta)(V)$, does not include $(-\infty, 1)$. Let

$$k(z) \equiv \sqrt{\alpha g(z) + \beta},$$

where the square root is chosen to map $\mathbb{C} - (-\infty, 0]$ into the right half-plane.

Note that $1/k \in H^\infty(V)$, so that $(1/k)(g_\lambda)^m \in H_m^\infty(V)$.

By (2) and (4) of Theorem 2.8, for $\lambda \notin V$,

$$(\lambda - A)^{-m} = k(A)\left(\frac{1}{k}(g_\lambda)^m\right)(A).$$

This implies that $((1/k)(g_\lambda)^m)(A)$ is injective. Thus, if

$$h \equiv \left(\frac{1}{k}(g_\lambda)^m\right)^2,$$

then h satisfies the hypotheses of Definition 3.2 for g, and thus $g \in \mathrm{EXT}_2(H_m^\infty(V))$ as desired.

For particular choices of V, $\mathrm{EXT}_2(V)$ includes much more than $\mathcal{H}(V)$.

For example, suppose, for some $\varepsilon > 0$, that V equals the horizontal strip $H_\varepsilon \equiv \{x + iy \mid x, y \in \mathbb{R},\ |y| < \varepsilon\}$. Then, for any polynomial p and for t real, $p(z)$ and $e^{tp(z)}$ are in $\mathrm{EXT}_2(H_\varepsilon)$, since $p(z)e^{-z^2}$ and $e^{tp(z)}e^{-z^{2n}}$, where n is the degree of p, are in $H_m^\infty(H_\varepsilon)$. The operator $(e^{-z^{2n}})(A)$ is shown to be injective by using Lemma 1.7 and the fact that $\{(e^{-tz^{2n}}(2\varepsilon i + z)^{-m})(A)\}_{t\geq 0}$ extends to a $(2\varepsilon i + A)^{-m}$-regularized analytic semigroup.

For V equal to the sector $S_\theta \equiv \{re^{i\phi} \mid r > 0,\ |\phi| < \theta\}$, with $0 < \theta < \pi/2$, the functions z and e^{tz}, for t real, are in $\mathrm{EXT}_2(S_\theta)$, since, for $\alpha\theta < \pi/2$ with $\alpha > 1$, ze^{-z^α} and $e^{tz}e^{-z^\alpha}$ are in $H_m^\infty(S_\theta)$. If $n\theta < \pi/2$ and p is a polynomial of degree n, then a similar argument shows that $p(z)$, $e^{tp(z)} \in \mathrm{EXT}_2(S_\theta)$.

(b) The same argument as in Example 2.5(b) shows that $\mathrm{EXT}_2(C_0(\mathbb{R}))$ includes $C(\mathbb{R})$. Or, given continuous g, we could have used $h(s) \equiv (1+|g(s)|)^{-1}(1+|s|)^{-1}$ in Definition 3.2.

(c) As in (b), $\mathrm{EXT}_2(B(\mathbb{R}))$ includes all Borel measurable functions.

(d) As in Example 2.5(d), let $\mathcal{F}$ be the set of Fourier transforms of L^1 functions. Then the same argument as in Example 2.5(d) shows that $C^1(\mathbb{R}) \subseteq \mathrm{EXT}_2(\mathcal{F})$.

DEFINITION 3.4. Suppose A has an $\mathcal{F}$ functional calculus. For $g \in \mathrm{EXT}_2(\mathcal{F})$, let h be as in Definition 3.2. Then $g(A) \equiv (h(A))^{-1}(gh)(A)$.

To see that this is well defined, that is, independent of which h is chosen, note that, for any $k \in \mathcal{F}$ such that $k(A)$ is injective,

$$\begin{aligned}((kh)(A))^{-1}(gkh)(A) &= (h(A))^{-1}(k(A))^{-1}k(A)(gh)(A)\\ &= (h(A))^{-1}(gh)(A).\end{aligned}$$

It is clear from Hypotheses 3.1 that this definition of $f(A)$ extends the $\mathcal{F}$ functional calculus.

Remark 3.5. This definition does not appear to involve regularized semigroups, except by analogy. However, in practice, the difficult part of applying this construction is in verifying that $h(A)$ is injective. Both Lemma 1.7 and the construction of the previous section will provide many such h, including our choices in the examples in the next section.

Before going further, we should verify that this construction agrees with the construction of the previous section, when they are both defined.

PROPOSITION 3.6. *Suppose A has an $\mathcal{F}$ functional calculus, $\mathcal{F}$ satisfies Hypotheses 2.1 and $g \in \mathrm{EXT}_1(\mathcal{F}) \cap \mathrm{EXT}_2(\mathcal{F})$. Then the $g(A)$ of Construction One equals the $g(A)$ of Construction Two.*

Proof. For $j = 1, 2$, let $g(j, A)$ be the $g(A)$ of Construction j. Suppose h_1 is as in Definition 2.4 and h_2 is as in Definition 3.2. Let $h \equiv h_1h_2$, $t \geq 0$,

$$W_1(t) \equiv (e^{tg}h_1)(A), \quad W(t) \equiv (e^{tg}h)(A) = W_1(t)h_2(A).$$

By Lemma 1.4(a), $g(1, A)$ is the generator of the regularized semigroup $\{W(t)\}_{t\geq 0}$.

Since $g(2, A)h_2(A) = (gh_2)(A) \in B(X)$, we have, by Lemma 2.2, for $t \geq 0$,

$$\begin{aligned}\int_0^t W_1(s)g(2,A)h_2(A)\,ds &= \int_0^t (e^{sg}gh)(A)\,ds\\ &= (e^{tg}h)(A) - h(A) = W(t) - W(0).\end{aligned}$$

Differentiating at $t = 0$ gives us

$$h_1(A)g(2, A)h_2(A) = g(1, A)h(A),$$

thus, by Lemma 1.4(b),

$$g(1, A) = (h(A))^{-1}g(1, A)h(A) = (h_2(A))^{-1}g(2, A)h_2(A) = g(2, A). \blacksquare$$

THEOREM 3.7. *Suppose A has an $\mathcal{F}$ functional calculus. Then the map $f \mapsto f(A)$ from Definition 3.4 is a map from $\mathrm{EXT}_2(\mathcal{F})$ into the space of all closed operators on X, extending the $\mathcal{F}$ functional calculus, such that*

(1) *$f_0 \in \mathrm{EXT}_2(\mathcal{F})$ and $f_0(A) = I$;*

(2) *if $g_\lambda \in \mathrm{EXT}_2(\mathcal{F})$, then $\lambda - A$ is injective and $g_\lambda(A)$ is an injective extension of $(\lambda - A)^{-1}$, with*

$$g_\lambda(A) = (h(A))^{-1}(\lambda - A)^{-1}h(A),$$

where h is as in Definition 3.2 for g_λ;

(3) *if $f \in \mathcal{F}$ and $g \in \mathrm{EXT}_2(\mathcal{F})$, then $fg \in \mathrm{EXT}_2(\mathcal{F})$, with*

$$f(A)g(A) \subseteq g(A)f(A) = (fg)(A);$$

(4) *if $f, g \in \mathrm{EXT}_2(\mathcal{F})$, then $fg \in \mathrm{EXT}_2(\mathcal{F})$, and*

$$f(A)g(A) \subseteq (fg)(A),$$

with

$$\mathcal{D}(f(A)g(A)) = \mathcal{D}((fg)(A)) \cap \mathcal{D}(g(A));$$

(5) *if $f, g \in \mathrm{EXT}_2(\mathcal{F})$, then $(f+g) \in \mathrm{EXT}_2(\mathcal{F})$, and*

$$f(A) + g(A) \subseteq (f+g)(A),$$

where $\mathcal{D}(f(A) + g(A)) \equiv \mathcal{D}(f(A)) \cap \mathcal{D}(g(A))$;

(6) *suppose $p(s) \equiv \sum_{k=0}^N a_k s^k \in \mathrm{EXT}_2(\mathcal{F})$ and there exists a complex λ such that, if $\lambda_1, \ldots, \lambda_N$ are the roots of $\lambda - p$, then for $1 \leq k \leq N$, $\lambda_k \in \varrho(A)$ and $g_{\lambda_k} \in \mathrm{EXT}_2(\mathcal{F})$; under these assumptions,*

$$p(A) = \sum_{k=0}^N a_k A^k$$

and $\mathcal{D}(p(A)) = \mathcal{D}(A^N)$;

(7) *if both f and $1/f$ are in $\mathrm{EXT}_2(\mathcal{F})$, then $f(A)$ is injective and*

$$\mathrm{Im}(f(A)) = \mathcal{D}((1/f)(A));$$

and

(8) *if $f \in \mathrm{EXT}_2(\mathcal{F})$ and λ is a complex number such that $(\lambda - f)^{-1} \in \mathcal{F}$, then $\lambda \in \varrho(f(A))$, with*

$$(\lambda - f(A))^{-1} = ((\lambda - f)^{-1})(A).$$

Proof. For (1), choose any $h \in \mathcal{F}$ such that $h(A)$ is injective (such an h is guaranteed by Hypothesis 3.1).

(2) There exists $h \in \mathcal{F}$ such that $h(A)$ is injective and $g_\lambda h \in \mathcal{F}$. This implies (see Definition 1.2(1)) that, for all $x \in X$, $(g_\lambda h)(A)x \in \mathcal{D}(A)$ with

$$(\lambda - A)(g_\lambda h)(A)x = h(A)x,$$

and for $x \in \mathcal{D}(A)$,

$$h(A)x = (g_\lambda h)(A)(\lambda - A)x,$$

so that $\lambda - A$ is injective. For $x \in \mathcal{D}(g_\lambda(A))$, we have

$$g_\lambda(A)x \equiv (h(A))^{-1}(g_\lambda h)(A)x = (h(A))^{-1}(\lambda - A)^{-1}h(A)x,$$

an extension of $(\lambda - A)^{-1}$. Thus

$$(\lambda - A)^{-1} \subseteq g_\lambda(A) \subseteq (h(A))^{-1}(\lambda - A)^{-1}h(A);$$

the first inclusion implies that

$$(h(A))^{-1}(\lambda - A)^{-1}h(A) \subseteq (h(A))^{-1}g_\lambda(A)h(A) = g_\lambda(A)$$

(see the comments after Definition 3.4).

Thus $g_\lambda(A) = (h(A))^{-1}(\lambda - A)^{-1}h(A)$. Since $h(A)$ and $(\lambda - A)^{-1}$ are injective, so is $g_\lambda(A)$.

(3) Let h be as in Definition 3.2 for g. For the containment, suppose $x \in \mathcal{D}(g(A))$. Then

$$\begin{aligned} h(A)(f(A)g(A)x) &= f(A)h(A)(h(A))^{-1}(gh)(A)x \\ &= f(A)(gh)(A)x = (gh)(A)(f(A)x). \end{aligned}$$

This implies that $f(A)x \in \mathcal{D}(g(A))$, with $g(A)f(A)x = f(A)g(A)x$, as desired.

It is clear that $fg \in \mathrm{EXT}_2$, since $(fg)h \in \mathcal{F}$. For the equality, suppose $x \in \mathcal{D}((fg)(A))$; then

$$(gh)(A)f(A)x = (ghf)(A)x = (fg)(A)h(A)x = h(A)((fg)(A)x).$$

This implies that $f(A)x \in \mathcal{D}(g(A))$, with $g(A)f(A)x = (fg)(A)x$.

Thus $(fg)(A) \subseteq g(A)f(A)$. Conversely, suppose $x \in \mathcal{D}(g(A)f(A))$. Then

$$(fg)(A)h(A)x = (fgh)(A)x = (gh)(A)f(A)x = h(A)(g(A)f(A)x).$$

This implies that $g(A)f(A) \subseteq (fg)(A)$.

(4) Let h_1 be as in Definition 3.2 for f, let h_2 be as in Definition 3.2 for g, and let $h \equiv h_1h_2$. Then $h(A)$ is injective whenever A has an $\mathcal{F}$ functional calculus, and $(fg)h \in \mathcal{F}$. Thus $fg \in \mathrm{EXT}_2$.

Suppose $x \in \mathcal{D}(f(A)g(A))$. By (3), $(fgh)(A) = f(A)(gh)(A)$, thus

$$\begin{aligned} (fg)(A)h(A)x &= (fgh)(A)x = f(A)(gh)(A)x \\ &= f(A)h(A)g(A)x = h(A)f(A)g(A)x. \end{aligned}$$

This implies that $x \in \mathcal{D}((fg)(A))$, with $(fg)(A)x = f(A)g(A)x$.

Thus $f(A)g(A) \subseteq (fg)(A)$ and $\mathcal{D}(f(A)g(A)) \subseteq \mathcal{D}((fg)(A)) \cap \mathcal{D}(g(A))$. Conversely, if $x \in \mathcal{D}((fg)(A)) \cap \mathcal{D}(g(A))$, then using the fact that $x \in \mathcal{D}(g(A))$, (3) of this theorem, and the fact that $x \in \mathcal{D}((fg)(A))$, in that order, we have

$$\begin{aligned} (fh)(A)(g(A)x) &= f(A)h(A)(g(A)x) \\ &= f(A)(gh)(A)x = (fhg)(A)x = h(A)((fg)(A)x). \end{aligned}$$

This implies that $g(A)x \in \mathcal{D}(f(A))$, that is, $x \in \mathcal{D}(f(A)g(A))$.

Thus $\mathcal{D}(f(A)g(A)) = \mathcal{D}((fg)(A)) \cap \mathcal{D}(g(A))$, as desired.

(5) Let h_1, h_2 and h be as in the proof of (4). Then $h(A)$ is injective whenever A has an $\mathcal{F}$ functional calculus, and $(f+g)h \in \mathcal{F}$. Thus $(f+g) \in \mathrm{EXT}_2(\mathcal{F})$.

Suppose $x \in \mathcal{D}(f(A)) \cap \mathcal{D}(g(A))$. Then since $x \in \mathcal{D}(f(A))$,

$$(fh)(A)x = h_2(A)(fh_1)(A)x \in \mathrm{Im}(h(A)),$$

and since $x \in \mathcal{D}(g(A))$,

$$(gh)(A)x = h_1(A)(gh_2)(A)x \in \mathrm{Im}(h(A)),$$

thus $((f+g)h)(A)x \in \mathrm{Im}(h(A))$, which implies that $x \in \mathcal{D}((f+g)(A))$, with

$$\begin{aligned} (f+g)(A)x &= (h(A))^{-1}((f+g)h)(A)x \\ &= (h(A))^{-1}((fh)(A)x + (gh)(A)x) = f(A)x + g(A)x, \end{aligned}$$

as desired.

(6) Without loss of generality, suppose the leading coefficient of p is one. Let $p_1(A)$ be the series definition.

Since

$$\lambda - p_1(A) = \prod_{k=1}^{N}(\lambda_k - A),$$

by (2), $\lambda \in \varrho(p_1(A))$, with

$$(\lambda - p_1(A))^{-1} = \prod_{k=1}^{N}(\lambda_k - A)^{-1} = \prod_{k=1}^{N} g_{\lambda_k}(A).$$

For $x \in \mathcal{D}(p(A))$, by (1), (4) and (2),

$$\prod_{k=1}^{N}(\lambda_k - A)^{-1}(\lambda - p(A))x = x,$$

so

$$(\lambda - p(A))x = \prod_{k=1}^{N}(\lambda_k - A)x = (\lambda - p_1(A))x,$$

thus $\lambda - p_1(A)$ is an extension of $\lambda - p(A)$. But by (4),

$$(\lambda - p(A)) \prod_{k=1}^{N} (\lambda_k - A)^{-1} = I,$$

so $\lambda - p(A)$ is surjective; since $\lambda - p_1(A)$ is injective, it follows that $\lambda - p(A)$ and $\lambda - p_1(A)$ must be equal.

(7) follows from (4) and (1).

(8) Let $B \equiv ((\lambda - f)^{-1})(A)$. By (1), (3) and (5),

$$B(\lambda - f(A)) \subseteq (\lambda - f(A))B = I. \ \blacksquare$$

IV. Examples

EXAMPLE 4.1. Here we will consider operators whose resolvent set contains a half-line, and whose norm is polynomially bounded. We will construct an unbounded analogue of the Riesz–Dunford functional calculus.

DEFINITION. Suppose V is an open set in the complex plane, whose boundary is a finite union of smooth (possibly unbounded) curves, with a half-line contained in the complement of V. We will say that an operator, A, is of *α-type* V if the spectrum of A is contained in V, with

$$\|(z - A)^{-1}\| \le M(1 + |z|)^{\alpha}, \qquad \forall z \notin V,$$

for some constant M, $\alpha \ge -1$.

For $\lambda \notin V$, let $m \equiv [\alpha] + 2$, $H_m^\infty(V) \equiv \{f \mid z \mapsto (\lambda - z)^m f(z) \in H^\infty(V)\}$.

Then it may be shown (see [4]) that A has an $H_m^\infty(V)$ functional calculus, given by

$$f(A) \equiv \int_{\partial V} f(w)(w - A)^{-1} \frac{dw}{2\pi i}.$$

DEFINITION. Let $\mathcal{H}(V)$ be the set of all functions holomorphic on V whose image $f(V)$ omits a half-line.

PROPOSITION. *Construction Two defines a map, $f \mapsto f(A)$, from $\mathrm{EXT}_2(H_m^\infty(V))$ into the set of all closed operators on X, such that $\mathcal{H}(V) \subseteq \mathrm{EXT}_2(H_m^\infty(V))$ and*

(1) $f_0(A) = I$;

(2) *for any $\lambda \notin V$, $g_\lambda(A) = (\lambda - A)^{-1}$;*

(3) *for $f \in H_m^\infty(V)$, $g \in \mathrm{EXT}_2(H_m^\infty(V))$,*

$$f(A)g(A) \subseteq g(A)f(A) = (fg)(A);$$

(4) *for $f, g \in \mathrm{EXT}_2(H_m^\infty(V))$,*

$$f(A)g(A) \subseteq (fg)(A),$$

with

$$\mathcal{D}(f(A)g(A)) = \mathcal{D}((fg)(A)) \cap \mathcal{D}(g(A));$$

(5) *if $f, g \in \mathrm{EXT}_2(H_m^\infty(V))$, then $(f+g) \in \mathrm{EXT}_2(H_m^\infty(V))$, and*

$$f(A) + g(A) \subseteq (f+g)(A),$$

where $\mathcal{D}(f(A) + g(A)) \equiv \mathcal{D}(f(A)) \cap \mathcal{D}(g(A))$;

(6) *if $p(s) \equiv \sum_{k=0}^{N} a_k s^k \in \mathcal{H}(V)$, then*

$$p(A) = \sum_{k=0}^{N} a_k A^k$$

and $\mathcal{D}(p(A)) = \mathcal{D}(A^N)$;

(7) *if h is holomorphic on V and for all $z \in V$, $h(z) \notin (-\infty, 0]$, then $h(A)$ is injective; and*

(8) *if $f \in \mathcal{H}(V)$ and there exists a positive constant c so that*

$$|\lambda - f(w)| \ge c(1 + |w|^m), \qquad \forall w \in V,$$

then $\lambda \in \varrho(f(A))$.

Proof. We have shown in Example 3.3(a) that $\mathcal{H}(V) \subseteq \mathrm{EXT}_2(H_m^\infty(V))$, thus we may apply Theorem 3.7.

For (6), choose complex $\lambda \notin p(V)$. Let $\lambda_1, \ldots, \lambda_N$ be the zeroes of $p - \lambda$. Then for $1 \le k \le N$, λ_k is outside V, thus by (2) of this Proposition and (6) of Theorem 3.7, the result follows.

For (7), note that $1/h \in \mathcal{H}(V)$, since it also avoids $(-\infty, 0]$. $\blacksquare$

Remarks. Suppose iA generates a strongly continuous group. Then A is of 0-type H_ε, for some $\varepsilon > 0$ (choose ε greater than the exponential type of e^{itA}), where H_ε is the horizontal strip $\{x + iy \mid x, y \in \mathbb{R},\ |y| < \varepsilon\}$. For any polynomial p and for t real, the functions $p(z)$ and $e^{tp(z)}$ are in $\mathrm{EXT}_2(H_\varepsilon)$ (see Example 3.3(a)), thus we may use Construction Two to define $p(A)$ and $e^{tp(A)} \equiv (e^{tp})(A)$.

Similarly, when $-A$ generates a strongly continuous holomorphic semigroup, then there exists a real ω, and $\theta < \pi/2$, so that $A - \omega$ is of (-1)-type S_θ, where S_θ is the sector $\{re^{i\phi} \mid r > 0,\ |\phi| < \theta\}$, so that we may use our construction to define e^{tA}, for t real. If $n\theta < \pi/2$ and p is a polynomial of degree n, then we may define $p(A)$ and $e^{tp(A)} \equiv (e^{tp})(A)$ by our construction (see Example 3.3(a)).

Perhaps of particular interest in this construction is the case when V is the disjoint union of two (both possibly unbounded) open sets Ω_1 and Ω_2. Then $1_{\Omega_j}(A)$, $j = 1, 2$, define unbounded projections, analogous to spectral projections. For example, when A is the generator of translation on L^1 of the unit circle, we may choose $\Omega_1 \equiv \{z \mid \mathrm{Re}(z) > -1/2\}$, $\Omega_2 \equiv \{z \mid \mathrm{Re}(z) < -1/2\}$; then $1_{\Omega_1}(A)$ is an unbounded analogue of a spectral projection onto H^1 of the unit disc (it is well known that there is no such bounded projection).

EXAMPLE 4.2. Let A be a spectral operator of scalar type (see [6], [7]). This includes, but is not limited to, self-adjoint operators on a Hilbert space. We may choose $\mathcal{F} = B(\mathbb{R})$, the space of bounded Borel measurable functions on the real line with the supremum norm. The $B(\mathbb{R})$ functional calculus for A is given by

$$f(A)x = \int_{\mathbb{R}} f(s)\,dE(s)x \quad (x \in X),$$

where E is the projection-valued measure for A.

We have shown in Example 3.3(c) that $\mathrm{EXT}_2(B(\mathbb{R}))$ contains all Borel measurable functions on the real line. Thus Theorem 3.7 gives us the following.

PROPOSITION. *Construction Two defines a map, $f \mapsto f(A)$, from the space of Borel measurable functions on the real line, into the set of all closed, densely defined operators on X, such that*

(1) $f_0(A) = I$;
(2) *for any* $\lambda \notin \mathbb{R}$, $g_\lambda(A) = (\lambda - A)^{-1}$;
(3) *for* $f \in B(\mathbb{R})$, *g Borel measurable,*

$$f(A)g(A) \subseteq g(A)f(A) = (fg)(A);$$

(4) *for f, g Borel measurable,*

$$f(A)g(A) \subseteq (fg)(A),$$

with

$$\mathcal{D}(f(A)g(A)) = \mathcal{D}((fg)(A)) \cap \mathcal{D}(g(A));$$

(5) *if f and g are Borel measurable, then*

$$f(A) + g(A) \subseteq (f+g)(A),$$

where $\mathcal{D}(f(A)+g(A)) \equiv \mathcal{D}(f(A)) \cap \mathcal{D}(g(A))$;
(6) *if* $p(s) \equiv \sum_{k=0}^N a_k s^k$, *then*

$$p(A) = \sum_{k=0}^N a_k A^k$$

and $\mathcal{D}(p(A)) = \mathcal{D}(A^N)$;
(7) *if f is Borel measurable and never zero, then $f(A)$ is injective and has dense range; and*
(8) *if f is Borel measurable and $\lambda \notin \overline{f(\mathbb{R})}$, then $\lambda \in \varrho(f(A))$.*

Proof. This is primarily a consequence of Theorem 3.7 and Example 3.3(c). For (6), choose complex λ outside of $p(\mathbb{R})$, then apply Theorem 3.7(6).

The density of $\mathcal{D}(f(A))$, for all Borel measurable f, follows from Lemma 1.7, since, given Borel measurable f, if $h(s) \equiv e^{-|f(s)|^2}$, then $\mathrm{Im}(h(A)) \subseteq \mathcal{D}(f(A))$, and $\{(e^{-z|f(s)|^2})(A)\}_{\mathrm{Re}(z)>0}$ is a strongly continuous holomorphic semigroup.

The density of $\mathrm{Im}(f(A))$, in (7), now follows from (7) of Theorem 3.7. ■

Remark. For Borel measurable functions that are bounded on bounded sets, one may also use the projection-valued measure E, for A, to define the functional calculus of this example:

$$f(A)x \equiv \lim_{N\to\infty} \int_{-N}^{N} f(s)\,dE(s)x,$$

with maximal domain, that is,

$$\mathcal{D}(f(A)) \equiv \Big\{x \in X \;\Big|\; \lim_{N\to\infty} \int_{-N}^{N} f(s)\,dE(s)x \text{ exists}\Big\}.$$

We emphasize that, in our construction, f may not even be locally bounded.

EXAMPLE 4.3. Suppose iA generates a bounded strongly continuous group, $\{e^{itA}\}_{t\in\mathbb{R}}$.

As in Example 2.5(d), let $\mathcal{F}$ be the set of Fourier transforms of L^1 functions. Then it is well known (see [3]) that A has an $\mathcal{F}$-functional calculus, given by

$$f(A)x \equiv \int_{\mathbb{R}} e^{-itA}xF(t)\,dt, \qquad \forall x \in X.$$

By Example 3.3(d), $C^1(\mathbb{R}) \subseteq \mathrm{EXT}_2(\mathcal{F})$. Thus Theorem 3.7 gives us the following.

PROPOSITION. *Construction Two defines a map, $f \mapsto f(A)$, from $C^1(\mathbb{R})$ into the set of all closed, densely defined operators on X, such that*

(1) $f_0(A) = I$;
(2) *for any* $\lambda \notin \mathbb{R}$, $g_\lambda(A) = (\lambda - A)^{-1}$;
(3) *for* $f \in \mathcal{F}$, $g \in C^1(\mathbb{R})$,

$$f(A)g(A) \subseteq g(A)f(A) = (fg)(A);$$

(4) *for* $f, g \in C^1(\mathbb{R})$,

$$f(A)g(A) \subseteq (fg)(A),$$

with

$$\mathcal{D}(f(A)g(A)) = \mathcal{D}((fg)(A)) \cap \mathcal{D}(g(A));$$

(5) *if* $f, g \in C^1(\mathbb{R})$, *then*

$$f(A) + g(A) \subseteq (f+g)(A),$$

where $\mathcal{D}(f(A)+g(A)) \equiv \mathcal{D}(f(A)) \cap \mathcal{D}(g(A))$;

(6) *if* $p(s) \equiv \sum_{k=0}^{N} a_k s^k$, *then*

$$p(A) = \sum_{k=0}^{N} a_k A^k,$$

and $\mathcal{D}(p(A)) = \mathcal{D}(A^N)$;

(7) *if* $f \in C^1(\mathbb{R})$ *is never zero then* $f(A)$ *is injective and has dense range; and*

(8) *if* $f \in C^1(\mathbb{R})$ *and* $\lambda \notin \overline{f(\mathbb{R})}$, *then* $\lambda \in \varrho(f(A))$.

Proof. This is primarily a consequence of Theorem 3.7 and Example 3.3(d). For (6), choose complex λ outside $p(\mathbb{R})$, then apply Theorem 3.7(6).

The density of $\mathcal{D}(f(A))$, for all $f \in C^1(\mathbb{R})$, follows from Lemma 1.7. Let h be as in Example 2.4(d). Then $\mathrm{Im}(h(A)) \subseteq \mathcal{D}(f(A))$, and the argument given in Definition 2.4(d) to show that $h(A)$ is injective also shows that $\mathrm{Im}(h(A))$ is dense.

The density of $\mathrm{Im}(f(A))$, in (7), now follows from (7) of Theorem 3.7. ■

More generally, we could consider iA to be a spectral distribution of degree k (see [1]; this includes, for example, the Laplacian on $C_0(\mathbb{R}^n)$ or $C(\Omega)$, for appropriate $\Omega \subseteq \mathbb{R}^n$). Then $\mathrm{EXT}_2(\mathcal{F})$ would include all $(k+1)$-times continuously differentiable functions on the real line.

This could also be generalized to functions of n commuting generators of bounded strongly continuous groups (see [5, Chapter XII]).

V. Exponentials and evolution equations. Consider the abstract Cauchy problem

$$\frac{d}{dt}u(t,x) = p(A)u(t,x) \quad (t \geq 0), \qquad u(0,x) = x, \tag{5.1}$$

where p is a polynomial of degree n, $p(s) \equiv \sum_{k=0}^{n} a_k s^k$, and

$$p(A)x \equiv \sum_{k=0}^{n} a_k A^k x, \qquad \mathcal{D}(p(A)) \equiv \mathcal{D}(A^n).$$

As in Section I, by a *mild solution* of (5.1) we mean that $t \mapsto u(t,x) \in C([0,\infty), X)$ and $\int_0^t u(s,x)\,ds \in \mathcal{D}(p(A))$, for all $t \geq 0$, with

$$u(t,x) = p(A)\Big(\int_0^t u(s,x)\,ds\Big) + x \qquad (t \geq 0).$$

By a *strong solution* we mean that $t \mapsto u(t,x) \in C^1([0,\infty), X) \cap C([0,\infty), \mathcal{D}(p(A)))$ and u satisfies (5.1).

THEOREM 5.2. *Suppose A is of α-type V (see Example 4.1), $\lambda \notin V$ and there exists h such that $z \mapsto e^{tp(z)}h(z) \in H^\infty(V)$, for all $t \geq 0$, and either*

(a) $h(V) \cap (-\infty, 0]$ *is empty, or*

(b) $h = e^k$, *for some holomorphic k such that $k(V) \subseteq \omega - S_\theta$, for some $\theta < \pi/2$, $\omega \in \mathbb{R}$.*

Let $m \equiv [\alpha] + 2$. Then

(1) *the family of operators $\{(e^{tp}hg_\lambda^m)(A)\}_{t\geq 0}$ is an $(hg_\lambda^m)(A)$-regularized semigroup generated by $p(A)$;*

(2) *(5.1) has a unique strong solution whenever $x \in (h(A))(\mathcal{D}(A^{m+n}))$;*

(3) *(5.1) has a unique mild solution whenever $x \in (h(A))(\mathcal{D}(A^m))$; and*

(4) *the set of all mild solutions of (5.1) equals the set of all x in $\bigcap_{t\geq 0}\mathcal{D}(e^{tp(A)})$ such that $t \mapsto e^{tp(A)}x$ is continuous.*

The mild solution of (5.1) then equals

$$u(t,x) = e^{tp(A)}x,$$

where $e^{tp(A)} \equiv (e^{tp})(A)$ is defined by Construction Two.

Proof. Since $h \in H^\infty(V)$, $h(A)$ is defined by Construction Two, as in Example 4.1. By (7) of the Proposition for Example 4.1 (for (a)), or Lemma 1.7 (for (b)), $h(A)$ is injective. For any $t \geq 0$, since $e^{tp}h \in H^\infty(V)$, it follows that $e^{tp}hg_\lambda^m \in H_m^\infty(V)$ (see Example 4.1). Thus $e^{tp} \in \mathrm{EXT}_2(H_m^\infty(V))$, so that $(e^{tp})(A)$ is defined by Construction Two.

For $f \in H_m^\infty(V)$, we have

$$f(A) \equiv \int_{\partial V} f(w)(w-A)^{-1}\frac{dw}{2\pi i}$$

(see Example 4.1).

It is clear from this representation of $W(t) \equiv (e^{tp}hg_\lambda^m)(A)$, and dominated convergence, that $t \mapsto W(t) \in C([0,\infty), B(X))$. The fact that $f \mapsto f(A)$, from $H_m^\infty(V)$ into $B(X)$, is an algebra homomorphism, implies that $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup, where $C \equiv (hg_\lambda^m)(A)$.

By Lemma 2.2, for any $t \geq 0$,

$$\int_0^t W(s)\,ds = \Big(\int_0^t e^{sp}hg_\lambda^m\,ds\Big)(A).$$

Since

$$p\Big(\int_0^t e^{sp}hg_\lambda^m\,ds\Big) = e^{tp}hg_\lambda^m - hg_\lambda^m,$$

by Definition 1.2(1), it follows that, for any $x \in X$ and $t \geq 0$, $\int_0^t W(s)x\,ds \in \mathcal{D}(p(A)) = \mathcal{D}(A^n)$, with

$$p(A)\int_0^t W(s)x\,ds = W(t)x - Cx.$$

Since $p(A)$ is closed and $Cp(A) \subseteq p(A)C$, it follows that an extension of $p(A)$ generates $\{W(t)\}_{t\geq 0}$. By [5, Proposition 3.9], with $G \equiv (\lambda - A)^{-n}$, $p(A)$ itself is the generator.

This proves (1). Assertions (2)–(4) now follow from (1) and Proposition 1.6, since $(e^{tp})(A) = (W(0))^{-1}W(t)$. ∎

OPEN QUESTION 5.3. It is clear from the proof of Theorem 5.2 that, whenever $e^{tp}h \in H_m^\infty(V)$, and $x \in \mathrm{Im}(h(A))$, then (5.1) has a mild solution. Does the set of all x for which (5.1) has a mild solution equal

$$\bigcup\{\mathrm{Im}(h(A)) \mid z \mapsto e^{tp(z)}h(z) \in H_m^\infty(V)\}?$$

Remark 5.4. Suppose iA generates a strongly continuous group. Then A is of 0-type $H_\varepsilon \equiv \{x + iy \mid x, y \in \mathbb{R},\ |y| < \varepsilon\}$, for all $\varepsilon > 0$; thus for *any* polynomial p, by choosing $h(z) \equiv e^{-z^{2n}}$, where n is the degree of p, we may apply Theorem 5.2, with $m = 2$.

Similarly, when $-A$ generates a strongly continuous holomorphic semigroup of angle $\pi/2$, then there exists a real ω such that $A - \omega$ is of (-1)-type $S_\theta \equiv \{re^{i\phi} \mid |\phi| < \theta, r > 0\}$, for all $\theta > 0$, so that we may apply Theorem 5.2, with $m = 1$ (use the same h as in the previous paragraph).

In both these cases, by Lemma 1.7, $\mathrm{Im}(h(A))$, hence $(h(A))(\mathcal{D}(A^{m+n}))$, is dense. Thus (5.1) has a strong solution for all initial data x in a dense set. With this approach we could obtain, for example, solutions of the backwards heat equation (see [9]) for all initial data in a dense set.

Remark 5.5. We could similarly treat the second order abstract Cauchy problem

$$\left(\frac{d}{dt}\right)^2 u(t) = p(A)u(t) \quad (t \geq 0), \qquad u(0) = x, \quad u'(0) = y,$$

by replacing e^{tp} with $\cos(it\sqrt{p})$.

Or, we could treat the time dependent abstract Cauchy problem

$$\frac{d}{dt}u(t) = (p(t))(A)u(t) \quad (t \geq s), \qquad u(s) = x,$$

for $s \geq 0$, $\{p(t)\}_{t\geq s}$ a family of polynomials, by defining

$$u(t) \equiv \Big(\exp\Big(\int_s^t p(r)\,dr\Big)\Big)(A)x.$$

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Received October 25, 1993
Revised version January 3, 1995
(3179)