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# The Cauchy problem and self-similar solutions for a nonlinear parabolic equation

by

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**Abstract.** The existence of solutions to the Cauchy problem for a nonlinear parabolic equation describing the gravitational interaction of particles is studied under minimal regularity assumptions on the initial conditions. Self-similar solutions are constructed for some homogeneous initial data.

**1. Introduction.** Our aim in this paper is to construct local and global-in-time solutions to the Cauchy problem for the parabolic equation

$$u_t = \Delta u + \nabla \cdot (u\nabla\varphi), \tag{1}$$

in $\mathbb{R}^n \times \mathbb{R}^+$, where the coefficient $\nabla\varphi$ is determined from $u$ via the potential

$$\varphi = E_n * u, \tag{2}$$

$E_n$ being the fundamental solution of the Laplacian in $\mathbb{R}^n$. Since $\Delta\varphi = u$, the equation (1) can be rewritten as a parabolic equation with a nonlocal coefficient $\nabla\varphi$:

$$u_t = \Delta u + u^2 + \nabla u \cdot \nabla\varphi. \tag{1'}$$

The physical interpretations of the equation (1) with an initial (nonnegative) condition

$$u(x, 0) = u_0(x) \tag{3}$$

come from nonequilibrium statistical mechanics. In particular, (1)–(3) is an evolution version of the Chandrasekhar equation for the gravitational equilibrium of polytropic stars. Here $u$ is the density of particles in $\mathbb{R}^n$ interacting with themselves through the gravitational potential $\varphi$. Another motivation for studying the above system is presented in the introduction

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to [6], where this equation is considered in a bounded domain of $\mathbb{R}^n$ and the problem (1)–(3) is supplemented with appropriate (nonlinear, no-flux) boundary conditions.

The local-in-time weak solutions of the initial-boundary value problem have been constructed in [6, Th. 2(i)–(iv)] for the densities $u_0 \geq 0$ in $L^p(\Omega)$, $p > n/2$. They enjoy some regularity properties, the crucial one being $u \in L^\infty_{\mathrm{loc}}((0,T); L^\infty(\Omega))$.

On the other hand, for the semilinear parabolic equation

$$u_t = \Delta u + u^2 \tag{4}$$

in an open subset $\Omega$ of $\mathbb{R}^n$ the authors of [1–2] proved that local nonnegative solutions exist if and only if $u_0$ satisfies a certain regularity assumption as well as a size condition. The above-mentioned regularity assumption is satisfied when $u_0 \in L^{n/2}(\Omega)$, and necessarily $u_0$ belongs to the Morrey space $M^{n/2}(\Omega)$. In fact, Proposition 3.2 in [2] concerning the equation $u_t = \Delta u + u^\gamma$ is formulated in terms of capacities associated with the Sobolev space $W^{2/\gamma,\gamma'}$, so for $\gamma = \gamma' = 2$ the condition (74) for $u_0$ in [2] coincides with that in the definition of the $M^{n/2}$ norm which will be given in Section 2.

The equation (1′) is slightly more complicated than (4); it contains a quasilinear term $\nabla u \cdot \nabla \varphi$. Nevertheless, nonlinear terms in (1′) are of degree 2 and (formally) of order 0 like in (4). However, the methods introduced in [1–2] cannot be applied to (1′) because the semilinear structure of the nonlinearity $u^2$ was heavily used there.

This suggested the question: What are minimal regularity assumptions on $u_0$ guaranteeing the local-in-time existence of solutions to the problem (1)–(3) (and their instantaneous regularization: $u(t) \in L^\infty(\Omega)$ for $t > 0$)?

It was conjectured in [6] that the optimal hypothesis on the initial condition is that $u_0$ belongs to a subspace of $M^{n/2}(\Omega)$ containing $L^{n/2}(\Omega)$, and $u_0$ satisfies a size condition in the Morrey norm. This conjecture has been strongly supported by an analysis of the existence and regularity of stationary solutions to (1)–(2) in [6, Th. 1], and by results on the nonglobal existence (i.e. blow-up in a finite time) of solutions caused by high concentration phenomena as announced in [6, Th. 2(v)] and described in more detail in [4].

Here the system (1)–(3) is studied in the whole space $\mathbb{R}^n$, but the results in Section 2 hold true also for the problem (1)–(3) with homogeneous boundary conditions imposed for $u$ on $\partial\Omega$. Sufficient conditions for the existence (both local and global in time) are regularity assumptions like belonging of $u_0$ to a Morrey space, and suitable size conditions. The methods used are



some modifications and adaptations to the Morrey space framework of ideas in Kato's papers [16–17] for the Navier–Stokes equations, and in Weissler's abstract parabolic semilinear equation approach in [25].

For $n \geq 3$ the equations (1)–(2) have a singular stationary solution

$$\widetilde{u}(x) = 2(n-2)|x|^{-2} \tag{5}$$

(a structure result for some other singular (homogeneous) stationary solutions to a renormalized version of (1)–(2) is given in [3]). This shows that the regularizing effect of the Laplacian in (1) alluded to in [6, Th. 2(ii)] is too weak in order to smooth out the singularity of (5) for $t > 0$.

In Section 3 there is shown the existence of solutions defined globally in time and satisfying the self-similarity relation

$$u(x,t) = \lambda^2 u(\lambda x, \lambda^2 t) \tag{6}$$

for all $\lambda > 0$, which emanate from small initial data homogeneous of degree $-2$, hence having singularities like (5): $\varepsilon|x|^{-2}$, but of smaller size: $\varepsilon \ll 1$.

The methods and techniques applied in Section 3 are motivated by those recently used by Y. Meyer and his collaborators ([8–11]) who solved a long standing problem of J. Leray concerning the existence of self-similar solutions to the Navier–Stokes system. The importance of such solutions, pointed out e.g. by C. Foiaş and R. Temam, is connected with the problem of the determination of long time asymptotics of arbitrary solutions to the Navier–Stokes system, hence the description of turbulence phenomena.

Likewise, the self-similar solutions to (1)–(2) describe the main term of an asymptotic expansion for global-in-time solutions, since the limit $\lim_{\lambda\to\infty} \lambda^2 u(\lambda x, \lambda^2 t)$ (of course, if it exists) is necessarily a self-similar solution. They are sought for in some "exotic" (from the point of view of classical P.D.E. theory) spaces like homogeneous Besov spaces $\dot{B}^s_{p,q}$, symbol spaces $E^{\varrho,m}$ and the (*ad hoc* introduced for $n = 2, 3$) spaces $\mathcal{X}_p$.

Recently there appeared several papers devoted to the analysis of evolution equations of parabolic type (mainly Navier–Stokes) where the functional framework is based on Besov and Morrey spaces. These spaces seem to be well adapted to such purposes thanks to possibilities of performing a frequency analysis and using local geometric norms. The list [8–11], [12], [14], [15], [16–17], [18], [19], [22] is by no means exhaustive, but this gives a flavour of recent advances in P.D.E. theory connected with harmonic analysis and function spaces approach (which has been so successful for elliptic operators).

**2. The Cauchy problem.** As was explained in the introduction, we are looking for solutions of (1)–(2) with nonsmooth initial data (3). The case of the whole space $\Omega = \mathbb{R}^n$, $n \geq 2$, is chosen here in order to simplify the notation. Actually, the same methods apply to the problem (1)–(3) with some homogeneous boundary conditions imposed on $u$ and/or $\varphi$ on $\partial\Omega$, so that a semigroup of linear operators associated with the heat equation can be defined. In fact, the existence results are interesting even if the local regularity with respect to the space variable $x$ is studied, since they outline a threshold for the regularizing effect for (1) to take place (compare again (5)).

A generalization of the Lebesgue and Sobolev spaces framework, usual for parabolic initial value problems, can be given within the scale of Morrey(-Campanato) spaces. Such a generalization is motivated by results in [6, Th. 1, 2], [4, the end of the proof of Th. 1], where regular stationary solutions were characterized as those with small Morrey space $M^{n/2}(\Omega)$ norm, and large values of $M^{n/2}$ norm were obstructions for the continuation of solutions in time.

As a standard practice we will work with the integral formulation of (1)–(2),

$$u(t) = e^{t\Delta}u_0 + \int_0^t (\nabla e^{(t-s)\Delta})\cdot(u\nabla\varphi)(s)\,ds, \tag{7}$$

obtained from the variation of parameter formula, using an obvious commutation property $\nabla e^{t\Delta} = e^{t\Delta}\nabla$. Here $\varphi = E_n * u$ and $e^{t\Delta}$ is the heat semigroup on $\mathbb{R}^n$ defined (for tempered distributions) by the convolution with the Gauss–Weierstrass kernel

$$p_t(z) = (4\pi t)^{-n/2}\exp(-|z|^2/(4t)) = \mathcal{F}^{-1}(\exp(-t|\xi|^2)). \tag{8}$$

The solutions of (7) are called *mild* solutions. It can be checked that mild solutions constructed in Proposition 1 and Theorems 1, 2 below are *weak* solutions in the sense of [6]. In particular, they enjoy the parabolic regularizing effect, becoming locally bounded in $x$, $t$ for $t > 0$.

Now we recall the definition of the Morrey spaces and some properties of the heat semigroup which are collected in [22], [14], [17], [21], [23–24].

We denote by $M^p = M^p(\mathbb{R}^n)$ the *Morrey space* of locally integrable functions such that the norm

$$\|f; M^p\| \equiv \sup_{x\in\mathbb{R}^n,\, 0<R\le 1} R^{n(1/p-1)} \int_{B_R(x)} |f|$$

is finite. $\dot{M}^p = \dot{M}^p(\mathbb{R}^n)$ is the *homogeneous Morrey space*, where in the definition the supremum is taken over all $0 < R < \infty$. Several authors also considered *Morrey spaces of measures* $\mu$ with an obvious generalization of the above definition when $f$ is replaced by $d\mu$.



More general Morrey spaces include

$$M^p_q = M^p_q(\mathbb{R}^n) = \Big\{ f \in L^q_{\mathrm{loc}}(\mathbb{R}^n) : \|f; M^p_q\|^q \equiv \sup_{x\in\mathbb{R}^n,\, 0<R\le 1} R^{n(q/p-1)} \int_{B_R(x)} |f|^q < \infty \Big\}, \tag{9}$$

where $1 \le q \le p \le \infty$, and its homogeneous version

$$\dot{M}^p_q = \dot{M}^p_q(\mathbb{R}^n) = \Big\{ f \in L^q_{\mathrm{loc}}(\mathbb{R}^n) : \|f; \dot{M}^p_q\|^q \equiv \sup_{x\in\mathbb{R}^n,\, 0<R<\infty} R^{n(q/p-1)} \int_{B_R(x)} |f|^q < \infty \Big\}. \tag{10}$$

The $\dot{M}^p_q(\mathbb{R}^n)$ norm has the same type of scaling as the $L^p(\mathbb{R}^n)$ norm, namely $\|f(\lambda\cdot); \dot{M}^p_q\| = \lambda^{-n/p}\|f; \dot{M}^p_q\|$, which explains the adjective "homogeneous". (Caution: the notation for the homogeneous Morrey spaces used in [22] is $\mathcal{M}^p_q$ instead of $\dot{M}^p_q$. Another convention for the parameters $p$, $q$ can be found elsewhere, e.g. in [17].)

It is well known that the Morrey spaces contain not only the usual Lebesgue spaces $L^p$ but also the Marcinkiewicz weak $L^p$ spaces. We also note the inclusions

$$L^p_{\mathrm{unif}} = M^p_p \subset M^p_q \subset M^p \qquad (L^p_{\mathrm{unif}} \subset L^p_{\mathrm{loc}}), \tag{11}$$

and an equivalent characterization of $M^p$ in terms of the heat kernel:

$$f \in M^p \Leftrightarrow e^{t\Delta}|f| \le Ct^{-n/(2p)} \tag{12}$$

for $0 < t \le 1$ (see [22, Ch. 3]).

We will use in the sequel the estimates for the heat semigroup $e^{t\Delta}$ and for the operator $\nabla e^{t\Delta}$ which can be found in [22, Th. 3.8, (3.71)–(3.75), (4.18)] :

$$\|e^{t\Delta}f; M^{p_2}_{q_2}\| \le Ct^{-n(1/p_1-1/p_2)/2}\|f; M^{p_1}_1\|, \tag{13}$$

valid for $1 < p_1 < p_2 < \infty$, $q_2 < p_2/p_1$, and

$$\|\nabla e^{t\Delta}f; M^{p_2}_{q_2}\| \le Ct^{-(1+n/p_1-n/p_2)/2}\|f; M^{p_1}_{q_1}\| \tag{14}$$

when $1 < p_1 < p_2 < \infty$, $q_1 > 1$ and $q_2/q_1 = p_2/p_1$ provided $p_1 \le n$, $q_2/q_1 < p_2/p_1$ otherwise.

An analogue of the Sobolev inequality for Morrey spaces reads (cf. [22, Th. 3.8(ii)]): if $\varphi = E_n * f$, $1/p_1 - 1/p_2 = 1/n$, then

$$\|\nabla\varphi; M^{p_2}_{q_2}\| \le C\|f; M^{p_1}_{q_1}\| \tag{15}$$

with $q_2/q_1 = p_2/p_1$ if also $p_1 \le n$, and $q_2/q_1 < p_2/p_1$ otherwise.

The limit case $p_1 = p_2$ in (13) also holds true: $e^{t\Delta} : M^p_q \to M^p_q$ is a bounded operator. However, $e^{t\Delta}$ is not a strongly continuous semigroup on $M_q^p$; cf. [17, Ex. 3.4]. This prevents us from applying directly the scheme of the existence proof in [25], which uses spaces of vector-valued functions continuous with respect to time. A remedy for this is either to weaken the usual definition of mild solution to the equation (7), or to consider a subspace of $M_q^p$ on which $e^{t\Delta}$ forms a strongly continuous semigroup.

The first way (similar to that in [22], [8–9]) to circumvent this technical problem is to replace the space $C([0,T];\mathcal{B})$ of (norm) continuous functions with values in a Banach space $\mathcal{B}$ of tempered distributions on $\mathbb{R}^n$ by the space $\mathcal{C}([0,T];\mathcal{B})$ which is defined as the subspace of $C([0,T];\mathcal{S}'(\mathbb{R}^n))$ such that $u(t)\in\mathcal{B}$ for each $t\in[0,T]$ and $\{u(t):t\in[0,T]\}$ is bounded in $\mathcal{B}$. Hence $\mathcal{C}([0,T];\mathcal{B})$ is the space of weakly continuous (in the sense of distributions) functions which are bounded in the norm of $\mathcal{B}$. Note that if $\mathcal{B}=X^*$ is the dual of a Banach space $X$, then $\mathcal{C}([0,T];\mathcal{B})$ coincides with the space of $\mathcal{B}$-valued functions which are continuous in the weak* topology $\sigma(\mathcal{B},X)$ of $\mathcal{B}$; cf. [8, I, Déf. 2.2].

The second manner is to consider (following [17, (1.1b), Ch. 3]) the space

$$\ddot{M}_q^p=\{f\in M_q^p:\|\tau_y f-f;M_q^p\|\to 0 \text{ as } |y|\to 0\},\tag{16}$$

$y\in\mathbb{R}^n$, $\tau_y f(x)=f(x-y)$. $\ddot{M}_q^p$ is the maximal closed subspace of $M_q^p$ on which the family of translations $\tau_y$ forms a strongly continuous group and, at the same time, the maximal closed subspace on which $e^{t\Delta}$ is a strongly continuous semigroup. It is of interest to note that $L^p\subset\ddot{M}_q^p\subset\mathring{M}_q^p\subset M_q^p$, where

$$\mathring{M}_q^p=\Big\{f\in M_q^p:\limsup_{R\to 0}R^{n(q/p-1)}\int_{B_R(x)}|f|^q=0\Big\}\tag{17}$$

(see [17, Ch. 3, Cor. 3.3] and [22, (4.14)]).

We begin with a simple result on the solvability of the Cauchy problem in $M_q^p$ with $p>n/2$ whose proof will show the elementary estimates (13)–(15) in action.

PROPOSITION 1. *Given $u_0\in M_q^p$ with $n\ge 2$, $n/2<p\le n$, $2-p/n\le q\le p$, there exists $T=T(u_0)>0$ and a unique solution of the integral equation (7) with $\varphi$ given by (2) in the space $\mathcal{X}=\mathcal{C}([0,T];M_q^p)$.*

Proof. Here and in the sequel the constants independent of functions in $M_q^p$ spaces will be denoted by the same letter $C$, even if they may vary from line to line, and may depend inessentially on $T$ (i.e. $\sup_{0\le t\le T}C<\infty$ for each $T>0$).

Define for $u\in\mathcal{X}$ and $0\le t\le T$ the nonlinear operator

$$N(u)(t)=e^{t\Delta}u_0+\int_0^t(\nabla e^{(t-s)\Delta})\cdot(u\nabla\varphi)(s)\,ds.$$



It is easy to estimate $N(u)$ in $\mathcal{X}$:

$$\begin{aligned}\|N(u)\|_{\mathcal{X}}&=\sup_{0\le t\le T}\|N(u)(t);M_q^p\|\\&\le C\|u_0;M_q^p\|\\&\quad+C\sup_{0\le t\le T}\int_0^t(t-s)^{-(1+n/p_3-n/p)/2}\|(u\nabla\varphi)(s);M_{q_3}^{p_3}\|\,ds\\&\le C\|u_0;M_q^p\|+C\sup_{0\le t\le T}\int_0^t(t-s)^{-n/(2p)}\|u(s);M_q^p\|^2\,ds\\&\le C\|u_0;M_q^p\|+CT^{1-n/(2p)}\|u\|_{\mathcal{X}}^2,\end{aligned}$$

where $1/p_2=1/p_3-1/p=1/p-1/n$, $q_2/q=p_2/p$, $1/q+1/q_2=1/q_3$; (13)–(14) have been used with $p_1=p$, $q_1=q$, and $\|u\nabla\varphi;M_{q_3}^{p_3}\|\le C\|u;M_q^p\|^2$ followed from the Hölder inequality.

Next, for $u,v\in\mathcal{X}$, $\varphi=E_n*u$, $\psi=E_n*v$, the local Lipschitz property of $N$ is shown:

$$\begin{aligned}\|N(u)-N(v)\|_{\mathcal{X}}&\le C\sup_{0\le t\le T}\int_0^t(t-s)^{-n/(2p)}\\&\qquad\times(\|(u-v)\nabla\varphi;M_{q_3}^{p_3}\|+\|v(\nabla\varphi-\nabla\psi);M_{q_3}^{p_3}\|)\,ds\\&\le C\sup_{0\le t\le T}\int_0^t(t-s)^{-n/(2p)}\\&\qquad\times(\|u(s);M_q^p\|+\|v(s);M_q^p\|)\|(u-v)(s);M_q^p\|\,ds\\&\le CT^{1-n/(2p)}(\|u\|_{\mathcal{X}}+\|v\|_{\mathcal{X}})\|u-v\|_{\mathcal{X}}.\end{aligned}$$

It is clear that for $r$ sufficiently large (e.g. $r\ge 2C\|u_0;M_q^p\|$) and $T>0$ small enough (so that $2CrT^{1-n/(2p)}<1$), the operator $N$ is a contraction in the ball $B_r(u_0)$ in $\{u\in\mathcal{X}:u(0)=u_0\}$ of radius $r$ centered at the constant function $u(t)\equiv u_0$. The fixed point $u=N(u)$ of the operator $N$ in $B_r(u_0)$ solves (7), and this is the (locally in time) unique solution of (7) in $\mathcal{X}$.

Note that for $p=q>n/2$ we recover an analogue of the local existence result from [6, Th. 2] proved for the problem (1)–(3) with the no-flux boundary condition imposed on $u$, since $u(t)\in L^\infty$ for $t>0$ by [6, Th. 2(ii)].

Now we state the main result (Theorem 1) in this section, namely the existence of local and global-in-time solutions to (7) with the initial data $u_0$ in the Morrey space $M_q^{n/2}$, $n\ge 3$, satisfying some additional size and/or regularity assumptions. The idea of looking for solutions in a space of vector-valued functions endowed with two norms, the first being a rough one, the second controlling the balance of smoothing property of the semigroup $e^{t\Delta}$ against the formation of singularities by the nonlinear term, goes back to H. Fujita and T. Kato. They applied it in the early sixties to the Navier–Stokes system, but an elegant abstract form of this idea was given by F. Weissler in [25]. Variations of this method can be found in [8–9], [16–17], [22], where sophisticated (more involved than simple contraction arguments) iteration schemes are also used. The choice of function spaces is, of course, crucial in order to apply [25, Th. 2] to our equation (7). The same technique will be used to obtain the solvability of (7) for $n=2$ with $u_0$ being a measure with finite total mass in Theorem 2. The results below are in a sense parallel to those in [6, Th. 2(ii), (iii)] for the initial-boundary value problem for the system (1)–(2).

THEOREM 1. *Let $u_0 \in M_q^{n/2}$, $n \geq 3$, $3/2 \leq q \leq n/2$. There exists $\varepsilon > 0$ such that if $l(u_0) \equiv \limsup_{t\to 0} t^{1/4}\|e^{t\Delta}u_0; M_{4q/3}^{2n/3}\| < \varepsilon$, then there exist $T>0$ and a local-in-time solution $u$ of (7), $0 \leq t \leq T$, which is unique in the space*

$$\mathcal{X} = \mathcal{C}([0,T]; M_q^{n/2}) \cap \{u : [0,T] \to M_q^{n/2} : \sup_{0<t\leq T} t^{1/4}\|u(t); M_{4q/3}^{2n/3}\| < \infty\}.$$

*Moreover, if $\sup_{t>0} t^{1/4}\|e^{t\Delta}u_0; M_{4q/3}^{2n/3}\| < \varepsilon$, then this solution can be extended to a global one.*

Proof. We begin with an estimate of the nonlinear term $u\nabla\varphi$ in (7) for $u \in M_{4q/3}^{2n/3}$. The inequality (15) gives $\nabla\varphi \in M_{4q}^{2n}$, and by the Hölder inequality $u\nabla\varphi \in M_q^{n/2}$ with

$$\|u\nabla\varphi; M_q^{n/2}\| \leq C\|u; M_{4q/3}^{2n/3}\|^2. \tag{18}$$

Similarly, if $q \geq 3/2$, $u \in M_q^{n/2}$ implies $u\nabla\varphi \in M^{n/3} \subset L^1_{\text{loc}}$, which enables us to apply the heat kernel to $u\nabla\varphi$.

Define for $u \in \mathcal{X}$, as in the proof of Proposition 1, the nonlinear operator

$$N(u)(t) = e^{t\Delta}u_0 + \int_0^t (\nabla e^{(t-s)\Delta}) \cdot (u\nabla\varphi)(s)\,ds.$$

We keep the previous notation $\mathcal{X}$ for spaces of vector-valued functions, but now our space $\mathcal{X}$ is endowed with the norm

$$\|u\|_{\mathcal{X}} = \max(\sup_{0\leq t\leq T}\|u(t); M_q^{n/2}\|, \sup_{0<t\leq T} t^{1/4}\|u(t); M_{4q/3}^{2n/3}\|).$$

For $u, v \in \mathcal{X}$ such that $\sup_{0\leq t\leq T}\|u(t); M_q^{n/2}\| \leq r$, $\sup_{0<t\leq T} t^{1/4}\|u(t); M_{4q/3}^{2n/3}\| \leq 2\varepsilon$ (where $\varepsilon > \limsup_{t\to 0} t^{1/4}\|e^{t\Delta}u_0; M_{4q/3}^{2n/3}\|$) and $\varphi = E_n * u$, $\psi = E_n * v$, we have, from the estimate (18),



$$\begin{aligned}\|N(u)(t); M_q^{n/2}\| &\leq C\|u_0; M_q^{n/2}\| + C\int_0^t (t-s)^{-1/2}\|u(s); M_{4q/3}^{2n/3}\|^2\,ds \\ &\leq C\|u_0; M_q^{n/2}\| + C\int_0^t (t-s)^{-1/2}s^{-1/2}(2\varepsilon)^2\,ds,\end{aligned}$$

which, for small $T>0$, is bounded from above by $C\|u_0; M_q^{n/2}\| + C\varepsilon^2$.

Moreover, we obtain

$$\begin{aligned}&\|N(u)(t) - N(v)(t); M_q^{n/2}\| \\ &\quad\leq C\int_0^t (t-s)^{-1/2}s^{-1/2}4\varepsilon(s^{1/4}\|u(s)-v(s); M_{4q/3}^{2n/3}\|)\,ds \leq C\varepsilon^2.\end{aligned}$$

Here we used the fact that

$$\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds = \text{const}\ (= \Gamma(1/2)^2 = \pi). \tag{19}$$

For the second ingredient of the norm in $\mathcal{X}$ we calculate

$$\begin{aligned}&t^{1/4}\|N(u)(t); M_{4q/3}^{2n/3}\| \\ &\quad\leq t^{1/4}\|e^{t\Delta}u_0; M_{4q/3}^{2n/3}\| + Ct^{1/4}\int_0^t (t-s)^{-1/2-1/4}\|u(s); M_{4q/3}^{2n/3}\|^2\,ds,\end{aligned}$$

which, for small $T>0$, is less than $\varepsilon + C\varepsilon^2$ since

$$t^{1/4}\int_0^t (t-s)^{-3/4}s^{-1/2}\,ds = \text{const}\ (= \Gamma(1/4)\Gamma(1/2)\Gamma(3/4)^{-1}). \tag{20}$$

Similarly we arrive at

$$\begin{aligned}&t^{1/4}\|N(u)(t) - N(v)(t); M_{4q/3}^{2n/3}\| \\ &\quad\leq Ct^{1/4}\int_0^t (t-s)^{-1/2-1/4}s^{-1/2}4\varepsilon(s^{1/4}\|u(s)-v(s); M_{4q/3}^{2n/3}\|)\,ds \leq C\varepsilon^2.\end{aligned}$$

Taking $r \geq 2C\|u_0; M_q^{n/2}\|$ and a suitably small $\varepsilon > 0$ we see that $N$ leaves invariant the box

$$B_{r,\varepsilon} = \{u \in \mathcal{X} : \|u(t); M_q^{n/2}\| \leq r\} \times \{u \in \mathcal{X} : \sup_{0<t\leq T} t^{1/4}\|u(t); M_{4q/3}^{2n/3}\| \leq 2\varepsilon\}$$

and $N$ is a strict contraction on $B_{r,\varepsilon}\cap\{u\in\mathcal{X}: u(0)=u_0\}$. Hence the local existence of solutions follows.

Concerning the global existence observe that under the assumption

$$\sup_{t>0} t^{1/4}\|e^{t\Delta}u_0; M^{2n/3}_{4q/3}\|<\varepsilon,$$

the Lipschitz constant of the operator $N$ is at most $C\varepsilon$ with $C$ *independent* of $T$ (a specific property of the heat semigroup on $\mathbb{R}^n$). Therefore the local-in-time construction from the proof above can be repeated for each $T>0$ and this gives a global-in-time solution.

We note that among the conditions defining the space $\mathcal{X}$ in Theorem 1 the second is more important. In particular, the proof via the contraction argument is not sensitive to the length $r$ of the $N$-invariant box $B_{r,\varepsilon}\subset\mathcal{X}$.

Remarks. For $n=3$ the results above concern only the usual Lebesgue space $L^{3/2}$, while for $n\ge 4$ a variety of Morrey spaces is admitted.

The assumptions on $u_0$ can be interpreted as a sort of (weak) supplementary regularity of an element of $M^{n/2}_q$. Namely, for an interpretation of the hypothesis $l(u_0)<\varepsilon$ note that for $u_0\in L^{n/2}$, or more generally for $u_0\in \mathring{M}^{n/2}_q$, $l(u_0)=0$. The proof follows from a reasoning leading to the characterization (12) (see [22, (3.4) and the proof of Th. 4.3] or [17, Cor. 3.3]). So the assumption $u_0\in\mathring{M}^{n/2}_q$ yields the local existence of solution *independently* of the size of $\|u_0; M^{n/2}_q\|$.

The sufficient condition for the global existence, $\sup_{t>0}t^{1/4}\|e^{t\Delta}u_0; M^{2n/3}_{4q/3}\|<\varepsilon$, is satisfied e.g. when the (homogeneous!) norm $\|u_0;\dot{M}^{n/2}_q\|$ is small enough.

All these remarks explain that for local/global existence of solutions to (7) it suffices that $u_0$ is regular enough and satisfies a size condition imposed on the Morrey norm. The example of the Chandrasekhar solution (5) shows that the results in Theorem 1 are qualitatively optimal; $\widetilde{u}$ from (5) is not smoothed out like $u$'s in Theorem 1 which satisfy the condition $u(t)\in M^{2n/3}$ for $t>0$. In fact, $\widetilde{u}\in M^{n/2}$ but $\widetilde{u}\notin L^{n/2}$, $\widetilde{u}\notin M^p$, $p>n/2$.

For $n=2$ the initial condition can be taken as a finite Borel measure $u_0\in\mathcal{M}(\mathbb{R}^2)$ satisfying a weak additional regularity condition. Global existence is obtained when the total mass of $u_0$ is sufficiently small, which is a counterpart for the Cauchy problem of the result in [6, Th. 2(iv)].

Below, $|\cdot|_p$ denotes the usual norm in the Lebesgue space $L^p(\mathbb{R}^2)$.

THEOREM 2. *There exists $\varepsilon>0$ such that given $u_0\in\mathcal{M}(\mathbb{R}^2)$ satisfying the condition $l(u_0)\equiv\limsup_{t\to0}t^{1/4}|e^{t\Delta}u_0|_{4/3}<\varepsilon$, there exists a local-in-time solution $u$ of* (7) *belonging to (and unique in) the space*

$$\mathcal{X}=\mathcal{C}([0,T];\mathcal{M}(\mathbb{R}^2))\cap\{u:[0,T]\to\mathcal{M}(\mathbb{R}^2): \sup_{0<t\le T}t^{1/4}|u(t)|_{4/3}<\infty\}.$$

*Moreover, if $\sup_{t>0}t^{1/4}|e^{t\Delta}u_0|_{4/3}<\varepsilon$, then the solution can be continued to a global one.*

Proof. The scheme of the demonstration is similar to that of Theorem 1. The counterparts of the inequalities (13)–(15) read

$$|e^{t\Delta}\mu|_1\le\|\mu\|_{\mathcal{M}},\qquad |\nabla e^{t\Delta}f|_{4/3}\le Ct^{-1/2}|f|_1,$$
$$|\nabla\varphi|_4\le C|f|_{4/3}\quad\text{ if } \varphi=E_2*f.$$

The crucial points in the proof include the estimate $|u\nabla\varphi|_1\le C|u|^2_{4/3}$, and again (19)–(20). The remaining part is standard.

Remarks. If $u_0$ is an integrable function then for the functional $l$ in Theorem 2, $l(u_0)=0$, hence a local solution emanating from $u_0$ can be constructed.

The global existence of solutions is assured when the initial measure $u_0$ is sufficiently small. Recall that global solutions for the initial-boundary value problem in [6, Th. 2(iv)] exist for nonnegative $u_0$, $|u_0|_1<4\pi$, $\int_\Omega u_0\log u_0<\infty$; the last condition is necessary for the finiteness of a (physically motivated) Lyapunov function. We return to this question in the Remark after Proposition 3(i).

**3. Self-similar solutions.** Due to homogeneity properties of (1)–(2), if $u$ solves this system, then the rescaled function $u_\lambda(x,t)=\lambda^2u(\lambda x,\lambda^2t)$, $\lambda>0$, is also a solution of (1)–(2). So it is natural to consider solutions which satisfy the scaling property (6): $u_\lambda\equiv u$. They are called *forward* self-similar solutions. By this definition they are global in time, and heuristically they describe large time behavior of general solutions to (1)–(2) (as was explained in the introduction). Indeed, if $\lim_{\lambda\to\infty}\lambda^2u(\lambda x,\lambda^2t)=U(x,t)$ in an appropriate sense, then $tu(xt^{1/2},t)\to U(x,1)$ as $t\to\infty$ (take $t=1$, $\lambda=t^{1/2}$), and $U$ satisfies (6). Hence this is a forward self-similar solution, and

$$U(x,t)=t^{-1}U(xt^{-1/2},1)\tag{21}$$

is determined by a function of $n$ variables $U(y)\equiv U(y,1)$, via the Boltzmann substitution $y=xt^{-1/2}$. Note that *backward* self-similar solutions of the form $(T-t)^{-1}U(x(T-t)^{-1/2})$ (are expected to) describe solutions blowing up in finite time $T$. Radial self-similar solutions which blow up are mentioned in [5].

Observe that if $u_0(x)=\lim_{t\to0}t^{-1}U(xt^{-1/2})$ exists, then $u_0$ is necessarily homogeneous of degree $-2$. Such $u_0\not\equiv0$ defined on $\mathbb{R}^n$, $n\ge3$, cannot have finite mass. Similarly, for the Navier–Stokes system self-similar solutions are of the form $u(x,t) = t^{-1/2}U(xt^{-1/2})$, hence the corresponding $u_0(x)$'s are homogeneous of degree $-1$, and their energy is infinite ([14], [10]).

The usual functional framework for finite mass (resp. energy) solutions to (1)–(2) (resp. Navier–Stokes) deals with spaces that do not contain self-similar solutions. A direct approach to these solutions via an elliptic equation obtained from (1) by substituting the particular form (21) seems to be very hard. The same difficulty appears for the Navier–Stokes system which also contains derivatives in the nonlinearities. This is in opposition to the case of semilinear parabolic equations and their self-similar solutions, which can be well understood by P.D.E./O.D.E. techniques.

Self-similar solutions to (1)–(2) can be obtained from Theorem 1 in Section 2 with suitably small $u_0$ homogeneous of degree $-2$ (the Morrey space $M_q^{n/2}$, $1 \le q < n/2$, does contain such $u_0$) and from Theorem 2 with $u_0 = M\delta$—a small multiple of the Dirac measure; cf. [14, Ch. 5], where an analogous construction is explained for the Navier–Stokes system. However, we are also interested in function spaces other than the Morrey ones, e.g. Besov or symbol spaces. The purpose of such a generalization is that sufficient conditions (imposed on the size of $u_0$) for the existence of self-similar solutions might be weaker than those for the global existence part of the conclusions of Theorems 1 and 2. The idea of converting (1)–(2) (interpreted as the equation (7)) with the particular form (21) of $u$ into an integral equation is that used by the authors of [8], [10], [11] for the Navier–Stokes system.

Let $\mathcal{B} \subset \mathcal{S}'(\mathbb{R}^n)$ be a Banach space of tempered distributions and let $v \in \mathcal{X} = \mathcal{C}([0,\infty);\mathcal{B})$. Define the nonlinear operator $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{N}(v)(t) = \int_0^t (\nabla e^{(t-s)\Delta}) \cdot (v\nabla\psi)(s)\,ds, \tag{22}$$

where $\psi(t) = E_n * v(t)$ at time $t$. We are looking for self-similar solutions of (7), i.e. $U$ of the form (21) satisfying the equation

$$U = V_0 + \mathcal{N}(U), \quad \text{where } V_0 = e^{t\Delta}u_0. \tag{23}$$

We begin with the observation that the equation (23) is well adapted to a study of self-similar solutions via an iterative algorithm.

LEMMA 1. (i) *If $u_0 \in \mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree $-\gamma$, i.e. $u_0(\lambda x) = \lambda^{-\gamma}u_0(x)$, then $V_0 \equiv e^{t\Delta}u_0 = t^{-\gamma/2}U_0(xt^{-1/2})$.*

(ii) *If $U$ is of the form (21), i.e. $U = t^{-1}U(xt^{-1/2})$ and $\mathcal{N}(U) \in \mathcal{S}'(\mathbb{R}^n)$ is well defined, then $\mathcal{N}(U)$ is again of the form (21): $\mathcal{N}(U) = t^{-1}V(xt^{-1/2})$ for some $V$.*

Proof. We use a Fourier transform argument. However, a direct one based on the representation (8) for the heat kernel is also possible.



(i) Passing to the Fourier transforms we have

$$\begin{aligned}\widehat{V}_0(\xi,t) &= \exp(-t|\xi|^2)\widehat{u}_0(\xi) = \exp(-|t^{1/2}\xi|^2)t^{-\gamma/2+n/4}\widehat{u}_0(t^{1/2}\xi)\\ &= \mathcal{F}(t^{-\gamma/2}U_0(xt^{-1/2})) \quad \text{for } U_0 = e^{\Delta}u_0.\end{aligned}$$

(ii) Observe that the gradient of the potential associated with the self-similar density $U$ is of the form $t^{-1/2}(\nabla\Phi)(xt^{-1/2})$ for some (vector-valued) function $\nabla\Phi$. Then $U\nabla\Phi = t^{-3/2}G(xt^{-1/2})$, and therefore

$$\begin{aligned}\mathcal{F}(\mathcal{N}(U)(t))(\xi) &= i\int_0^t \xi\exp(-(t-s)|\xi|^2)s^{(n-3)/2}\widehat{G}(s^{1/2}\xi)\,ds\\ &= i\int_0^1 \xi\exp(-(1-\lambda)t|\xi|^2)t^{(n-3)/2}\lambda^{(n-3)/2}\\ &\quad\times \widehat{G}(\lambda^{1/2}t^{1/2}\xi)t\,d\lambda\\ &= i\int_0^1 \exp(-(1-\lambda)|t^{1/2}\xi|^2)t^{n/2-1}\lambda^{(n-3)/2}(t^{1/2}\xi)\\ &\quad\times \widehat{G}(\lambda^{1/2}t^{1/2}\xi)\,d\lambda\\ &= t^{n/2-1}\widehat{H}(t^{1/2}\xi) = \mathcal{F}(t^{-1}H(xt^{-1/2})).\end{aligned} \tag{24}$$

Lemma 1 implies that it suffices to consider the equation (23) in $\mathcal{X}$ for $t = 1$, so this reduces the study of (23) to the space $\mathcal{B}$.

If we wanted to solve (23) by the iterative application of $\mathcal{N}$:

$$V_{n+1} = V_0 + \mathcal{N}(V_n); \tag{25}$$

then for $u_0$ homogeneous of degree $-2$ all $V_n$'s would be of the self-similar form (21). Hence the iterative algorithm is entrapped in the set of self-similar functions. If we showed the convergence of this algorithm, the limit would automatically be a self-similar solution of (7).

Next we prove the existence of solutions to (23) under natural assumptions on $\mathcal{N}$ (cf. [8, I, Lemme 2.3, IV, Lemme 2.9]).

LEMMA 2. *Suppose that $\mathcal{N} : \mathcal{B} \to \mathcal{B}$ is a nonlinear operator defined on a Banach space $(\mathcal{B}, \|\cdot\|)$ by $\mathcal{N}(U) = B(U,U)$ with a bilinear continuous form $B : \mathcal{B}\times\mathcal{B} \to \mathcal{B}$ such that $\|B(U,V)\| \le K\|U\|\,\|V\|$ for some constant $K$. If $\|V_0\| < 1/(4K)$, then the equation (23) has a solution which can be obtained as the limit of $V_n$ defined by the recursive algorithm (25).*

Proof. The simplest case $\mathcal{B} = \mathbb{R}$ suggests the idea of the proof, and shows that this solution, in general, is not unique (but this is the only one which is stable).

It is easy to verify that the operator $N(U) = V_0 + \mathcal{N}(U)$ leaves invariant the ball $\{U \in \mathcal{B} : \|U\| \le r\}$ for

$$r \in [(1-(1-4K\|V_0\|)^{1/2})/(2K), (1+(1-4K\|V_0\|)^{1/2})/(2K)].$$

Since

$$\begin{aligned}\|N(U)-N(V)\| &= \|B(U,U)-B(V,V)\|\\ &\le \|B(U-V,U)\| + \|B(V,U-V)\|\\ &\le K(\|U\|+\|V\|)\|U-V\|,\end{aligned}$$

the operator $N$ is a contraction for each

$$r \in [(1-(1-4K\|V_0\|)^{1/2})/(2K), 1/(2K)),$$

so it has a unique fixed point in this ball.

The above lemmas show that a good functional framework to study (23) should satisfy the following conditions:

(i) $u_0 \in \mathcal{B}$ is a distribution homogeneous of degree $-2$ (for $n = 2$ and positive $u_0$ this means that $u_0$ is a multiple of the Dirac measure $\delta$),

(ii) $\mathcal{N}$ defined by (22) and represented by (24) for $t = 1$ is a continuous quadratic form on $\mathcal{B}$.

In particular, for $n \ge 3$, $u_0$ cannot be of finite mass unless $u_0 \equiv 0$.

Below, we give a choice of function spaces (necessarily different from usual $L^p$ or Sobolev spaces) which satisfy these conditions. Besides homogeneous Besov spaces and spaces containing functions related to symbols of classical pseudodifferential operators which have been used in [8–11] for the Navier–Stokes system, the $\mathcal{X}_p$ spaces in Proposition 2 are specifically adapted to our problem for $n = 2, 3$.

Concerning the interpretation of $\lim_{t\to 0} t^{-1}U(xt^{-1/2})$ note that solutions of (23) enjoy the same continuity properties as those considered in Section 2. Therefore the initial condition in (23) is attained in the sense of distributions, and the curve $t \mapsto U$ in (21) is bounded in $\mathcal{B}$. Moreover, if $\mathcal{B}$ is a dual space, then continuity is meant with respect to the weak* topology of $\mathcal{B}$.

We can interpret $V_0$ in (23) as the main (tendency) term and $\mathcal{N}(U)$ as a fluctuation around the drift of $u_0$ described by the heat equation.

Here we collect the definition and some basic facts concerning homogeneous Besov spaces which will be used in the proof of Theorem 3 giving the first solvability result for (23) with $n \ge 4$. We begin with the analysis in Besov spaces defined via the Littlewood–Paley theory because then the calculations leading to the proof of continuity of the quadratic operator $\mathcal{N}$ are relatively simple. The advantage of Besov spaces is an easy frequency analysis since their abstract definition using a dyadic decomposition is particularly well suited for convolution and potential estimates of Hardy–Littlewood–Sobolev type, (see [7], [13], [15], [20], [23–24]).



Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$, $\mathcal{Z} = \{v \in \mathcal{S} : D^\alpha \widehat{v}(0) = 0$ for every multiindex $\alpha\}$ and $\widehat{\mathcal{D}}_0 = \{v \in \mathcal{S} : \widehat{v} \in C_0^\infty(\mathbb{R}^n \setminus \{0\})\}$. Since $\widehat{\mathcal{D}}_0$ is dense in $\mathcal{Z}$ and $\mathcal{Z}$ is a closed subspace of $\mathcal{S}$, the inclusion $\mathcal{Z} \subset \mathcal{S}$ induces a surjective map $\pi : \mathcal{S}' \to \mathcal{Z}'$ such that $\ker \pi = \mathcal{P}$, the space of polynomials, so $\mathcal{Z}' = \mathcal{S}'/\mathcal{P}$.

Let $\widehat{\psi} \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \le \widehat{\psi} \le 1$, $\widehat{\psi}(\xi) = 1$ for $|\xi| \le 1$, $\widehat{\psi}(\xi) = 0$ for $|\xi| \ge 2$, and define for any $k \in \mathbb{Z}$,

$$\widehat{\phi}_k(\xi) = \widehat{\psi}(2^{-k}\xi) - \widehat{\psi}(2^{-(k+1)}\xi).$$

Evidently $\operatorname{supp} \widehat{\phi}_k \subset A_k \equiv \{\xi : 2^{k-1} \le |\xi| \le 2^{k+1}\}$, $\sum_k \widehat{\phi}_k(\xi) = 1$ for any $\xi \ne 0$, with at most two nonzero terms in the series. The convolutions $\phi_k * v$ are meaningful not only for $v \in \mathcal{S}'$ but also for all $v \in \mathcal{Z}'$. The homogeneous Besov space $\dot{B}^s_{pq} \subset \mathcal{Z}'$ is defined for $s \in \mathbb{R}$, $1 \le p, q \le \infty$, by the condition

$$\|v; \dot{B}^s_{pq}\| \equiv \Big(\sum_k 2^{ksq} |\phi_k * v|_p^q\Big)^{1/q} < \infty, \tag{26}$$

with obvious modification if $q = \infty$:

$$\|v; \dot{B}^s_{p\infty}\| \equiv \sup_k 2^{ks} |\phi_k * v|_p < \infty,$$

where $|\cdot|_p$ is the $L^p(\mathbb{R}^n)$ norm.

Of course, $\mathcal{F}(\phi_k * v) = \widehat{\phi}_k \widehat{v}$, so the Besov norms control the size of the Fourier transform $\widehat{v}$ over the dyadic annuli $A_k$, and the parameter $s$ measures the smoothness of the function $v$.

We will be mainly interested in the (nonseparable) Banach spaces $\mathcal{B}_n = \dot{B}^{n/2-2}_{2\infty}(\mathbb{R}^n)$, $n \ge 4$, whose elements can be realized as tempered distributions (hence simpler to interpret than elements of $\mathcal{Z}' = \mathcal{S}'/\mathcal{P}$); see [7], [15, Appendix] and [20, p. 181]. It is easy to verify that $|x|^{-2} \in \mathcal{B}_n$ (since $\mathcal{F}(|x|^{-2}) = 2^{n-2}\pi^{n/2}\Gamma(n/2-1)|\xi|^{2-n}$), and functions homogeneous of degree $-2$ belong to $\mathcal{B}_n$ provided they are smooth enough on the unit sphere of $\mathbb{R}^n$ (cf. related results in [8, IV, Th. 2.1], [9]).

THEOREM 3. *If $u_0 \in \dot{B}^{n/2-2}_{2\infty}$ is homogeneous of degree $-2$ and the norm $\|u_0; \dot{B}^{n/2-2}_{2\infty}\|$ is small enough, then there exists a solution $U$ of the equation* (23). *This solution is unique in the class of distributions satisfying $\|U; \dot{B}^{n/2-2}_{2\infty}\| \le r$, $r$ given at the end of the proof of Lemma* 2.

Proof. According to Lemma 2 it suffices to prove that the operator $\mathcal{N}$ in (24) is defined by a bounded bilinear form on $\mathcal{B}_n \times \mathcal{B}_n$. By the Plancherel formula the norm in $\mathcal{B}_n$ is equivalent to $\|v\| \equiv \sup_k 2^{k(n/2-2)}|\widehat{\phi}_k \widehat{v}|_2 \sim \sup_k 2^{k(n/2-2)} |\widehat{v}|_{L^2(A_k)}$.

Let $U$, $V \in \mathcal{B}_n$, and $\Phi$, $\Psi$ be the corresponding potentials. Due to the regularizing effect of the heat semigroup expressed for (24) as

$$|\xi| \int_0^1 \exp(-(1-\lambda)|\xi|^2)\,d\lambda = |\xi|^{-1}\exp(-|\xi|^2)(\exp(|\xi|^2)-1) \le \min(|\xi|, |\xi|^{-1}),$$

it suffices to show that

$$\sup_k 2^{-k}\{2^{k(n/2-2)}|\phi_k * (U\nabla\Psi)|_2\} \le C\|U\|\,\|V\|. \tag{27}$$

Indeed, applying the dilations with $\lambda \in [0,1]$ we obtain, for each $m \in \mathbb{Z}$,

$$|\widehat{\phi}_m (U\nabla\Psi)\widehat{\ }(\lambda^{1/2}\xi)|_2^2 \sim \int_{A_m} |\widehat{G}(\lambda^{1/2}\xi)|^2\,d\xi = \lambda^{-n/2}\int_{\lambda^{1/2}A_m} |\widehat{G}(\eta)|^2\,d\eta \sim \lambda^{-n/2}|\phi_k * (U\nabla\Psi)|_2^2$$

if $\lambda^{1/2}2^m \sim 2^k$, and $\lambda^{-n/4}2^{k(3-n/2)} \sim \lambda^{(3-n)/2}2^{m(3-n/2)}$, so the factor $\lambda^{(3-n)/2}$ will cancel out the one in the integrand of (24).

Passing to convolutions in (27) we should show

$$\sup_k 2^{k(n/2-3)}\Big|\widehat{\phi}_k(\xi)\Big(\sum_j \int_{A_j} \widehat{U}(\xi-\eta)\eta|\eta|^{-2}\widehat{V}(\eta)\,d\eta\Big)\Big|_2 \le C\|U\|\,\|V\|.$$

To do this we decompose the sum $\sum_j$ into three pieces:

$$\sum_{j\le k-3} + \sum_{j=k-2}^{k+2} + \sum_{j\ge k+3} \equiv I_1 + I_2 + I_3,$$

and estimate them separately.

For the first term, by the Young inequality we obtain

$$\begin{aligned}
2^{k(n/2-3)}|I_1|_2 &\le 2^{k(n/2-3)}|\widehat{U}|_{L^2(A_{k-1}\cup A_k\cup A_{k+1})} \sum_{j\le k-3} |\widehat{V}(\eta)|\eta|^{-1}|_{L^1(A_j)} \\
&\le 2^{k(n/2-3)}|\widehat{U}|_{L^2(A_{k-1}\cup A_k\cup A_{k+1})} \sum_{j\le k-3} 2^{j(n/2-1)}|\widehat{V}|_{L^2(A_j)} \\
&\le C2^{-k}\{2^{k(n/2-2)}|\widehat{U}|_{L^2(A_{k-1}\cup A_k\cup A_{k+1})}\} \\
&\quad \times \Big\{\sum_{j\le k-3} 2^{j(n/2-2)}2^j|\widehat{V}|_{L^2(A_j)}\Big\} \\
&\le C2^{-k}2^k\|U\|\,\|V\| \le C\|U\|\,\|V\|.
\end{aligned}$$

For the second term note that $A_k - A_{k-2} \subset \bigcup_{j\le k+1} A_j, \ldots, A_k - A_{k+2} \subset \bigcup_{j\le k+3} A_j$, so again by the Young inequality we can write



$$\begin{aligned}
|I_2|_2 &\le \sum_{j=k-2}^{k+2}\sum_{i\le k+3} |\widehat{U}|_{L^1(A_i)}|\widehat{V}(\eta)|\eta|^{-1}|_{L^2(A_j)} \\
&\le C\sum_{j=k-2}^{k+2} 2^{-j}|\widehat{V}|_{L^2(A_j)} \sum_{i\le k+3} |\widehat{U}|_{L^2(A_i)}2^{in/2} \\
&\le C2^{-k}2^{-k(n/2-2)} \\
&\quad \times \Big\{2^{k(n/2-2)}\sum_{j=k-2}^{k+2} |\widehat{V}|_{L^2(A_j)}\Big\}\Big\{\sum_{i\le k+3} |\widehat{U}|_{L^2(A_i)}2^{i(n/2-2)}2^{2i}\Big\} \\
&\le C2^{-k-k(n/2-2)+2k}\|U\|\,\|V\| = C2^{k(3-n/2)}\|U\|\,\|V\|.
\end{aligned}$$

For the last term we begin with a pointwise estimate for $\xi \in A_k$:

$$\begin{aligned}
|I_3(\xi)| &\le \sum_{j\ge k+3} |\widehat{U}|_{L^2(\xi-A_j)}2^{-j}|\widehat{V}|_{L^2(A_j)} \\
&\le \sum_{j\ge k+3} |\widehat{U}|_{L^2(A_{j-1}\cup A_j\cup A_{j+1})}2^{-j}|\widehat{V}|_{L^2(A_j)} \\
&\le C\sum_{j\ge k+3} \{2^{j(n/2-2)}|\widehat{U}|_{L^2(A_{j-1})}\}\{2^{j(n/2-2)}|\widehat{V}|_{L^2(A_j)}\}2^{-j(n-3)} \\
&\le C2^{-k(n-3)}\|U\|\,\|V\| \quad \text{since } n-3>0.
\end{aligned}$$

After integrating the square of this expression over $A_k$ we have

$$2^{k(n/2-3)}|I_3|_2 \le C2^{k(n/2-3)}2^{k(3-n)}2^{kn/2}\|U\|\,\|V\| \le C\|U\|\,\|V\|.$$

These estimates put together imply for $U = V$ the bound $\|\mathcal{N}(U)\| \le K\|U\|^2$ for some $K \ge 0$, as claimed.

Observe that this does not work when $n = 3$ and $\mathcal{B}_3$ is the Besov space of negative order $\dot{B}_{2\infty}^{-1/2} \supset \dot{B}_{3/2\,\infty}^{0}$, the latter space being perhaps a better candidate for solving (23) in this situation.

The uniqueness of solutions constructed in Theorem 3 can be inferred from the proof of Lemma 2.

Remarks. It can be easily proved that $V_0 \in L^p(\mathbb{R}^n)$ for each $p > n/2$ but $V_0 \notin L^{n/2}(\mathbb{R}^n)$ unless $V_0 = 0$.

It is of interest to note that the potential $\Phi$ associated with $U$ is an element of $\dot{B}_{2\infty}^{n/2} \subset BMO$ (see [7, p. 53]).

Another class of spaces suitable to study the equation (23) is the scale of spaces $E^{\varrho,m}$ considered in [8], [10]. These spaces consist of functions from $C^m(\mathbb{R}^n)$ satisfying natural decay estimates at infinity, and for their homogeneous counterparts $\dot{E}^{\varrho,m}$, estimates of the singularity at the origin (like symbols of classical pseudo-differential operators).

For $\varrho > 0$ and $m \in \mathbb{N}$ define the following Banach spaces of functions on $\mathbb{R}^n$:

$$(28)\quad \begin{aligned} E^{\varrho,m} &= E^{\varrho,m}(\mathbb{R}^n) \\ &= \{v \in C^m(\mathbb{R}^n) : |D^\alpha v(x)| \le C(1+|x|)^{-\varrho-|\alpha|},\ |\alpha| \le m\} \end{aligned}$$

and

$$(29)\quad \begin{aligned} \dot{E}^{\varrho,m} &= \dot{E}^{\varrho,m}(\mathbb{R}^n) \\ &= \{v \in C^m(\mathbb{R}^n \setminus \{0\}) : |D^\alpha v(x)| \le C|x|^{-\varrho-|\alpha|},\ |\alpha| \le m\}, \end{aligned}$$

with the norms of $v$ defined as the least constants $C$ in (28), (29) respectively.

Evidently, for $\varrho < n$, $\dot{E}^{\varrho,m} \subset L^1_{\mathrm{loc}}$, since the singularity at the origin is integrable.

We look for solutions to (23) of the form $U = V_0 + \mathcal{N}(U) \in E^{2,m}$, with $u_0 \in \dot{E}^{2,m}$, and $m+2 < n$ in order to avoid nonintegrable singularities in further analysis. Note that for the three-dimensional Navier–Stokes system in [8], [10], $U \in E^{1,m}$, $u_0 \in \dot{E}^{1,m}$ has a weaker singularity than in our case but the nonlinearity is more difficult to study.

The spaces $E^{\varrho,m}$ are more natural (in their interpretation) to study (23) than Besov spaces, but the estimates (30) (see below) of (24) (without the use of Fourier transforms) are slightly more subtle than the frequency bound (27) particularly well adapted to the Littlewood–Paley decomposition of functions.

THEOREM 4. *Let $n \ge 3$, and $m \le n-3$ be an integer. If $u_0 \in \dot{E}^{2,m}$ is homogeneous of degree $-2$ and has a sufficiently small norm, then there exists a self-similar solution $t^{-1}U(xt^{-1/2})$ with $U \in E^{2,m}$ and $\mathcal{N}(U) \in E^{\varrho,m}$, $\varrho < 3$. Such a solution is unique among those satisfying $\|U; E^{2,m}\| \le r$, with $r$ from the end of the proof of Lemma 2.*

Proof. First we verify that $V_0 = e^{t\Delta}u_0 \in E^{2,m}$, $t > 0$. The crucial point is, of course, the estimate for $m = 0$. We can represent $u_0$ as $u_0(y) = f(y)|y|^{-2}$ with a function $f \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and using (8) we write

$$\begin{aligned} |x|^2(e^{t\Delta}u_0)(x) &= (4\pi t)^{-n/2}|x|^2 \int \exp(-|x-y|^2/(4t)) f(y)|y|^{-2}\,dy \\ &= C \int \exp(-|X-Y|^2/4)|X|^2|Y|^{-2} f(t^{1/2}Y)\,dY. \end{aligned}$$

This integral is bounded, which can be seen by decomposing the integral over $\mathbb{R}^n$ into $\int_{B_1} + \int_{\mathbb{R}^n \setminus B_1}$. In other words, we use the rapid decay of the Gauss–Weierstrass kernel and the fact that $|Y|^{-2} \in L^{n/2-\varepsilon} + L^{n/2+\varepsilon}$ for each $\varepsilon > 0$.

It is easy to verify that if $U, V \in E^{\varrho,m}$, then $\nabla\Psi \in E^{\varrho-1,m+1}$, $G = U\nabla\Psi \in E^{2\varrho-1,m}$ and $\nabla\cdot(U\nabla\Psi) \in E^{2(\varrho-1),m-1}$. However, to retrieve derivatives of order $m$ we should take into account the regularizing effect of the heat kernel in (24).



To estimate the $E^{2,m}$ norm of $\mathcal{N}(U)$ we calculate, for $|\alpha| \le m < n-2$,

$$(30)\quad \begin{aligned} &|x|^{|\alpha|+2}\Big|D^\alpha_x\Big(\int_0^t (\nabla e^{(t-s)\Delta})\cdot s^{-3/2}G\Big(\frac{x-\cdot}{s^{1/2}}\Big)\,ds\Big)\Big| \\ &\le C\int_0^t\int (t-s)^{-n/2-1}|y|\exp\Big(\frac{-|y|^2}{4(t-s)}\Big)(|x-y|^{|\alpha|+2}+|y|^{|\alpha|+2}) \\ &\quad\times s^{-3/2}\Big|D^\alpha_x\Big(G\Big(\frac{x-y}{s^{1/2}}\Big)\Big)\Big|\,dy\,ds \equiv J_1 + J_2. \end{aligned}$$

The derivatives of $G = U\nabla\Phi$ can be bounded by a multiple of $\|U\|^2$ by recalling the definition (28).

After a change of variables in convolutions ($x = t^{1/2}(1-\lambda)^{1/2}X$, $y = t^{1/2}(1-\lambda)^{1/2}Y$, $s = t\lambda$), the first integral in (30) is majorized by

$$\begin{aligned} J_1 &\le C\int_0^t\int |x-y|^{|\alpha|+3}(t-s)^{-n/2-1}\exp\Big(\frac{-|x-y|^2}{4(t-s)}\Big) \\ &\quad\times \frac{s^{-3/2-|\alpha|/2}}{1+|ys^{-1/2}|^{|\alpha|+3}}\,dy\,ds \\ &= C\int_0^1\int |x-y|^{|\alpha|+3}t^{-n/2-1}(1-\lambda)^{-n/2-1}\exp\Big(\frac{-|x-y|^2}{4t(1-\lambda)}\Big) \\ &\quad\times \frac{t^{-3/2-|\alpha|/2}\lambda^{-3/2-|\alpha|/2}t}{1+|yt^{-1/2}\lambda^{-1/2}|^{|\alpha|+3}}\,dy\,d\lambda \\ &= C\int_0^1\int |X-Y|^{|\alpha|+3}\exp\Big(\frac{-|X-Y|^2}{4}\Big) \\ &\quad\times \frac{(1-\lambda)^{(|\alpha|+1)/2}\lambda^{(|\alpha|+3)/2}}{1+|Y(1-\lambda)^{1/2}\lambda^{-1/2}|^{|\alpha|+3}}\,dY\,d\lambda \\ &= C\int_0^1\int |X-Y|^{|\alpha|+3}\exp\Big(\frac{-|X-Y|^2}{4}\Big) \\ &\quad\times \frac{(1-\lambda)^{(|\alpha|+1)/2}}{\lambda^{(|\alpha|+3)/2}+|Y(1-\lambda)^{1/2}|^{|\alpha|+3}}\,dY\,d\lambda. \end{aligned}$$

For $k = |\alpha| + 3 \le n$ we have $|Y|^{k-1}\lambda^{1/2}(1-\lambda)^{(k-1)/2} \le C(\lambda^{k/2} + |Y|^k(1-\lambda)^{k/2})$. Hence

$$\begin{aligned} J_1 &\le C\int_0^1 \lambda^{-1/2}(1-\lambda)^{-1/2}\,d\lambda \\ &\quad\times \int |X-Y|^k\exp(-|X-Y|^2/4)|Y|^{1-k}\,dY < \infty; \end{aligned}$$

compare (19) and $|Y|^{1-k} \in L^1 + L^\infty$.

The estimates for the second integral are similar to the preceding ones:

$$J_2 \le C \int_0^t \int |x-y|(t-s)^{-n/2-1} \exp\left(\frac{-|x-y|^2}{4(t-s)}\right)$$
$$\times \frac{s^{-3/2-|\alpha|/2}|y|^{|\alpha|+2}}{1+|ys^{-1/2}|^{|\alpha|+3}}\,dy\,ds$$
$$= C \int_0^1 \int |x-y| t^{-n/2-1}(1-\lambda)^{-n/2-1} \exp\left(\frac{-|x-y|^2}{4t(1-\lambda)}\right)$$
$$\times \frac{t^{-3/2-|\alpha|/2}\lambda^{-3/2-|\alpha|/2} t|y|^{|\alpha|+2}}{1+|yt^{-1/2}\lambda^{-1/2}|^{|\alpha|+3}}\,dy\,d\lambda$$
$$= C \int_0^1 \int |X-Y| \exp\left(\frac{-|X-Y|^2}{4}\right)$$
$$\times \frac{(1-\lambda)^{(|\alpha|+1)/2}\lambda^{(|\alpha|+3)/2}|Y|^{|\alpha|+2}}{1+|Y(1-\lambda)^{1/2}\lambda^{-1/2}|^{|\alpha|+3}}\,dY\,d\lambda$$
$$= C \int_0^1 \int |X-Y| \exp\left(\frac{-|X-Y|^2}{4}\right)$$
$$\times \frac{(1-\lambda)^{(|\alpha|+1)/2}|Y|^{|\alpha|+2}}{\lambda^{(|\alpha|+3)/2}+|Y(1-\lambda)^{1/2}|^{|\alpha|+3}}\,dY\,d\lambda.$$

Finally, by an estimate analogous to that for $J_1$ we obtain

$$J_2 \le C \int_0^1 \lambda^{-1/2}(1-\lambda)^{-1/2}\,d\lambda \int |X-Y| \exp(-|X-Y|^2/4)\,dY < \infty.$$

Concerning the decay and regularity of the fluctuation term $\mathcal{N}(U)$ for the solution $U \in E^{2,m}$ of (23) observe that for $\varrho < 3$ and $m = 0$, $||x|^{\varrho}\mathcal{N}(U)(x)|$ can be estimated as above by

$$C\|U\|^2 t^{\varrho/2-1} \int_0^1 \lambda^{(\varrho-3)/2}(1-\lambda)^{-1/2}\,d\lambda$$
$$\times \int (|X-Y|^{1+\varrho}|Y|^{-\varrho} + |X-Y|) \exp(-|X-Y|^2/4)\,dY.$$

This implies that $\mathcal{N}(U) \in E^{\varrho}$ for each $\varrho < 3$. The case $m \le n-3$ is similar, which ends the proof of Theorem 4.

The remaining case $n = 2$ will be treated with the use of spaces $\mathcal{X}_p$, which appear to be also suitable for an alternative approach for $n = 3$. Define for $n = 2, 3$ the Banach space

$$\mathcal{X}_p = \{v \in L^p(\mathbb{R}^n) : \widehat{v}(\xi)/|\xi| \in L^1(\mathbb{R}^n)\}$$



endowed with the norm $\|v\| \equiv \|v\|_{\mathcal{X}_p} = \max(|v|_p, A|\widehat{v}(\xi)/|\xi||_1)$ with a constant $A > 0$ to be chosen later. Since $|w|_\infty \le (2\pi)^{-n}|\widehat{w}|_1$, we have $\mathcal{X}_p \subset \{v \in L^p : \nabla\psi \in L^\infty\}$, where $\psi = E_n * v$, so we will work with sufficiently regular solutions to (23).

PROPOSITION 2. (i) *If $n = 2$, $1 < p < 2$, and $u_0 = M\delta$ with $M > 0$ small enough, then there exists a self-similar solution obtained by application of Lemma 2 to* (23).

(ii) *If $n = 3$, $3/2 < p \le 2$, and $u_0$ is a sufficiently small function homogeneous of degree $-2$, then the conclusion of* (i) *holds true.*

Proof. For $U, V \in \mathcal{X}_p$ the bilinear form $B$ reads

$$B(U,V)(x) = (4\pi)^{-n/2} \int_0^1 (2(1-\lambda)^{n/2+1})^{-1} x$$
$$\times \exp\left(\frac{-|x|^2}{4(1-\lambda)}\right) * \lambda^{-3/2}(U\nabla\Phi)(\cdot\lambda^{-1/2})\,d\lambda.$$

Therefore we can write (cf. the $L^p$ analogues of (13)–(15))

$$|B(U,V)|_p \le 2^{-1}(4\pi)^{-n/2}\Big(\int_0^1 (1-\lambda)^{-1/2}\lambda^{-3/2}\lambda^{n/(2p)}\,d\lambda\Big)$$
$$\times \Big(\int_{\mathbb{R}^n} |x| \exp(-|x|^2/4)\,dx\Big)\|U\|A^{-1}(2\pi)^{-n}\|V\|$$
$$= A^{-1}B(1/2, n/(2p)-1/2)2^{-1-2n}\pi^{-3n/2}$$
$$\times \Big(\sigma_n \int_0^\infty \varrho^n e^{-\varrho^2/4}\,d\varrho\Big)\|U\|\,\|V\|$$
$$\equiv A^{-1}c_1\|U\|\,\|V\|,$$

where $B(a,b) = \Gamma(a)\Gamma(b)\Gamma(a+b)^{-1}$ is the Euler Beta function.

For the second term defining the norm in $\mathcal{X}_p$ we recall (24) and estimate (using the Hausdorff–Young inequality, $1/p+1/p' = 1$, $\widehat{L^p} * L^1 \subset L^{p'} * L^1 \subset L^{p'}$)

$$A|\mathcal{F}(B(U,V))(\xi)/|\xi||_1 \le A \int_0^1 |\exp(-(1-\lambda)|\xi|^2)|_p \lambda^{(n-3)/2}$$
$$\times |(\widehat{U} * \widehat{\nabla\Psi})(\lambda^{1/2}\xi)|_{p'}\,d\lambda$$
$$\le A \int_0^1 (1-\lambda)^{-n/(2p)}\lambda^{(n-3)/2-n/(2p')}\,d\lambda$$
$$\times \Big(\int_{\mathbb{R}^n} \exp(-p|\xi|^2)\,d\xi\Big)^{1/p}(2\pi)^{n(1-1/p)}\|U\|A^{-1}\|V\|$$

$$= B(1-n/(2p), n/(2p))(2\pi)^{n(1-1/p)}$$
$$\times \Big(\sigma_n \int_0^\infty \varrho^{n-1} e^{-\varrho^2}\, d\varrho\, p^{-1/2}\Big)^{1/p} \|U\|\,\|V\|$$
$$\equiv c_2 \|U\|\,\|V\|.$$

The above inequalities imply that $\|B(U,V)\| \le K\|U\|\,\|V\|$ with $K = \max(A^{-1}c_1, c_2)$ showing Proposition 2.

Remark. In the two-dimensional case we can produce an explicit bound for $M$. Namely, for $u_0 = M\delta$, $V_0(x,1) = Me^{\Delta}\delta = M(4\pi)^{-1}\exp(-|x|^2/4)$, so

$$|V_0|_p = M(4\pi)^{-1}\Big(2\pi \int_0^\infty \varrho e^{-\varrho^2}\, d\varrho\Big)^{1/p} 2^{2/p} p^{-1/p} \equiv M d_1,$$

$$A|\widehat{V}_0(\xi)/|\xi||_1 = AM \int_{\mathbb{R}^2} \exp(-|\xi|^2)|\xi|^{-1}\, d\xi = AM2\pi \int_0^\infty e^{-\varrho^2}\, d\varrho \equiv AM d_2.$$

Now the solvability condition in Lemma 2 reads

$$M < (4\max(A^{-1}c_1, c_2)\max(d_1, Ad_2))^{-1} = (4\max(c_1 d_2, c_2 d_1))^{-1}$$

after the optimization of $A > 0$.

Note that the fluctuation term $\mathcal{N}(U)$ is uniformly bounded in $\mathbb{R}^n$, $n = 2, 3$, which can be easily proved by a repeated use of estimates of $B(U,V)$ at the beginning of the proof.

Forward self-similar solutions which are radial can also be studied by O.D.E. techniques. If a positive solution $u$ is radial, then passing to the integrated density $Q(r,t) = \sigma_n \int_0^r \varrho^{n-1} u(\varrho, t)\, d\varrho$, $r = |x|$, we obtain the equation (cf. [5, (6)])

$$Q_t = Q_{rr} - (n-1)r^{-1}Q_r + \sigma_n^{-1} r^{1-n} Q Q_r. \tag{31}$$

For a positive self-similar $U$ of the form (21), $Q(r,t) = \sigma_n t^{n/2-1}\zeta(r^2/t)$, where the nondecreasing function $\zeta = \zeta(y)$, $y = r^2/t$, satisfies the equation

$$\zeta'' + \frac{1}{4}\zeta' - \frac{n-2}{2y}\zeta' - \frac{n-2}{8y}\zeta + \frac{1}{2y^{n/2}}\zeta\zeta' = 0, \tag{32}$$

with the prime standing for $d/dy$, together with the condition $\zeta(0) = 0$ (cf. an analogous problem for backward self-similar solutions in [5, (23)]).

The Chandrasekhar solution (5) corresponds to $\widetilde{\zeta}(y) = 2y^{n/2-1}$, $n \ge 3$.

Moreover, if we change the variables $y = e^s$ and $z(s) = \zeta(y)$, then

$$\ddot{z} - \frac{n}{2}\dot{z} + \frac{1}{4}e^s\dot{z} - \frac{n-2}{8}e^s z + \frac{1}{2}e^{(1-n/2)s} z\dot{z} = 0, \tag{33}$$

with the dot standing for $d/ds$, and $z(-\infty) = 0$, $z$ nondecreasing, is a useful form of the problem (32).



The following simple result describes how big the total mass of a radial self-similar solution can be.

PROPOSITION 3. (i) *If $n = 2$, then for each $M \in [0, 4\pi(1+e^{-2}))$ there is a nondecreasing solution of the equation*

$$\zeta'' + \zeta'/4 + \zeta\zeta'/(2y) = 0 \tag{34}$$

(*i.e.* (32) *for $n = 2$*) *such that $\zeta(0) = 0$ and $\lim_{y\to\infty}\zeta(y) = M/(2\pi)$. For $M > 8\pi$ there is no such solution.*

(ii) *If $n \ge 3$, then for each nondecreasing solution $\zeta \not\equiv 0$ of* (32), $\lim_{y\to\infty}\zeta(y) = \infty$.

Proof. (i) Drop the term $e^s\dot{z}/4$ on the left-hand side of (33) and note that for any *nondecreasing* solution $z$ we have $(\dot{z} - z + z^2/4)^{\cdot} \le 0$. Therefore, by $\dot{z}(-\infty) = 0$, we obtain $\dot{z} \le z(1 - z/4)$. Since $\dot{z}$ is negative for $z \notin [0,4]$, the nonexistence part of (i) follows.

Now we consider the initial value problem for (32) with $\zeta(0) = 0$ and $\zeta'(0) = a \ge 0$. From the concavity of $\zeta$ we have $\zeta(y) \le \min(ay, 4)$. This upper bound put into the equation (34) leads to a lower bound for the derivative $\zeta'$:

$$\zeta'(y) \ge a\exp(-(1/4 + a/2)y) \quad \text{for } 0 \le y \le 4/a$$

and

$$\zeta'(y) \ge \zeta'(4/a)\exp\Big(\int_{4/a}^{y} (1/4 + 2/\eta)\, d\eta\Big)$$
$$\ge 16a^{-1}e^{-2}e^{-t/4}t^{-2} \quad \text{for } y \ge 4/a.$$

After an integration we obtain

$$\zeta(y) \ge a(1/4 + a/2)^{-1}(1 - e^{-1/a-2}) + 16a^{-1}e^{-2}e^{-y/4}(a/4 - 1/y),$$

and (taking $y = a^{-1/2}$ and $a \to \infty$) $\sup_{a,y}\zeta(y) \ge 2(1 + e^{-2})$.

(ii) Suppose that for a solution $\zeta \ge 0$, $\zeta' \ge 0$, $\zeta \not\equiv 0$, there is a constant $A$ such that $\zeta(y) \le A$. Then for $\xi = \zeta' + \zeta/4 = e^{-y/4}(e^{y/4}\zeta)'$, (32) implies the differential inequality

$$\xi' - \frac{n-2}{2y}\xi + \frac{A}{2y^{n/2}}\xi \ge 0,$$

i.e.

$$(y^{1-n/2}\exp(-A(n-2)^{-1}y^{1-n/2})\xi)' \ge 0.$$

After two integrations we have

$$\zeta(y) \ge C(y_0)e^{-y/4}\int_{y_0}^{y} \eta^{n/2-1}e^{\eta/4}\, d\eta \to \infty$$

as $y \to \infty$, a contradiction.

Remark. Note that for $n = 2$ the solution of the Cauchy problem (1)–(2) with $u_0 = M\delta$ is a radial self-similar solution. Therefore Proposition 3(i) expresses an improved explicit sufficient condition for the global-in-time solvability of a particular case of the problem studied in Theorem 2 and in Proposition 2(i).

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**Addendum.** The result in Proposition 3(i) can be improved: Solutions of (34) exist if and only if $M \in [0, 8\pi)$. The proof of this will appear in the paper *Growth and accretion of mass in an astrophysical model*, accepted for publication in Applicationes Math.

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