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A necessary and sufficient condition for the existence of a father wavelet

by

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Abstract. It is proved that if $\{2^{-m/2}\psi(2^{-m}\bullet - k)\}_{m,k\in\mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{R};\mathbb{C})$, then the mother wavelet ψ is obtained from a multiresolution generated by a father wavelet if and only if $\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}|\widehat{\psi}(2^p(\bullet+k))|^2 > 0$ a.e.

1. Introduction. The question studied in this paper is the following: If we suppose that ψ is an orthonormal mother wavelet, that is, $2^{-m/2}\psi(2^{-m}\bullet - k)_{m,k\in\mathbb{Z}}$ (where $\bullet$ denotes a generic argument) is an orthonormal basis in $L^2(\mathbb{R};\mathbb{C})$, when is it true that ψ is obtained from an orthonormal multiresolution generated by a father wavelet φ?

By an *orthonormal multiresolution generated by* φ we mean a pair $(\{V_m\}_{m\in\mathbb{Z}}, \varphi)$ that satisfies the following properties:

(1) $\varphi \in L^2(\mathbb{R};\mathbb{C})$ and V_m is, for each $m \in \mathbb{Z}$, the closed subspace of $L^2(\mathbb{R};\mathbb{C})$ spanned by $\{\varphi(2^{-m}\bullet - k)\}_{k\in\mathbb{Z}}$,

(2) $V_m \subset V_{m-1}$, $m \in \mathbb{Z}$,

(3) $\lim_{m\to-\infty} V_m = L^2(\mathbb{R};\mathbb{C})$, i.e. $\lim_{m\to-\infty} P_m f = f$ for every $f \in L^2(\mathbb{R};\mathbb{C})$, where P_m is the orthogonal projection of $L^2(\mathbb{R};\mathbb{C})$ onto V_m,

(4) $\{\varphi(\bullet - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis in V_0.

The definition of a multiresolution (or a *multiresolution analysis* as it is also called) is often given in slightly different forms (but with exactly the same content) (see e.g., [3], [6], [8], [11], and [12]).

From a multiresolution one obtains an orthonormal mother wavelet that generates an orthonormal basis in $L^2(\mathbb{R};\mathbb{C})$ as follows. By (2) and (4) we have

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$$\varphi = 2\sum_{k\in\mathbb{Z}} \alpha(k)\varphi(2\bullet - k), \tag{5}$$

where the filter α is given by

$$\alpha(k) = \int_{\mathbb{R}} \varphi(x)\overline{\varphi(2x-k)}\,dx, \qquad k \in \mathbb{Z}.$$

Having found the filter α we can obtain an orthonormal mother wavelet ψ from the multiresolution by

$$\psi = 2\sum_{k\in\mathbb{Z}} \beta(k)\varphi(2\bullet - k), \tag{6}$$

where

$$\beta(k) = (-1)^k\overline{\alpha(1-k)}, \qquad k \in \mathbb{Z}.$$

For more details, see [3], [6], [8], [11], and [12].

The question of when an orthonormal mother wavelet is associated with a multiresolution is a natural one, and a partial answer is given in [9] and [10], where it is shown that this is the case if ψ has compact support and is Hölder continuous. Another result, announced in [1], is that it suffices to assume that $\widehat{\psi}$ is continuous and decays sufficiently rapidly at infinity. Here we shall extend these results and prove that a necessary and sufficient assumption on ψ is that

$$\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^p(\bullet + k))|^2 > 0 \quad \text{a.e.}$$

Here the Fourier transform $\widehat{\psi}$ is defined by $\widehat{\psi} = \int_{\mathbb{R}} e^{-i2\pi t\bullet}\psi(t)\,dt$ (so that the factor 2π will in this paper appear in the argument of the exponent function only). As a corollary we deduce that a sufficient assumption is that the Lebesgue measure of the set $\{\omega \in \mathbb{R} \mid \widehat{\psi}(\omega) = 0\}$ is zero, which, of course, is the case when ψ has compact support, or, more generally, if ψ decays e.g. exponentially (cf. [7]). Another corollary deals with the case, considered in [1], where $\widehat{\psi}$ is continuous and $\sup_{|\omega|\geq\bullet} |\widehat{\psi}(\omega)| \in L^2(\mathbb{R}^+;\mathbb{R})$.

2. Statement of results. First we state a well-known characterization in terms of Fourier transforms of when a function ψ is an orthonormal mother wavelet.

THEOREM 1. *Let $\psi \in L^2(\mathbb{R};\mathbb{C})$ be such that $\|\psi\|_{L^2(\mathbb{R})} = 1$. Then $\{2^{-m/2}\psi(2^{-m}\bullet - k)_{k,m\in\mathbb{Z}}\}$ is an orthonormal basis in $L^2(\mathbb{R};\mathbb{C})$ if and only if*

$$\sum_{m\in\mathbb{Z}} |\widehat{\psi}(2^m\bullet)|^2 = 1 \quad \text{a.e.}, \tag{7}$$

and

$$\sum_{p=0}^{\infty} \widehat{\psi}(2^p\bullet)\overline{\widehat{\psi}(2^p(\bullet + k))} = 0 \quad \text{a.e. for all odd integers } k. \tag{8}$$

This result can be found at least in [7, p. 230] and [8, p. 29], and the sufficiency part is given in more general form (that is, as a statement about frame bounds) in [5, Thm. 2.9]. For completeness, and because in [5] and in [7] an additional technical assumption on ψ is used, we give a proof of this theorem below.

THEOREM 2. *Let $\psi \in L^2(\mathbb{R};\mathbb{C})$ be an orthonormal mother wavelet, i.e. $\{2^{-m/2}\psi(2^{-m}\bullet - k)\}_{k,m\in\mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{R};\mathbb{C})$. Then ψ is obtained (in the sense of* (6)*) from a multiresolution satisfying* (1)–(4) *if and only if*

$$\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^p(\bullet + k))|^2 > 0 \quad \text{a.e.} \tag{9}$$

In this case it is actually true that

$$\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^p(\bullet + k))|^2 = 1 \quad \text{a.e.} \tag{10}$$

In the proof of the sufficiency of (9) the relation (8) will be of crucial importance, whereas (7) is of surprisingly little direct use. Using a result about quasianalyticity that can be found e.g. in [2, p. 406], we get the following result that, of course, is applicable to wavelets with compact support.

COROLLARY 3. *Let $\psi \in L^2(\mathbb{R};\mathbb{C})$ be an orthonormal mother wavelet, i.e. $\{2^{-m/2}\psi(2^{-m}\bullet - k)\}_{k,m\in\mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{R};\mathbb{C})$. Then ψ is obtained (in the sense of* (6)*) from a multiresolution satisfying* (1)–(4) *if the measure of the set $\{\omega \in \mathbb{R} \mid \widehat{\psi}(\omega) = 0\}$ is zero, in particular, if*

$$\int_1^{\infty} |\psi(t)|\,dt < \infty \quad \text{and} \quad \int_1^{\infty} \tau^{-2}\Big|\log\Big(\int_{\tau}^{\infty} |\psi(t)|\,dt\Big)\Big|\,d\tau = \infty,$$

or

$$\int_{-\infty}^{-1} |\psi(t)|\,dt < \infty \quad \text{and} \quad \int_{-\infty}^{-1} \tau^{-2}\Big|\log\Big(\int_{-\infty}^{\tau} |\psi(t)|\,dt\Big)\Big|\,d\tau = \infty.$$

The following corollary is essentially the same as the result in [1].

COROLLARY 4. *Let $\psi \in L^2(\mathbb{R};\mathbb{C})$ be an orthonormal mother wavelet, i.e. $\{2^{-m/2}\psi(2^{-m}\bullet - k)\}_{k,m\in\mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{R};\mathbb{C})$. Then ψ is obtained (in the sense of* (6)) *from a multiresolution satisfying* (1)–(4) *if*

$$\widehat{\psi} \in C(\mathbb{R};\mathbb{C}), \tag{11}$$

and

$$\sup_{|\omega|\ge\bullet} |\widehat{\psi}(\omega)| \in L^2(\mathbb{R}^+;\mathbb{R}). \tag{12}$$

If ψ has compact support, then it follows, as shown in [9] and [10], that φ has compact support as well. In general, one can show that φ and ψ have similar smoothness and decay properties, but this question will not be studied here.

An example (taken from [8]) of an orthonormal mother wavelet that is not generated by a multiresolution is the function ψ with Fourier transform

$$\widehat{\psi} = \chi_{[-4/7,-2/7)} + \chi_{[2/7,3/7)} + \chi_{[12/7,16/7)}.$$

It is easy to check that the conditions (7) and (8) are satisfied but that

$$\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^p(\bullet + k))|^2 = 0 \quad \text{in } (2/7,3/7) \text{ and in } (4/7,5/7).$$

3. Proof of Theorem 1. We use the notation $\psi_{m,k} = 2^{-m/2}\psi(2^{-m}\bullet - k)$ and note that $\widehat{\psi}_{m,k} = 2^{m/2}e^{-i2\pi 2^m k\bullet}\widehat{\psi}(2^m\bullet)$.

According to a standard result in Hilbert space theory, a set $\{g_j\}_{j\in\mathbb{I}}$ is an orthonormal basis in H if and only if $\|g_j\|_H = 1$ for every $j\in\mathbb{I}$ and $\|x\|_H^2 = \sum_{j\in\mathbb{I}}|\langle f, g_j\rangle_H|^2$ for every $f\in H$. Thus the set $\{\psi_{m,k}\}_{m,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R};\mathbb{C})$ if and only if for each $f\in L^2(\mathbb{R};\mathbb{C})$ we have

$$\|f\|^2_{L^2(\mathbb{R})} = \sum_{\substack{m\in\mathbb{Z}\\ k\in\mathbb{Z}}} |\langle f, \psi_{m,k}\rangle|^2, \tag{13}$$

where $\langle\bullet,\bullet\rangle$ denotes the inner product in $L^2(\mathbb{R};\mathbb{C})$.

Let $f\in L^2(\mathbb{R};\mathbb{C})$ be such that $\widehat{f}$ is bounded and vanishes outside a compact set in $\mathbb{R}\setminus\{0\}$. First we let $m\in\mathbb{Z}$ be arbitrary, and we use Plancherel's theorem, a change of variables, and the fact that the Fourier transform is an isometry from $L^2(\mathbb{T};\mathbb{C})$ to $\ell^2(\mathbb{Z};\mathbb{C})$ (where $\mathbb{T}$ denotes $\mathbb{R}/\mathbb{Z}$, i.e., functions defined on $\mathbb{T}$ are periodic functions on $\mathbb{R}$ with period 1) to get

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}|\langle f,\psi_{m,k}\rangle|^2 &= \sum_{k\in\mathbb{Z}}\Big|\int_{\mathbb{R}} 2^{m/2}\widehat{f}(\omega)e^{i2\pi 2^m k\omega}\overline{\widehat{\psi}(2^m\omega)}\,d\omega\Big|^2\\
&= 2^{-m}\sum_{k\in\mathbb{Z}}\Big|\int_{\mathbb{R}} e^{i2\pi k\omega}\widehat{f}(2^{-m}\omega)\overline{\widehat{\psi}(\omega)}\,d\omega\Big|^2\\
&= 2^{-m}\sum_{k\in\mathbb{Z}}\Big|\int_0^1 e^{i2\pi k\omega}\sum_{j\in\mathbb{Z}}\widehat{f}(2^{-m}(\omega+j))\overline{\widehat{\psi}(\omega+j)}\,d\omega\Big|^2\\
&= 2^{-m}\int_0^1\Big|\sum_{j\in\mathbb{Z}}\widehat{f}(2^{-m}(\omega+j))\overline{\widehat{\psi}(\omega+j)}\Big|^2\,d\omega\\
&= 2^{-m}\int_0^1\sum_{j\in\mathbb{Z}}|\widehat{f}(2^{-m}(\omega+j))|^2|\widehat{\psi}(\omega+j)|^2\,d\omega\\
&\quad + 2^{-m}\int_0^1\sum_{\substack{j_1,j_2\in\mathbb{Z}\\ j_1\ne j_2}}\widehat{f}(2^{-m}(\omega+j_1))\overline{\widehat{f}(2^{-m}(\omega+j_2))}\\
&\quad \times\overline{\widehat{\psi}(\omega+j_1)}\widehat{\psi}(\omega+j_2)\,d\omega\\
&= 2^{-m}\int_{\mathbb{R}}|\widehat{f}(2^{-m}\omega)|^2|\widehat{\psi}(\omega)|^2\,d\omega\\
&\quad + 2^{-m}\int_0^1\sum_{\substack{j,j_*\in\mathbb{Z}\\ j\ne 0}}\widehat{f}(2^{-m}(\omega+j_*))\overline{\widehat{f}(2^{-m}(\omega+j_*+j))}\\
&\quad \times\overline{\widehat{\psi}(\omega+j_*)}\widehat{\psi}(\omega+j_*+j)\,d\omega,\\
&= \int_{\mathbb{R}}|\widehat{f}(\omega)|^2|\widehat{\psi}(2^m\omega)|^2\,d\omega\\
&\quad + 2^{-m}\int_{\mathbb{R}}\sum_{\substack{j\in\mathbb{Z}\\ j\ne 0}}\widehat{f}(2^{-m}\omega)\overline{\widehat{f}(2^{-m}(\omega+j))\widehat{\psi}(\omega)}\widehat{\psi}(\omega+j)\,d\omega.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|\langle f,\widehat{\psi}_{m,k}\rangle|^2 &= \int_{\mathbb{R}}|\widehat{f}(\omega)|^2\sum_{m\in\mathbb{Z}}|\widehat{\psi}(2^m\omega)|^2\,d\omega\\
&\quad + \sum_{m\in\mathbb{Z}}2^{-m}\int_{\mathbb{R}}\sum_{\substack{j\in\mathbb{Z}\\ j\ne 0}}\widehat{f}(2^{-m}\omega)\overline{\widehat{f}(2^{-m}(\omega+j))\widehat{\psi}(\omega)}\widehat{\psi}(\omega+j)\,d\omega.
\end{aligned}\tag{14}$$

Now we use the fact that every nonzero integer j can be written in the form $j = 2^p k$, where p is a nonnegative integer and k an odd integer. Provided the use of Fubini's theorem can be justified (which we will check later) we therefore get

$$\begin{aligned}
(15)\quad & \sum_{m\in\mathbb{Z}} 2^{-m}\int_{\mathbb{R}} \sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} \widehat{f}(2^{-m}\omega)\overline{\widehat{f}(2^{-m}(\omega+j))\widehat{\psi}(\omega)}\widehat{\psi}(\omega+j)\,d\omega \\
&= \sum_{\substack{k\in\mathbb{Z}\\ k \text{ odd}}} \int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} \sum_{p=0}^{\infty} 2^{-m}\widehat{f}(2^{-m}\omega)\overline{\widehat{f}(2^{-m}(\omega+k2^p))\widehat{\psi}(\omega)}\widehat{\psi}(\omega+k2^p)\,d\omega \\
&= \sum_{\substack{k\in\mathbb{Z}\\ k \text{ odd}}} \int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} \sum_{p=0}^{\infty} 2^{-m+p}\widehat{f}(2^{-m+p}\omega)\overline{\widehat{f}(2^{-m+p}(\omega+k))} \\
&\qquad \times \overline{\widehat{\psi}(2^p\omega)}\widehat{\psi}(2^p(\omega+k))\,d\omega \\
&= \sum_{\substack{k\in\mathbb{Z}\\ k \text{ odd}}} \int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} 2^{-m}\widehat{f}(2^{-m}\omega)\overline{\widehat{f}(2^{-m}(\omega+k))}\,\overline{\sum_{p=0}^{\infty}\widehat{\psi}(2^p\omega)\overline{\widehat{\psi}(2^p(\omega+k))}}\,d\omega.
\end{aligned}$$

To see that we may apply Fubini's theorem above we first invoke Cauchy's inequality three times and perform two changes of variables to get

$$\begin{aligned}
& \sum_{\substack{k\in\mathbb{Z}\\ k \text{ odd}}} \int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} 2^{-m}|\widehat{f}(2^{-m}\omega)||\widehat{f}(2^{-m}(\omega+k))| \\
&\qquad \times \sum_{p=0}^{\infty} |\widehat{\psi}(2^p\omega)||\widehat{\psi}(2^p(\omega+k))|\,d\omega \\
&\le \sum_{\substack{k\in\mathbb{Z}\\ k \text{ odd}}} \int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} 2^{-m}|\widehat{f}(2^{-m}\omega)||\widehat{f}(2^{-m}(\omega+k))| \\
&\qquad \times \Big(\sum_{p=0}^{\infty} |\widehat{\psi}(2^p\omega)|^2\Big)^{1/2}\Big(\sum_{p=0}^{\infty} |\widehat{\psi}(2^p(\omega+k))|^2\Big)^{1/2}\,d\omega \\
&\le \sum_{\substack{k\in\mathbb{Z}\\ k \text{ odd}}} \Big(\int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} 2^{-m}|\widehat{f}(2^{-m}\omega)||\widehat{f}(2^{-m}(\omega+k))| \sum_{p=0}^{\infty} |\widehat{\psi}(2^p\omega)|^2\,d\omega\Big)^{1/2} \\
&\qquad \times \Big(\int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} 2^{-m}|\widehat{f}(2^{-m}\omega)||\widehat{f}(2^{-m}(\omega+k))| \sum_{p=0}^{\infty} |\widehat{\psi}(2^p(\omega+k))|^2\,d\omega\Big)^{1/2} \\
&= \sum_{\substack{k\in\mathbb{Z}\\ k \text{ odd}}} \Big(\int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} 2^{-m}|\widehat{f}(2^{-m}\omega)||\widehat{f}(2^{-m}(\omega+k))| \sum_{p=0}^{\infty} |\widehat{\psi}(2^p\omega)|^2\,d\omega\Big)^{1/2} \\
&\qquad \times \Big(\int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} 2^{-m}|\widehat{f}(2^{-m}(\omega-k))||\widehat{f}(2^{-m}\omega)| \sum_{p=0}^{\infty} |\widehat{\psi}(2^p\omega)|^2\,d\omega\Big)^{1/2} \\
&\le \Big(\sum_{\substack{k\in\mathbb{Z}\\ k \text{ odd}}} \int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} 2^{-m}|\widehat{f}(2^{-m}\omega)||\widehat{f}(2^{-m}(\omega+k))| \sum_{p=0}^{\infty} |\widehat{\psi}(2^p\omega)|^2\,d\omega\Big)^{1/2} \\
&\qquad \times \Big(\sum_{\substack{k\in\mathbb{Z}\\ k \text{ odd}}} \int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} 2^{-m}|\widehat{f}(2^{-m}(\omega-k))||\widehat{f}(2^{-m}\omega)| \sum_{p=0}^{\infty} |\widehat{\psi}(2^p\omega)|^2\,d\omega\Big)^{1/2} \\
&= \sum_{\substack{k\in\mathbb{Z}\\ k \text{ odd}}} \int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} 2^{-m}|\widehat{f}(2^{-m}\omega)||\widehat{f}(2^{-m}(\omega+k))| \sum_{p=0}^{\infty} |\widehat{\psi}(2^p\omega)|^2\,d\omega.
\end{aligned}$$

In fact, by the same argument we have for a symmetrical set $\mathbb{I}\subset\mathbb{Z}$ and arbitrary set $\mathbb{J}\subset\mathbb{Z}$,

$$\begin{aligned}
(16)\quad & \sum_{k\in\mathbb{I}} \int_{\mathbb{R}} \sum_{m\in\mathbb{J}} 2^{-m}|\widehat{f}(2^{-m}\omega)||\widehat{f}(2^{-m}(\omega+k))| \\
&\qquad \times \sum_{p=0}^{\infty} |\widehat{\psi}(2^p\omega)||\widehat{\psi}(2^p(\omega+k))|\,d\omega \\
&\le \sum_{k\in\mathbb{I}} \int_{\mathbb{R}} \sum_{m\in\mathbb{J}} 2^{-m}|\widehat{f}(2^{-m}\omega)||\widehat{f}(2^{-m}(\omega+k))| \sum_{p=0}^{\infty} |\widehat{\psi}(2^p\omega)|^2\,d\omega.
\end{aligned}$$

It is clear that

$$|\widehat{f}(2^{-m}\bullet)||\widehat{f}(2^{-m}(\bullet+k))| = 0$$

when $2^{-m}|k| > \operatorname{diam}(\operatorname{supp}(\widehat{f}))$, where $\operatorname{supp}(\widehat{f})$ denotes the support of $\widehat{f}$ (i.e. the smallest closed set such that $\widehat{f}=0$ outside it). Moreover, if $\widehat{f}(\omega)=0$ when $|\omega|<\alpha$ or $|\omega|>\beta$, then there are, for each $\omega\neq 0$, at most $\lfloor \log_2(\beta/\alpha)+1\rfloor$ numbers m such that $\widehat{f}(2^{-m}\omega)\neq 0$. Thus

$$\begin{aligned}
(17)\quad & \sum_{\substack{k\in\mathbb{Z}\\ k \text{ odd}}} \sum_{m\in\mathbb{Z}} 2^{-m}|\widehat{f}(2^{-m}\omega)||\widehat{f}(2^{-m}(\omega+k))| \\
&\le \|\widehat{f}\|^2_{L^\infty(\mathbb{R})}\operatorname{diam}(\operatorname{supp}(\widehat{f})) \\
&\qquad \times \lfloor \log_2(\sup|\operatorname{supp}(\widehat{f})|) - \log_2(\inf|\operatorname{supp}(\widehat{f})|) + 1\rfloor,
\end{aligned}$$

where $|\Omega|$ is the set $\{|\omega| \mid \omega\in\Omega\}$. Since we also have

$$(18)\qquad \int_{\mathbb{R}} \sum_{p=0}^{\infty} |\widehat{\psi}(2^p\omega)|^2\,d\omega = \sum_{p=0}^{\infty} 2^{-p}\int_{\mathbb{R}} |\widehat{\psi}(\omega)|^2\,d\omega = 2,$$

we get from (16) our desired conclusion that

$$\sum_{\substack{k\in\mathbb{Z}\\ k\,\mathrm{odd}}}\int_{\mathbb{R}}\sum_{m\in\mathbb{Z}}2^{-m}|\widehat{f}(2^{-m}\omega)||\widehat{f}(2^{-m}(\omega+k))|\times\sum_{p=0}^{\infty}|\widehat{\psi}(2^p\omega)||\widehat{\psi}(2^p(\omega+k))|\,d\omega<\infty.$$

If we assume that (7) and (8) hold, we conclude from (14) and (15) that (13) holds for all $f\in L^2(\mathbb{R};\mathbb{C})$ with bounded Fourier transform supported in a compact set in $\mathbb{R}\setminus\{0\}$. This is a dense set, and therefore we deduce from Fatou's lemma that $\|f\|^2_{L^2(\mathbb{R})}\ge\sum_{m,k\in\mathbb{Z}}|\langle f,\psi_{m,k}\rangle|^2$ for every $f\in L^2(\mathbb{R};\mathbb{C})$. In particular, the set $\{\psi_{m,k}\}_{m,k\in\mathbb{Z}}$ is orthonormal. But then from our restricted version of (13) we see that $f=\sum_{m,k\in\mathbb{Z}}\langle f,\psi_{m,k}\rangle\psi_{m,k}$ for a dense set of functions f and this gives the desired conclusion.

Assume next that $\{\psi_{m,k}\}_{m,k\in\mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{R};\mathbb{C})$. We let $\omega_0\ne0$ be arbitrary and choose $f=f_\delta$, where $\widehat{f}_\delta=\delta^{-1/2}\chi_{(\omega_0,\omega_0+\delta)}$ and $0<\delta<|\omega_0|$. We see immediately that if $|\widehat{f}_\delta(2^{-m}\omega)||\widehat{f}_\delta(2^{-m}(\omega+k))|\ne0$, then $2^{-m}|k|<\delta$ and $|\omega|\ge2^m\min\{|\omega_0|,|\omega_0+\delta|\}$, which shows that

$$\lim_{\delta\downarrow0}\sum_{\substack{k\in\mathbb{Z}\\ k\,\mathrm{odd}}}\sum_{m\in\mathbb{Z}}2^{-m}|\widehat{f}(2^{-m}\omega)||\widehat{f}(2^{-m}(\omega+k))|=0,\qquad\omega\in\mathbb{R}.$$

If we now use (13)–(18) and the dominated convergence theorem, then we see that

$$\lim_{\delta\to0}\frac{1}{\delta}\int_{\omega_0}^{\omega_0+\delta}\sum_{m\in\mathbb{Z}}|\widehat{\psi}(2^{-m}\omega)|^2\,d\omega=1,$$

and we obtain (7).

To get (8) we choose an arbitrary number $\omega_0\notin\mathbb{Z}$, an odd integer k_0, and an arbitrary complex number α with $|\alpha|=1$ and take $f=f_\delta$ to be such that

$$\widehat{f}_\delta=\delta^{-1/2}\chi_{(\omega_0,\omega_0+\delta)}+\alpha\delta^{-1/2}\chi_{(\omega_0+k_0,\omega_0+k_0+\delta)},$$

where $\delta>0$ is so small that

(19) if $\omega_1,\omega_2\in(\omega_0,\omega_0+\delta)$ or $\omega_1,\omega_2\in(\omega_0+k_0,\omega_0+k_0+\delta)$, then $\omega_1\ne2\omega_2$.

It is easy to see that

$$\sum_{k=\pm k_0}\int_{\mathbb{R}}\widehat{f}_\delta(\omega)\overline{\widehat{f}_\delta(\omega+k)}\,\overline{\sum_{p=0}^{\infty}\widehat{\psi}(2^p\omega)\overline{\widehat{\psi}(2^p(\omega+k))}}\,d\omega$$
$$=2\Re\biggl(\frac{\alpha}{\delta}\int_{\omega_0}^{\omega_0+\delta}\sum_{p=0}^{\infty}\widehat{\psi}(2^p\omega)\overline{\widehat{\psi}(2^p(\omega+k_0))}\,d\omega\biggr).$$

In view of (7) and (13)–(15) it therefore suffices to show that

$$\lim_{\delta\downarrow0}\sum_{k=\pm k_0}\int_{\mathbb{R}}\sum_{m\in\mathbb{Z}\setminus\{0\}}2^{-m}|\widehat{f}_\delta(2^{-m}\omega)||\widehat{f}_\delta(2^{-m}(\omega+k))|\times\sum_{p=0}^{\infty}|\widehat{\psi}(2^p\omega)||\widehat{\psi}(2^p(\omega+k))|\,d\omega=0,\tag{20}$$

and

$$\lim_{\delta\downarrow0}\sum_{\substack{k\in\mathbb{Z}\setminus\{\pm k_0\}\\ k\,\mathrm{odd}}}\int_{\mathbb{R}}\sum_{m\in\mathbb{Z}}2^{-m}|\widehat{f}_\delta(2^{-m}\omega)||\widehat{f}_\delta(2^{-m}(\omega+k))|\times\sum_{p=0}^{\infty}|\widehat{\psi}(2^p\omega)||\widehat{\psi}(2^p(\omega+k))|\,d\omega=0.\tag{21}$$

In order to establish (20) we observe that if $\delta<|k_0|/2$ and $k=\pm k_0$, then $2^{-m}|k|\notin[|k_0|-\delta,|k_0|+\delta]$ when $m\ne0$, and this implies that $|\widehat{f}_\delta(2^{-m}\omega)||\widehat{f}_\delta(2^{-m}(\omega+k))|\ne0$ only in the case where $|2^{-m}k|\le\delta$, that is, only when both $2^{-m}\omega$ and $2^{-m}(\omega+k)$ belong to either $(\omega_0,\omega_0+\delta)$ or $(\omega_0+k_0,\omega_0+k_0+\delta)$. For each fixed ω this can, by (19), happen for at most 2 values of m, and since $2^{-m}\le\delta$ we see that it will not happen if δ is sufficiently small because we have either

$$\begin{aligned}&2^{-m}|\omega|\ge\min\{|\omega_0|,|\omega_0+\delta|,|\omega_0+k_0|,|\omega_0+k_0+\delta|\}>0\quad\text{or}\\&|\widehat{f}_\delta(2^{-m}\omega)|=0.\end{aligned}\tag{22}$$

Hence

$$\sum_{k=\pm k_0}\sum_{m\in\mathbb{Z}\setminus\{0\}}2^{-m}|\widehat{f}_\delta(2^{-m}\omega)||\widehat{f}_\delta(2^{-m}(\omega+k))|\le4,\qquad\omega\in\mathbb{R},\tag{23}$$

and

$$\lim_{\delta\downarrow0}\sum_{k=\pm k_0}\sum_{m\in\mathbb{Z}\setminus\{0\}}2^{-m}|\widehat{f}_\delta(2^{-m}\omega)||\widehat{f}_\delta(2^{-m}(\omega+k))|=0,\qquad\omega\in\mathbb{R}.\tag{24}$$

Combining (16), (18), (23) and (24) with the dominated convergence theorem we obtain (20).

In order to establish (21) we again note that by (19) there are, for each ω, at most 2 values for m such that $|\widehat{f}_\delta(2^{-m}\omega)||\widehat{f}_\delta(2^{-m}(\omega+k))|\ne0$. Thus we have to count for how many values of k this will be the case. One possibility is that $2^{-m}|k|<\delta$ and there are at most $\delta2^{m+1}$ such odd integers. Another possibility is that $2^{-m}|k|\in[|k_0|-\delta,|k_0|+\delta]$ and we see that there are at most $\delta2^{m+2}$ odd integers k for which this holds. But note that in both of these cases we must have $2^m\delta\ge1$; for the second case we invoke the facts that both k and k_0 are odd and $|k| \neq |k_0|$. Thus we deduce that

$$(25) \qquad \sum_{\substack{k\in\mathbb{Z}\setminus\{\pm k_0\}\\ k \text{ odd}}} \sum_{m\in\mathbb{Z}} 2^{-m}|\widehat{f}_\delta(2^{-m}\omega)||\widehat{f}_\delta(2^{-m}(\omega+k))| \le 12, \qquad \omega\in\mathbb{R},$$

and by (22) that

$$(26) \qquad \lim_{\delta\downarrow 0} \sum_{\substack{k\in\mathbb{Z}\setminus\{\pm k_0\}\\ k \text{ odd}}} \sum_{m\in\mathbb{Z}} 2^{-m}|\widehat{f}_\delta(2^{-m}\omega)||\widehat{f}_\delta(2^{-m}(\omega+k))| = 0, \qquad \omega\in\mathbb{R}.$$

We can again combine (16), (18), (25), and (26) with the dominated convergence theorem and (21) follows.

Thus the proof is complete. ∎

4. Proof of Theorem 2. The necessity of (9) is easy to establish, because if ψ is obtained from a multiresolution generated by φ, then $|\widehat{\varphi}(\bullet)|^2 = \sum_{p=1}^{\infty}|\widehat{\psi}(2^p\bullet)|^2$ (see [8, p. 31]) and the orthonormality of the functions $\{\varphi(\bullet-k)\}_{k\in\mathbb{Z}}$ is equivalent to the fact that $\sum_{k\in\mathbb{Z}}|\widehat{\varphi}(\bullet+k)|^2 = 1$ a.e., so we get (10), hence (9), immediately.

For the converse we start with some auxiliary results. Again we use the notation $\psi_{m,k} = 2^{-m/2}\psi(2^{-m}\bullet - k)$. We denote by W_m the closed subspace of $L^2(\mathbb{R};\mathbb{C})$ spanned by $\{\psi_{m,k}\}_{k\in\mathbb{Z}}$ and by Q_m the orthogonal projection onto W_m. Let $V_m = \bigoplus_{j=m+1}^{\infty} W_j$ and let P_m be the orthogonal projection onto V_m. If τ_s, $s\in\mathbb{R}$, is the translation operator $\tau_s f = f(\bullet - s)$, then a straightforward calculation shows that

$$(27) \qquad \tau_{2^m p} Q_m \tau_{-2^m p} = Q_m, \qquad m,\ p\in\mathbb{Z}.$$

Since $P_m = I - \sum_{j=-\infty}^{m} Q_j$ we conclude that

$$(28) \qquad \tau_{2^m p} P_m \tau_{-2^m p} = P_m, \qquad m,\ p\in\mathbb{Z}.$$

Next we prove an auxiliary result.

LEMMA 5. *Let $f\in L^2(\mathbb{R};\mathbb{C})$. Then*

$$\lim_{m\to\infty}\sup_{s\in\mathbb{R}} \|P_m(\tau_s f)\|_{L^2(\mathbb{R})} = 0.$$

Proof. Let $\varepsilon > 0$ be arbitrary. There exists a step function

$$g = \sum_{i=1}^{i_*} c_i \chi_{[a_i,b_i]}$$

such that

$$(29) \qquad \|f-g\|_{L^2(\mathbb{R})} \le \varepsilon/3.$$

It is easy to see that there exists a number n_* such that if $m\ge 1$ and $s\in\mathbb{R}$, then

$$(30) \qquad \tau_s g = \sum_{n=1}^{n_*} C_n \chi_{[2^m k_n + A_n, 2^m k_n + B_n]} + h,$$

where, for $1\le n\le n_*$,

$$(31) \qquad \begin{gathered} |C_n| \le \max_{1\le i\le i_*}|c_i|, \qquad k_n\in\mathbb{Z},\\ 0 < \tfrac{1}{2}B_n < A_n \le B_n \le 2^m, \qquad |B_n - A_n| \le \max_{1\le i\le i_*}|b_i - a_i|, \end{gathered}$$

and

$$(32) \qquad \|h\|_{L^2(\mathbb{R})} \le \varepsilon/3.$$

Since $P_m = \sum_{j=m+1}^{\infty} Q_j$ and the functions $\psi_{j,k}$ are orthonormal, it follows from (28), Cauchy's inequality, and from the fact that the intervals $[2^{-j}A_n, 2^{-j}B_n]$ are disjoint when $j > m$ by (31), that for each $n = 1,2,\ldots,n_*$,

$$\begin{aligned} (33) \qquad & \|P_m(C_n\chi_{[2^m k_n + A_n, 2^m k_n + B_n]})\|^2_{L^2(\mathbb{R})} \\ &= \|P_m(C_n\chi_{[A_n,B_n]})\|^2_{L^2(\mathbb{R})} \\ &= \sum_{j=m+1}^{\infty}\sum_{k\in\mathbb{Z}}\Big|\int_{\mathbb{R}} C_n\chi_{[A_n,B_n]}(t)\overline{\psi_{j,k}(t)}\,dt\Big|^2 \\ &\le |C_n|^2|B_n - A_n| \sum_{j=m+1}^{\infty}\sum_{k\in\mathbb{Z}} 2^{-j}\int_{A_n}^{B_n}|\psi(2^{-j}t - k)|^2\,dt \\ &= |C_n|^2|B_n - A_n| \sum_{j=m+1}^{\infty}\sum_{k\in\mathbb{Z}} \int_{2^{-j}A_n}^{2^{-j}B_n}|\psi(t-k)|^2\,dt \\ &= |C_n|^2|B_n - A_n| \sum_{k\in\mathbb{Z}} \int_0^1 \chi_{\bigcup_{j=m+1}^{\infty}[2^{-j}A_n, 2^{-j}B_n]}(t)|\psi(t-k)|^2\,dt. \end{aligned}$$

Since $\psi\in L^2(\mathbb{R};\mathbb{C})$ there exists a number $\delta > 0$ such that if $E\subset[0,1]$ satisfies $\mathrm{m}(E) < \delta$, then

$$\sum_{k\in\mathbb{Z}}\int_E |\psi(t-k)|^2\,dt \le \frac{\varepsilon^2}{9|C_n|^2\max_{1\le i\le i_*}|b_i - a_i| n_*^2}.$$

Now

$$\mathrm{m}\Big(\bigcup_{j=m+1}^{\infty}[2^{-j}A_n, 2^{-j}B_n]\Big) = 2^{-m}|B_n - A_n|,$$

and we conclude from (31) and (33) that if $2^{-m}\max_{1\le i\le i_*}|b_i-a_i|<\delta$, then

$$\Big\|P_m\Big(\sum_{n=1}^{n_*}C_n\chi_{[2^mk_n+A_n,2^mk_n+B_n]}\Big)\Big\|_{L^2(\mathbb{R})}\le\frac{\varepsilon}{3}.$$

Together with (29), (30), (32), and the fact that an orthogonal projection is a contraction, this completes the proof of Lemma 5. ∎

Next we show that

$$P_0f=\sum_{m=1}^{\infty}2^{-m}\sum_{p=1}^{2^m}\tau_pQ_m(\tau_{-p}f),\qquad f\in L^2(\mathbb{R};\mathbb{C}).\tag{34}$$

Let $j\ge1$ be some integer. By (28) we know that

$$P_0f=2^{-j}\sum_{p=1}^{2^j}\tau_pP_0(\tau_{-p}f),$$

and it follows from the fact that $P_0=\sum_{m=1}^{j}Q_m+P_j$ that

$$P_0f=\sum_{m=1}^{j}2^{-j}\sum_{p=1}^{2^j}\tau_pQ_m(\tau_{-p}f)+2^{-j}\sum_{p=1}^{2^j}\tau_pP_j(\tau_{-p}f).$$

By (27) we know that for each m between 1 and j we have

$$\sum_{p=1}^{2^j}\tau_pQ_m(\tau_{-p}f)=2^{j-m}\sum_{p=1}^{2^m}\tau_pQ_m(\tau_{-p}f),$$

and when we combine this result with the fact that Lemma 5 implies that

$$\begin{aligned}\Big\|2^{-j}\sum_{p=1}^{2^j}\tau_pP_j(\tau_{-p}f)\Big\|_{L^2(\mathbb{R})}&\le2^{-j}\sum_{p=1}^{2^j}\|P_j(\tau_{-p}f)\|_{L^2(\mathbb{R})}\\&\le\sup_{p\in\mathbb{Z}}\|P_j(\tau_{-p}f)\|_{L^2(\mathbb{R})}\to0\quad\text{as } j\to\infty,\end{aligned}$$

we conclude that (34) holds.

We proceed to take the Fourier transform of both sides of (34) and first we consider the term $\tau_pQ_m(\tau_{-p}f)$. Using Plancherel's theorem and the fact that $\widehat{\psi}_{m,k}=2^{m/2}e^{-i2\pi2^mk\bullet}\widehat{\psi}(2^m\bullet)$, we get

$$\begin{aligned}\tau_pQ_m(\tau_{-p}f)&=\sum_{k\in\mathbb{Z}}2^{-m}\Big(\int_{\mathbb{R}}f(t+p)\overline{\psi(2^{-m}t-k)}\,dt\Big)\psi(2^{-m}\bullet-2^{-m}p-k)\\&=\sum_{k\in\mathbb{Z}}\Big(\int_{\mathbb{R}}\widehat{f}(\omega)\overline{\widehat{\psi}(2^m\omega)}e^{i2\pi(p+2^mk)\omega}\,d\omega\Big)\psi(2^{-m}\bullet-2^{-m}p-k).\end{aligned}$$

Thus

$$\widehat{P_0f}=\sum_{m=1}^{\infty}\sum_{p=1}^{2^m}\sum_{k\in\mathbb{Z}}\Big(\int_{\mathbb{R}}\widehat{f}(\omega)\overline{\widehat{\psi}(2^m\omega)}e^{i2\pi(p+2^mk)\omega}\,d\omega\Big)e^{-i2\pi(p+2^mk)\bullet}\widehat{\psi}(2^m\bullet).$$

If we combine this result with the fact that

$$\begin{aligned}&\int_{\mathbb{R}}\widehat{f}(\omega)\overline{\widehat{\psi}(2^m\omega)}e^{i2\pi(p+2^mk)\omega}\,d\omega\\&\qquad=\int_0^1\sum_{j\in\mathbb{Z}}\widehat{f}(\omega+j)\overline{\widehat{\psi}(2^m(\omega+j))}e^{i2\pi(p+2^mk)\omega}\,d\omega,\end{aligned}$$

then, since every integer can be written in a unique way as $p+2^mk$, where $k\in\mathbb{Z}$ and $p=1,\dots,2^m$, we obtain

$$\widehat{P_0f}=\sum_{m=1}^{\infty}\widehat{\psi}(2^m\bullet)\sum_{j\in\mathbb{Z}}\widehat{f}(\bullet+j)\overline{\widehat{\psi}(2^m(\bullet+j))}.$$

An immediate consequence is that if $f\in V_0$, i.e., $P_0f=f$, then

$$\sum_{k\in\mathbb{Z}}|\widehat{f}(\bullet+k)|^2=\sum_{m=1}^{\infty}\Big|\sum_{k\in\mathbb{Z}}\widehat{f}(\bullet+k)\overline{\widehat{\psi}(2^m(\bullet+k))}\Big|^2\quad\text{a.e.}\tag{35}$$

For each $p\ge1$ we have $2^{-p}\psi(2^{-p}\bullet)\in W_p\subset V_0$ and it follows from (35) and Cauchy's inequality that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}|\widehat{\psi}(2^p(\bullet+k))|^2&=\sum_{m=1}^{\infty}\Big|\sum_{k\in\mathbb{Z}}\widehat{\psi}(2^p(\bullet+k))\overline{\widehat{\psi}(2^m(\bullet+k))}\Big|^2\quad\text{a.e.}\\&\le\sum_{k\in\mathbb{Z}}|\widehat{\psi}(2^p(\bullet+k))|^2\sum_{m=1}^{\infty}\sum_{k\in\mathbb{Z}}|\widehat{\psi}(2^m(\bullet+k))|^2\quad\text{a.e., }p\ge1.\end{aligned}\tag{36}$$

In order to simplify the arguments involving sets of measure zero we change the values of $\widehat{\psi}$ to 0 on a set with measure 0 so that (8) and (36) hold everywhere. For each $p\ge1$ let

$$E_p=\Big\{\omega\in\mathbb{R}\,\Big|\,\sum_{k\in\mathbb{Z}}|\widehat{\psi}(2^p(\omega+k))|^2>0\Big\}.$$

It is clear that $\omega\in E_p$ if and only if $\omega+1\in E_p$. Moreover, an equivalent formulation of assumption (9) is that

$$\mathrm{m}\Big(\mathbb{R}\setminus\bigcup_{p=1}^{\infty}E_p\Big)=0.\tag{37}$$

If $\omega \in E_p$ then we conclude from (36) that

$$\sum_{m=1}^{\infty}\sum_{k\in\mathbb{Z}}|\widehat{\psi}(2^m(\omega+k))|^2 \ge 1$$

and therefore, by (37), this inequality holds for almost every ω. But because $\|\psi\|_{L^2(\mathbb{R})} = 1$ it follows that

$$\int_0^1 \sum_{m=1}^{\infty}\sum_{k\in\mathbb{Z}}|\widehat{\psi}(2^m(\omega+k))|^2\,d\omega = 1, \tag{38}$$

and therefore we see that (10) holds. The conclusion we draw from this is that the application of Cauchy's inequality in (36) (almost) always gives an equality and it follows that for each $p \ge 1$ and $m \ge 1$ there exists a function $c_{m,p}$ such that

$$\begin{aligned} c_{m,p}(\omega+1) &= c_{m,p}(\omega), \\ \widehat{\psi}(2^m\omega) &= c_{m,p}(\omega)\widehat{\psi}(2^p\omega), \end{aligned} \qquad \omega \in E_p,\ m,p \ge 1, \tag{39}$$

where we have, if necessary, subtracted a set of measure 0 from each E_p (and redefined $\widehat{\psi}$ so that the original definition of E_p still holds).

Now we proceed to construct the function φ, or rather its Fourier transform. We write

$$\widehat{\psi} = |\widehat{\psi}(\bullet)|e^{i2\pi\Psi(\bullet)-i\pi(\bullet+1)},$$

and we observe that the function Ψ is not determined by this equation at the points where $\widehat{\psi}$ vanishes. Similarly we write

$$\widehat{\varphi} = |\widehat{\varphi}(\bullet)|e^{i2\pi\Phi(\bullet)}, \qquad \widehat{\alpha} = |\widehat{\alpha}(\bullet)|e^{i2\pi A(\bullet)},$$

where

$$|\widehat{\varphi}(\bullet)| = \Big(\sum_{p=1}^{\infty}|\widehat{\psi}(2^p\bullet)|^2\Big)^{1/2}, \tag{40}$$

$$\widehat{\varphi}(2\bullet) = \widehat{\alpha}(\bullet)\widehat{\varphi}(\bullet) \quad \text{a.e.}, \tag{41}$$

and Φ is to be determined. Note that (41) puts no restrictions on $\widehat{\alpha}$ at the points where $\widehat{\varphi}$ vanishes and also that since α must be a square summable sequence, $\widehat{\alpha}$ should be periodic with period 1.

Now φ and α must be chosen in such a way that we get (6), or after taking Fourier transforms,

$$\widehat{\psi}(2\bullet) = e^{-i2\pi\bullet+i\pi}\overline{\widehat{\alpha}(\bullet+1/2)}\widehat{\varphi}(\bullet) \quad \text{a.e.} \tag{42}$$

The absolute values of $\widehat{\varphi}$ and $\widehat{\alpha}$ are fixed by (40) and (41), so first we must check that

$$|\widehat{\alpha}(\bullet+1)| = |\widehat{\alpha}(\bullet)| \quad \text{a.e.}, \tag{43}$$

and that

$$|\widehat{\psi}(2\bullet)| = |\widehat{\alpha}(\bullet+1/2)||\widehat{\varphi}(\bullet)| \quad \text{a.e.} \tag{44}$$

Let $\omega \in \bigcup_{p=1}^{\infty} E_p$ be such that $|\widehat{\varphi}(\omega)| > 0$ (so that (41) determines $|\widehat{\alpha}(\omega)|$) and let $q \ge 1$ be such that $\widehat{\psi}(2^q\omega) > 0$. Then $\omega \in E_q$ and by (39) for each $k \in \mathbb{Z}$ we have

$$|\widehat{\varphi}(\omega+k)|^2 = |\widehat{\psi}(2^q(\omega+k))|^2\sum_{p=1}^{\infty}|c_{p,q}(\omega)|^2,$$

$$|\widehat{\varphi}(2(\omega+k))|^2 = |\widehat{\psi}(2^q(\omega+k))|^2\sum_{p=2}^{\infty}|c_{p,q}(\omega)|^2.$$

This shows that if $|\widehat{\varphi}(\omega+k)| > 0$, then $|\widehat{\alpha}(\omega+k)| = |\widehat{\alpha}(\omega)|$ by (41) and for the points where $|\widehat{\varphi}(\omega+k)| = 0$ we can choose $|\widehat{\alpha}(\omega+k)| = |\widehat{\alpha}(\omega)|$. Thus we have established (43).

Let ω be such that $\omega + 1/2$ and 2ω are in $\bigcup_{p=1}^{\infty} E_p$ and that (8) holds for the argument 2ω. Thus ω is arbitrary except for a set of measure 0. There must then exist an integer k so that $\sum_{p=1}^{\infty}|\widehat{\psi}(2^p(\omega+1/2+k))|^2 = |\widehat{\varphi}(\omega+1/2+k)|^2 > 0$. This implies by (43) that

$$|\widehat{\alpha}(\omega+1/2)|^2 = \frac{\sum_{p=1}^{\infty}|\widehat{\psi}(2^p(2\omega+2k+1))|^2}{\sum_{p=0}^{\infty}|\widehat{\psi}(2^p(2\omega+2k+1))|^2},$$

and in order to establish (44) we must show that

$$|\widehat{\psi}(2\omega)|^2\sum_{p=0}^{\infty}|\widehat{\psi}(2^p(2\omega+2k+1))|^2 = \sum_{p=1}^{\infty}|\widehat{\psi}(2^p(2\omega+2k+1))|^2\sum_{p=0}^{\infty}|\widehat{\psi}(2^p2\omega)|^2. \tag{45}$$

From (8) and (39) we obtain

$$\widehat{\psi}(2\omega)\overline{\widehat{\psi}(2\omega+2k+1)} = -\widehat{\psi}(2^q2\omega)\overline{\widehat{\psi}(2^q(2\omega+2k+1))}\sum_{p=1}^{\infty}|c_{p,q}(2\omega)|^2,$$

where $q \ge 1$ is such that $2\omega \in E_q$, and then again by (39), we get

$$|\widehat{\psi}(2\omega)|^2|\widehat{\psi}(2\omega+2k+1)|^2 = \sum_{p=1}^{\infty}|\widehat{\psi}(2^p(2\omega+2k+1))|^2\sum_{p=1}^{\infty}|\widehat{\psi}(2^p2\omega)|^2.$$

Now we obtain (45) by adding terms to both sides, and (44) is established.

Since (10) implies that $\sum_{k\in\mathbb{Z}}|\widehat{\varphi}(\bullet+k)|^2=1$ a.e., from a standard argument that uses (43) we get

$$
\begin{aligned}
(46)\qquad 1 &= \sum_{k\in\mathbb{Z}}|\widehat{\varphi}(2\bullet+k)|^2\\
&= \sum_{m\in\mathbb{Z}}|\widehat{\alpha}(\bullet+2m/2)|^2|\widehat{\varphi}(\bullet+2m/2)|^2\\
&\quad+\sum_{m\in\mathbb{Z}}|\widehat{\alpha}(\bullet+(2m+1)/2)|^2|\widehat{\varphi}(\bullet+(2m+1)/2)|^2\\
&= |\widehat{\alpha}(\bullet)|^2\sum_{m\in\mathbb{Z}}|\widehat{\varphi}(\bullet+m)|^2+|\widehat{\alpha}(\bullet+1/2)|^2\sum_{m\in\mathbb{Z}}|\widehat{\varphi}(\bullet+1/2+m)|^2\\
&= |\widehat{\alpha}(\bullet)|^2+|\widehat{\alpha}(\bullet+1/2)|^2 \quad \text{a.e.}
\end{aligned}
$$

We will get (42) from (44) once we have shown that

$$(47)\qquad \Psi(2\bullet)=-A(\bullet+1/2)+\Phi(\bullet) \bmod 1 \quad \text{a.e.}$$

The restrictions we have on A and Φ are that

$$(48)\qquad \begin{aligned} A(\bullet)&=A(\bullet+1) \bmod 1 \quad \text{a.e.},\\ A(\bullet)&=\Phi(2\bullet)-\Phi(\bullet) \bmod 1 \quad \text{a.e.}\end{aligned}$$

For each $p\ge 1$ let

$$F_p=\{\omega \mid \widehat{\psi}(2^{p+1}\omega)\widehat{\psi}(2\omega)\neq 0\},$$

and define the function B, first in $\bigcup_{p=1}^{\infty}F_p$, by

$$(49)\qquad B(\omega)=\Psi(2^{p+1}\omega)-\Psi(2\omega) \bmod 1,\qquad \omega\in F_p\setminus\bigcup_{q=1}^{p-1}F_q.$$

We claim that

$$(50)\qquad B(\omega)=B(\omega+k/2)\quad \text{if } \omega,\omega+k/2\in\bigcup_{p=1}^{\infty}F_p,\ k\in\mathbb{Z},$$

and once this result is established we can extend the definition of B to $\mathbb{R}$ by

$$B(\omega)=\begin{cases} B(\omega+k/2) & \text{if } \omega+k/2\in\bigcup_{p=1}^{\infty}F_p,\ k\in\mathbb{Z},\\ 0 & \text{otherwise,}\end{cases}$$

so that

$$(51)\qquad B(\bullet+1/2)=B(\bullet).$$

In order to establish (50) we assume that $\omega\in F_p$ and $\omega+k/2\in F_q$ and first we consider the case where k is even. We conclude that $\omega\in E_1$ and therefore by (39) that

$$\widehat{\psi}(2^{r+1}(\omega+k/2))\overline{\widehat{\psi}(2^{r+1}\omega)}=|c_{r+1,1}(\omega)|^2\widehat{\psi}(2(\omega+k/2))\overline{\widehat{\psi}(2\omega)}.$$

Now we know that $c_{r+1,1}(\omega)\neq 0$ for both $r=p$ and $r=q$, and therefore we can deduce that we can take $p=q$ and it is easy to see that (50) holds in this case. Next we consider the case where k is odd. We note that $2\omega\in E_1$ and it follows from (8) and (39) that

$$\widehat{\psi}(2\omega)\overline{\widehat{\psi}(2\omega+k)}+\Big(\sum_{r=1}^{\infty}|c_{r,1}(2\omega)|^2\Big)\widehat{\psi}(2^{r+1}\omega)\overline{\widehat{\psi}(2^r(2\omega+k))}=0,$$

and since $\sum_{r=1}^{\infty}|c_{r,1}(2\omega)|^2>0$ we again see that we can take $r=p=q$ and that (50) holds.

By (7), for almost all points ω there is an integer m so that $\widehat{\psi}(2^m\omega)\neq 0$, and starting from this value we can define $\Psi(\omega)$ at the points where $\widehat{\psi}(\omega)=0$ by

$$(52)\qquad \Psi(4\omega)=\Psi(2\omega)+B(\omega) \bmod 1.$$

We only have to prove the consistency of this definition, that is, we must show that if $\widehat{\psi}(2^M\omega)\neq 0$ and $\widehat{\psi}(2^m\omega)\neq 0$ for some ω and $m<M$, then

$$\Psi(2^M\omega)-\Psi(2^m\omega)=\sum_{j=m-1}^{M-2}B(2^j\omega) \bmod 1.$$

If $M=m+1$, this is the definition (49) and otherwise we may assume that $\widehat{\psi}(2^j)=0$ for $j=m+1,\ldots,M-1$. Thus we have to prove that $B(2^j\omega)=0$ when $j=m,\ldots,M-2$. Let j be one of these integers; without loss of generality, assume that $j=0$. Hence $\widehat{\psi}(2\omega)=0$ but $\widehat{\varphi}(\omega)\neq 0$ so that, by (44), we must have $\widehat{\alpha}(\omega+1/2)=0$ (for almost every ω) and hence also $|\widehat{\alpha}(\omega)|=1$ by (46). The fact that $\widehat{\alpha}(\omega+1/2)=0$ implies that if for some k we have $\widehat{\psi}(2\omega+2k+1)\neq 0$ then $\widehat{\psi}(2^p(2\omega+2k+1))=0$ for every $p\ge 1$. On the other hand, the fact that $|\widehat{\alpha}(\omega)|=1$ implies that for all k we have $\widehat{\psi}(2\omega+2k)=0$. Combining these two results we see that $\omega\notin\bigcup_{p=1}^{\infty}F_p+(k/2)\mathbb{Z}$, and therefore $B(\omega)=0$, and this was what we had to prove.

Now B can be written as a Fourier series

$$B=\sum_{k\in\mathbb{Z}}e^{i2\pi 2k\bullet}\widehat{B}(2k),$$

where we have used the fact that (51) implies that $\widehat{B}(k)=0$ for odd k. Next we shall solve for A the equation

$$(53)\qquad A(\bullet)+A(\bullet+1/2)-A(2\bullet+1/2)=B(\bullet).$$

It is easy to see that we get a solution by taking

$$A=\sum_{k\in\mathbb{Z}}e^{i2\pi k\bullet}\widehat{A}(k),$$

where

$$\widehat{A}(2k+1)=0,\qquad \widehat{A}(2k)=\tfrac{1}{2}((-1)^k\widehat{A}(k)+\widehat{B}(2k)),\quad k\in\mathbb{Z}.$$

We conclude that $\widehat{A}(0)=\widehat{B}(0)$ and that when p is odd and $n\geq 1$ we have

$$\widehat{A}(2^n p)=\sum_{j=1}^{n}2^{-n+j-1}\widehat{B}(2^j p).$$

Thus it follows from Cauchy's inequality that

$$|\widehat{A}(2^n p)|^2\leq\sum_{j=1}^{n}2^{-n+j-1}|\widehat{B}(2^j p)|^2,$$

and hence

$$\sum_{n=1}^{\infty}\sum_{p\,\mathrm{odd}}|\widehat{A}(2^n p)|^2\leq\sum_{n=1}^{\infty}\sum_{p\,\mathrm{odd}}|\widehat{B}(2^n p)|^2.$$

This implies that the function A is well defined, at least almost everywhere. Now the function A is periodic with period 1 (in fact, 1/2), and this implies by (43) that

$$\widehat{\alpha}(\bullet+1)=\widehat{\alpha}(\bullet)\quad\text{a.e.}\tag{54}$$

We define Φ by (47) and it follows from (52) and (53) that we get the second equation in (48). This result combined with the definitions of $|\widehat{\varphi}|$ and $|\widehat{\alpha}|$ implies (41).

Now the rest of the proof is straightforward. The orthonormality of the functions $\{\varphi(\bullet-k)\}_{k\in\mathbb{Z}}$ is a consequence of (10) and (40) (see [8, p. 28]). Since $\widehat{\alpha}\in L^2([0,1];\mathbb{C})$ satisfies (54) we see that α is a square integrable sequence that satisfies (5) by (41), and (2) follows. Since we obtain (6) by (44) and (47), we conclude that ψ is obtained from a multiresolution; note that (3) is a consequence of the assumption that $\{2^{-m/2}\psi(2^{-m}\bullet-k)\}_{k,m\in\mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{R};\mathbb{C})$. This completes the proof of Theorem 2. ■

5. Proof of Corollary 4. Let

$$\sigma=\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}|\widehat{\psi}(2^p(\bullet+k))|^2.$$

From (36) we conclude that for almost every $\omega\in\mathbb{R}$ we have either

$$\sigma(\omega)=0\quad\text{or}\quad\sigma(\omega)\geq 1.\tag{55}$$

If we can prove that the series defining σ converges uniformly on every interval $[\delta,1-\delta]$, where $\delta\in(0,1/2)$, then we conclude from (11) that σ is continuous on this interval. By letting $\delta\downarrow 0$ we conclude from (38) and (55) that σ cannot vanish on $(0,1)$ and this gives (9) since σ is a periodic function.

In order to establish the uniform convergence of the series we let

$$\eta=\sup_{|\omega|\geq\bullet}|\widehat{\psi}(\omega)|^2,$$

and by (12) we know that $\eta\in L^1(\mathbb{R}^+;\mathbb{R})$. We observe that

$$|\widehat{\psi}(2^p(\omega+k))|^2\leq\eta(2^p(\delta+\min\{|k|,|k+1|\})),\qquad \omega\in[\delta,1-\delta],\ k\in\mathbb{Z},\ p\geq 1,\tag{56}$$

where $\delta\in(0,1/2)$. The fact that η is nonincreasing and integrable implies that

$$\sum_{p=1}^{\infty}\sum_{k=0}^{\infty}\eta(2^p(\delta+k))\leq\sum_{p=1}^{\infty}\Bigg(2^{-p}\frac{1}{\delta}\int_0^{2^p\delta}\eta(t)\,dt+\sum_{k=1}^{\infty}2^{-p}\int_{2^p(\delta+k-1)}^{2^p(\delta+k)}\eta(t)\,dt\Bigg)\leq\frac{1}{\delta}\int_0^{\infty}\eta(t)\,dt<\infty.$$

Combined with (56), this gives the desired uniform convergence of the series defining σ, and the proof is complete. ■

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Pointwise ergodic theorems in Lorentz spaces $L(p,q)$ for null preserving transformations

by

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Abstract. Let $(X,\mathcal{F},\mu)$ be a finite measure space and τ a null preserving transformation on $(X,\mathcal{F},\mu)$. Functions in Lorentz spaces $L(p,q)$ associated with the measure μ are considered for pointwise ergodic theorems. Necessary and sufficient conditions are given in order that for any f in $L(p,q)$ the ergodic average $n^{-1}\sum_{i=0}^{n-1} f\circ\tau^i(x)$ converges almost everywhere to a function f^* in $L(p_1,q_1)$, where (p,q) and (p_1,q_1) are assumed to be in the set $\{(r,s): r=s=1,\ \text{or } 1<r<\infty \text{ and } 1\le s\le\infty,\ \text{or } r=s=\infty\}$. Results due to C. Ryll-Nardzewski, S. Gładysz, and I. Assani and J. Woś are generalized and unified.

1. Introduction and results. Let $(X,\mathcal{F},\mu)$ be a finite measure space and τ a *null preserving transformation* on $(X,\mathcal{F},\mu)$ (i.e., $\tau^{-1}\mathcal{F}\subset\mathcal{F}$ and $\mu(\tau^{-1}A)=0$ whenever $\mu(A)=0$). We define an operator T by putting

$$Tf = f\circ\tau.$$

T is said to satisfy *the pointwise ergodic theorem from* $L(p,q)$ *to* $L(p_1,q_1)$ if for any f in $L(p,q)$ the ergodic average

$$M_n(T)f = \frac{1}{n}\sum_{i=0}^{n-1} T^i f$$

converges a.e. to a function f^* in $L(p_1,q_1)$, where $L(p,q)$ and $L(p_1,q_1)$ are the Lorentz spaces associated with the measure μ. Throughout this paper we shall assume that (p,q) and (p_1,q_1) are in the set

$$\{(r,s): r=s=1,\ \text{or } 1<r<\infty \text{ and } 1\le s\le\infty,\ \text{or } r=s=\infty\}.$$

The basic properties of Lorentz spaces $L(p,q)$ are explained in Hunt [5]. In particular, the following are used in the argument below.

(I) $f\in L(p,q)$ if and only if $\|f\|_{pq}<\infty$, where

$$\|f\|_{pq} = \begin{cases} \left(q\int_0^\infty (\mu(\{|f|>t\}))^{q/p} t^{q-1}\,dt\right)^{1/q} & (q\ne\infty),\\ \sup_{t>0} t(\mu(\{|f|>t\}))^{1/p} & (q=\infty).\end{cases}$$

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