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# Banach space properties of strongly tight uniform algebras

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**Abstract.** We use the work of J. Bourgain to show that some uniform algebras of analytic functions have certain Banach space properties. If $X$ is a Banach space, we say $X$ is *strong* if $X$ and $X^*$ have the Dunford-Pettis property, $X$ has the Pełczyński property, and $X^*$ is weakly sequentially complete. Bourgain has shown that the ball-algebras and the polydisk-algebras are strong Banach spaces. Using Bourgain's methods, Cima and Timoney have shown that if $K$ is a compact planar set and $A$ is $R(K)$ or $A(K)$, then $A$ and $A^*$ have the Dunford–Pettis property. Prior to the work of Bourgain, it was shown independently by Wojtaszczyk and Delbaen that $R(K)$ and $A(K)$ have the Pełczyński property for special classes of sets $K$. We show that if $A$ is $R(K)$ or $A(K)$, where $K$ is arbitrary, or if $A$ is $A(D)$ where $D$ is a strictly pseudoconvex domain with smooth $C^2$ boundary in $\mathbb{C}^n$, then $A$ is a strong Banach space. More generally, if $A$ is a uniform algebra on a compact space $K$, we say $A$ is *strongly tight* if the Hankel-type operator $S_g : A \to C/A$ defined by $f \mapsto fg + A$ is compact for every $g \in C(K)$. Cole and Gamelin have shown that $R(K)$ and $A(K)$ are strongly tight when $K$ is arbitrary, and their ideas can be used to show $A(D)$ is strongly tight for the domains $D$ considered above. We show strongly tight uniform algebras are strong Banach spaces.

**Introduction.** Let $X$ and $Y$ be Banach spaces. If $T : X \to Y$ is a bounded linear operator, we say $T$ is *completely continuous* (or is a *Dunford–Pettis operator*) if $T$ takes weakly null sequences in $X$ to norm null sequences in $Y$. We say $X$ has the *Dunford–Pettis property* if every weakly compact operator $T : X \to Y$ is completely continuous. It may be shown that this is equivalent to saying weakly null sequences in $X$ tend to zero uniformly on weakly compact subsets of $X^*$. It follows easily from the latter characterization that $X$ has the Dunford–Pettis property whenever $X^*$ does.

Let $\{x_n\}$ be a sequence in $X$. We say $\{x_n\}$ is a *weakly unconditionally Cauchy* (*w.u.C.*) *series* if $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$ for every $x^* \in X^*$, and a bounded linear operator $T : X \to Y$ is an *unconditionally converging operator* if $T$ takes w.u.C. series to series that converge unconditionally in norm. This is equivalent to saying $T$ is never an isomorphic embedding

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when restricted to an isomorphic copy of $c_0$ in $X$ (see [1]). For example, completely continuous operators are unconditionally converging operators since the unit vector basis in $c_0$ is weakly null. We say $X$ has the *Pełczyński property*, or *property* "V", if every unconditionally converging operator on $X$ is weakly compact. This is equivalent to saying $X$ has the following property: if $N \subset X^*$ is a bounded subset such that every w.u.C. series $\{x_n\}$ tends to zero uniformly on $N$ (i.e., $\lim_{n\to\infty} \sup_{x^*\in N} |x^*(x_n)| = 0$), then $N$ is relatively weakly compact.

We say $\{x_n\}$ is a *weak Cauchy sequence* if $\lim_{n\to\infty} x^*(x_n)$ exists for all $x^*$ in $X^*$, and that $X$ is *weakly sequentially complete* if every weak Cauchy sequence is weakly convergent. It is not difficult to show that $X^*$ is weakly sequentially complete whenever $X$ has the Pełczyński property (see Lemma 1 of this paper). The converse of this statement is false. In [5], J. Bourgain and F. Delbaen construct a Banach space $X$ such that $X^*$ is isomorphic to $l^1$ and $X$ contains no copy of $c_0$. Since any space with the Pełczyński property that fails to contain $c_0$ is reflexive, $X$ cannot have the Pełczyński property. See J. Diestel's survey paper [14] for more information on the Dunford–Pettis and Pełczyński properties.

It was shown in Grothendieck's paper [19] that $C(K)$ spaces have the Dunford–Pettis property. Apparently, many uniform algebras of analytic functions enjoy this property as well. Chaumat showed the disk algebra has the Dunford–Pettis property in [6], and Bourgain did the the same for $H^\infty$ in [4]. Bourgain also showed in [3] that the ball-algebras and the polydisk-algebras all share this property. Using the methods in [3], Cima and Timoney found in [7] that $R(K)$ and $A(K)$ have the Dunford–Pettis property for every compact set $K \subset \mathbb{C}$ (see Section 7 for the definitions).

Pełczyński introduced his property "V" in [26], where it was shown that every $C(K)$ space has this property. It was shown independently by Kisliakov in [23] and by Delbaen in [12] that the disk algebra has the Pełczyński property. Wojtaszczyk and Delbaen showed in [29] and [13], respectively, that, for $K \subset \mathbb{C}$, all $P(K)$ spaces and some $R(K)$ and $A(K)$ spaces have the Pełczyński property. Finally, Bourgain proved in [2] that the Pełczyński property is enjoyed by the ball-algebras and the polydisk-algebras.

In [3], Bourgain considered the operator $S_g : A \to C(K)/A$ defined by $f \mapsto fg+A$ where $A$ is a uniform algebra on $K$, and $g \in C(K)$. It was shown that if $(S_g)^{**}$ is completely continuous for all $g$, then $A^*$ has the Dunford–Pettis property. This is actually a weaker form of what was proved, since something more sophisticated must be done to prove the polydisk-algebras have the Dunford–Pettis property. Cima and Timoney noted in their paper [7] that the set $A_B$ of all $g$ such that $(S_g)^{**}$ is completely continuous is a uniform algebra containing $A$, and used this observation to show that $A_B = C(K)$ when $A$ is $R(K)$ or $A(K)$. They also noted that the only



property needed was the fact that $R(K)$ and $A(K)$ are "T-invariant". T-invariant planar algebras are defined in Section 7. Using different methods, it was shown earlier by Cole and Gamelin in [8] that when $A$ is a T-invariant planar algebra, $S_g$ and hence $(S_g)^{**}$ is compact for all $g \in C(K)$.

In [8], Cole and Gamelin studied uniform algebras which have the property that $S_g$ is weakly compact for every $g$ in $C(K)$. Uniform algebras with this property were called *tight uniform algebras*. We say $A$ is *strongly tight* if $S_g$ is compact for all $g$. Although they were only concerned with tight uniform algebras, many of the algebras Cole and Gamelin considered, such as the T-invariant algebras, were shown to be strongly tight. For example, let $D$ be a bounded domain in $\mathbb{C}^n$, and let $A(D)$ be the uniform algebra of continuous functions on $\overline{D}$ that are analytic in $D$. It was noted in [8] that if $D$ is a smoothly bounded strictly pseudoconvex domain, then $A(D)$ is strongly tight. The idea involved estimates on the diagonal of Henkin's reproducing kernel. Cole and Gamelin also found that for arbitrary domains $D$, the solvability of the $\overline{\partial}$-problem is closely related to the property of $A(D)$ being tight.

We show the ideas in [8] concerning the $\overline{\partial}$-problem can also be used to show certain uniform algebras are strongly tight. Using Kerzman's estimates on solutions of the $\overline{\partial}$-problem, we show in Theorem 7 that $A(D)$ is strongly tight when $D$ is a strictly pseudoconvex domain with smooth $C^4$ boundary in some Stein manifold. When $D$ is a domain in $\mathbb{C}^n$, we may assume the boundary is $C^2$.

It follows immediately from the results of Bourgain on the Dunford–Pettis property that if $A$ is a strongly tight uniform algebra, then $A^*$ has the Dunford–Pettis property. We demonstrate that there is also a connection between strong tightness and Bourgain's work on the Pełczyński property. To show certain spaces have the Pełczyński property, Bourgain introduced the following concept in [2]: if $X$ is a closed subspace of $C(K)$, $X$ is said to be a *rich subspace* if there exists a probability measure $m$ on $K$ with the property that if $\{f_n\}$ is a bounded sequence in $X$ such that $\int |f_n|\,dm \to 0$, then $\|f_n g + X\| \to 0$ for all $g$ in $C(K)$. Bourgain shows that $X$ has the Pełczyński property whenever $X$ is a rich subspace.

Using Bourgain's work, we show in Theorem 4 that if $A$ is a strongly tight uniform algebra on $K$, then $A$ has the Pełczyński property. If $A$ is strongly tight on a metrizable space $K$, we show $A$ is a rich subspace.

The strongly tight property is also related to G. M. Henkin's work on the non-isomorphism of the ball-algebras and the polydisk-algebras. The idea in Henkin's work that we are interested in is the property of a Banach space being a separable distortion of an $L^1$-space (see Section 1). Henkin uses the concept of an analytic measure (see Section 6) to show the dual of the ball-algebra is a separable distortion of an $L^1$-space. It is not difficult to deduce from Henkin's proof that the ball-algebras are strongly tight. As a corollary to Theorem 4, we give a simple proof that the dual of a strongly tight uniform algebra on a metrizable space is a separable distortion on an $L^1$-space. Later, in Section 6, we discuss the relationship between the tight property and Henkin's analytic measures.

We also study a weaker version of Bourgain's property. We say $X$ is *strongly rich* if it rich in the above sense, and *weakly rich* if the norm convergence in $C/X$ is replaced with weak convergence (see Section 3 for the full definition). We say $m$ is a *weakly* or *strongly rich measure* if $X$ is weakly or strongly rich with respect to the measure $m$. When $A$ is a strongly tight uniform algebra, it is easy to see that any weakly rich measure is a strongly rich measure.

We show that for any uniform algebra $A$, the property of being weakly rich is equivalent to some ideas from pointwise bounded approximation theory and band theory. If $m \in M(K)$, let $m_a + m_s$ be the Lebesgue decomposition of $m$ with respect to $\mathscr{B}_{A^\perp}$, the band generated by $A^\perp$. We show in Proposition 6 that $m$ is weakly rich if and only if the natural projection $H^\infty(\mathscr{B}_{A^\perp}) \to H^\infty(m_a)$ is injective (see Section 1 for the relevant definitions). For example, a theorem by A. M. Davie (see [11] or [10]) asserts that when $A$ is a T-invariant uniform algebra in the plane, and $m = \lambda_Q =$ planar measure restricted to the non-peak points of $A$, the natural projection is a homeomorphism between the weak-star topologies and is an isometry. This is enough to show $\lambda_Q$ is a weakly rich measure. Since $A$ is strongly tight, this implies $\lambda_Q$ is strongly rich. However, the power of Davie's theorem is not necessary here. Showing $\lambda_Q$ is weakly rich involves some simple calculations with the Cauchy transform, and use of basic properties of T-invariant algebras.

Section 1 reviews some ideas from band theory.

Section 2 contains some results of Bourgain on the Dunford–Pettis property, and discusses the concept of a Bourgain algebra.

In Section 3, we study weakly and strongly rich measures, and their relations to Bourgain's work on the Pełczyński property.

In Section 4, we study tight and strongly tight uniform algebras, and prove in Theorem 4 that strongly tight uniform algebras have the Pełczyński property. As a corollary to Theorem 4, we show that when the underlying space is metrizable, the dual of a strongly tight uniform algebra is a separable distortion of an $L^1$-space.

In Section 5, we explore the connections between the weakly rich property and some ideas from pointwise bounded approximation theory.

Section 6 contains some miscellaneous ideas relating tightness to Henkin's analytic measures.



In Section 7, we study T-invariant planar uniform algebras, and show in Theorem 5 that they have the Pełczyński property. This proves $R(K)$ and $A(K)$ have the Pełczyński property for every compact planar set $K$, which is one of our main results.

In Section 8, we study the uniform algebra $A(D)$ and show in Theorem 7 that $A(D)$ has the Dunford–Pettis and Pełczyński properties when $D$ is strictly pseudoconvex with sufficiently smooth boundary. This is another of our main results.

**1. Band theory.** Let $K$ be a compact Hausdorff space, and let $M(K)$ equal $C(K)^*$, the space of regular, finite, complex Borel measures on $K$. If $\mathscr{B} \subset M(K)$, we say $\mathscr{B}$ is a *band of measures* if $\mathscr{B}$ is a closed subspace of $M(K)$ and has the property that when $\mu \in \mathscr{B}$, $\nu \in M(K)$, and $\nu \ll \mu$, then $\nu \in \mathscr{B}$. The Lebesgue decomposition theorem says that if $\mu \in M(K)$ then $\mu$ can be uniquely written as $\mu = \mu_a + \mu_s$ where $\mu_a \in \mathscr{B}$ and $\mu_s$ is singular to every element of $\mathscr{B}$. If $\mathscr{B}$ is a band, the *complementary band* $\mathscr{B}'$ of $\mathscr{B}$ is the collection of measures singular to every measure in $\mathscr{B}$. It follows that $M(K) = \mathscr{B} \oplus_{l^1} \mathscr{B}'$. It is a well-known fact that if $\mathscr{B}$ is a band, then there exists a measure space $(\Omega, \Sigma, \sigma)$ such that $\mathscr{B} \cong L^1(\sigma)$.

If $\mathscr{B}$ is a band, we define $L^\infty(\mathscr{B})$ to be the space of uniformly bounded families of functions $F = \{F_\nu\}_{\nu \in \mathscr{B}}$ where $F_\nu \in L^\infty(\nu)$ and $F_\nu = F_\mu$ a.e. $[d\nu]$ whenever $\nu \ll \mu$. The norm in $L^\infty(\mathscr{B})$ is given by $\|F\| = \sup_{\nu\in\mathscr{B}} \|F_\nu\|_{L^\infty(\nu)}$. The pairing $\langle \nu, F\rangle = \int F_\nu \, d\nu$ for $\nu \in \mathscr{B}$ and $F \in L^\infty(\mathscr{B})$ defines an isometric isomorphism between $L^\infty(\mathscr{B})$ and $\mathscr{B}^*$. If $E$ is a subset of $L^\infty(\mathscr{B})$, and $E_\mu = \{G_\mu \mid G \in E\}$ for $\mu \in \mathscr{B}$, it is not difficult to show that $F$ is in the weak-star closure of $E$ if and only if $F_\mu$ is in the weak-star closure of $E_\mu$ for every $\mu \in \mathscr{B}$. If $A$ is a uniform algebra on $K$, we define $H^\infty(\mathscr{B})$ to be the weak-star closure of $A$ in $L^\infty(\mathscr{B})$. If $\mu \in M(K)$, $H^\infty(\mu)$ is the weak-star closure of $A$ in $L^\infty(\mu)$. If $\mu \in \mathscr{B}$, there is a natural projection $H^\infty(\mathscr{B}) \to H^\infty(\mu)$ defined by $F \mapsto F_\mu$.

It is easy to see that the intersection of an arbitrary collection of bands is a band. If $\mathscr{S}$ is an arbitrary subset of $M(K)$, we define the *band generated by* $\mathscr{S}$ to be the smallest band containing $\mathscr{S}$. If $A$ is a uniform algebra on $K$, we define $\mathscr{B}_{A^\perp}$ to be the band generated by the measures in $A^\perp$, and $\mathscr{S}_{A^\perp}$ to be the band complement to $\mathscr{B}_{A^\perp}$. Since $A^* \cong M(K)/A^\perp$ it follows from the Lebesgue decomposition with respect to $\mathscr{B}_{A^\perp}$ that

$$A^* \cong \frac{\mathscr{B}_{A^\perp}}{A^\perp} \oplus_{l^1} \mathscr{S}_{A^\perp} \quad \text{and} \quad A^{**} \cong H^\infty(\mathscr{B}_{A^\perp}) \oplus_{l^\infty} L^\infty(\mathscr{S}_{A^\perp}),$$

where the above isomorphisms are isometries. For more information on band theory, see [8] or [10].

If $X$ is a Banach space, we say $X$ is a *separable distortion of an $L^1$-space* if there exists a separable Banach space $Y$ and a measure space $(\Omega, \Sigma, \sigma)$

such that $X$ is isomorphic to $Y \oplus_{l^1} L^1(\sigma)$. It follows from the above that when $A$ is a uniform algebra and $\mathscr{B}_{A^\perp}/A^\perp$ is separable, then $A^*$ is a separable distortion of an $L^1$-space. Many of the examples of uniform algebras we consider in this paper are shown to have the property that $\mathscr{B}_{A^\perp}/A^\perp$ is separable. The separable distortion property was an idea used in Henkin's work in [20] (also, see [27]) to prove some non-isomorphism results on the ball-algebras and polydisk-algebras. More precisely, it is shown, without considering $\mathscr{B}_{A^\perp}$, that the duals of the ball-algebras are separable distortions of $L^1$-spaces, but the duals of the polydisk-algebras are not isomorphic to separable distortions of $L^1$-spaces in dimensions greater than one.

**2. Bourgain algebras.** Let $K$ be a compact topological space, and let $X$ be a closed subspace of $C(K)$. If $g \in C(K)$, let $S_g : X \to C/X$ be the Hankel-type operator $f \mapsto gf + X$. We may identify the subspace $X^{\perp\perp}$ of $C^{**}$ with $X^{**}$ by standard duality theory. Note that $(S_g)^{**} : X^{**} \to C^{**}/X^{**}$ is the operator $x^{**} \mapsto gx^{**} + X^{**}$, where $gx^{**}$ is the action of the second adjoint of the multiplication operator on $x^{**}$. Let $X_b$ and $X_B$ be the sets of those $g \in C(K)$ such that $S_g$ and $(S_g)^{**}$, respectively, are completely continuous. These sets, called the *Bourgain algebras of $X$*, were first defined by Cima and Timoney in [7], where it was shown that $X_b$ and $X_B$ are uniformly closed subalgebras of $C(K)$.

The next result follows from the work of Bourgain in [3].

PROPOSITION 1. *Suppose $X$ is a closed subspace of $C(K)$.*

(a) *If $X_B = C(K)$, and $\{x_n^*\}$ is a bounded sequence in $X^*$ that fails to tend to zero on some weakly compact set in $X^{**}$, then $\{x_n^*\}$ fails to tend to zero uniformly on some w.u.C. series in $X$.*

(b) *If $X_b = C(K)$, then the same conclusion holds if we replace $X^{**}$ with $X$.*

The above results are connected to the Dunford–Pettis property by the following well-known lemma.

LEMMA 1. *Suppose $X$ is a Banach space, and $N \subset X^*$ is a bounded set with the property that every sequence in $N$ has a weak Cauchy subsequence. Then every w.u.C. series in $X$ tends to zero uniformly on $N$.*

Proof. Given a w.u.C. series $\{x_n\}$ in $X$, define an operator $T : X^* \to l^1$ by $T(x^*) = (x^*(x_k))_{k=1}^\infty$. Since $l^1$ has the Schur property, $T(N)$ is relatively norm compact, and hence totally bounded. It then follows that given any $\varepsilon > 0$, there exists an integer $J$ such that $\sum_{k=J}^\infty |x^*(x_k)| < \varepsilon$ for all $x^* \in N$, which proves the result. ∎



By applying the proposition, and using the lemma where $N$ is a weakly compact subset of $X^*$, Bourgain's results may now be stated in the following way.

THEOREM 1 (Bourgain, [3]). *Let $K$ be a compact space and let $X$ be a closed subspace of $C(K)$. Then $X_b = C(K)$ implies $X$ has the Dunford–Pettis property, and $X_B = C(K)$ implies $X^*$ has the Dunford–Pettis property.*

These results were used by Bourgain to show certain spaces of analytic and smooth functions have dual spaces with the Dunford–Pettis property. Cima and Timoney introduced the algebras $X_b$ and $X_B$ in [7], and used the connection with Bourgain's work to show certain planar uniform algebras have dual spaces with the Dunford–Pettis property.

**3. Rich subspaces.** Let $K$ be compact, and let $X$ be a closed subspace of $C(K)$. We say $X$ is a *weakly* (resp. *strongly*) *rich subspace* if there exists a measure $m \in M(K)$ with the following property: whenever $\{f_n\}$ is a bounded sequence in $X$ such that $\int |f_n|\, d|m| \to 0$, we have

$$S_g(f_n) = f_n g + X \xrightarrow{w} 0$$

respectively

$$\|S_g(f_n)\| = \|f_n g + X\| \to 0$$

for every $g \in C$. In this case we say $m$ is a *weakly* (resp. *strongly*) *rich measure* for $X$. If $A$ is a uniform algebra on $K$ which is a weakly or strongly rich subspace, we will say $A$ is a *weakly* or *strongly rich uniform algebra on* $K$, respectively.

The following proposition shows that when dealing with uniform algebras, in order to demonstrate a measure is weakly or strongly rich, it suffices only to look at a certain "part" of the measure.

PROPOSITION 2. *Let $A$ be a uniform algebra on a compact space $K$. Let $m \in M(K)$, and let $m_a + m_s$ be the Lebesgue decomposition of $m$ with respect to $\mathscr{B}_{A^\perp}$, the band generated by the measures in $A^\perp$. Then $m$ is weakly (resp. strongly) rich if and only if $m_a$ is weakly (resp. strongly) rich.*

Proof. Clearly if $m_a$ is strongly or weakly rich, then $m$ is strongly or weakly rich, respectively.

Assume $m_a$ is not strongly rich. We may then find functions $\{f_n\} \subset A$ with $\|f_n\| \le M$ for some $M$ and a $g \in C$ such that $\int |f_n|\, d|m_a| \to 0$ and $\|gf_n + A\| \ge \varepsilon > 0$ for all $n \ge 1$. Choose $\{\mu_n\} \subset A^\perp$ so that $\|\mu_n\| = 1$ and $\int gf_n\, d\mu_n \ge \varepsilon/2$. Let $\nu_1 = |m_a| + \sum(|\mu_n|/2^n)$ and $\nu_2 = |m_s|$, so $\nu_1 \perp \nu_2$.

CLAIM. *There exist functions $\{g_n\} \subset A$ with $\|g_n\| \le 1$ such that $g_n \to 1$ a.e. $[\nu_1]$ and $g_n \to 0$ a.e. $[\nu_2]$.*

To prove the claim, let $\nu = \nu_1 - \nu_2$. A direct computation shows that $\|\nu + A^\perp\| = \|\nu\|$. Hence, $\|\nu\|_{A^*} = \|\nu\|$, so we may find $\{h_n\} \subset A$ with $\|h_n\| \le 1$ so that $\int h_n\,d\nu \to \|\nu\|$. Let $G$ be a weak-star accumulation point of $\{h_n\}$ in $L^\infty(\nu)$, so $\|G\|_\infty \le 1$ and $\int G\,d\nu = \|\nu\|$. It follows that $G = d|\nu|/d\nu$, so $G = 1$ a.e. $[\nu_1]$ and $G = -1$ a.e. $[\nu_2]$. Since $G$ is a weak accumulation point of $\{h_n\}$ in $L^1(\nu)$, after taking convex combinations and passing to a subsequence if necessary, we may assume $h_n \to G$ pointwise a.e. $[\nu]$. Letting $g_n = (h_n + 1)/2$ proves the claim.

We may now pass to a subsequence $\{g_{k_n}\}$ so that $|\int g_{k_n} g f_n\,d\mu_n| \ge \varepsilon/4$ for $n \ge 1$, and $\lim_{n\to\infty} \int |g_{k_n} f_n|\,d|m_a| = \lim_{n\to\infty} \int |g_{k_n} f_n|\,d|m_s| = 0$. It now follows that $\|g_{k_n} f_n g + A\| \ge \varepsilon/4$, $\lim_{n\to\infty} \int |g_{k_n} f_n|\,d|m| = 0$, $g_{k_n} f_n \in A$, and $\|g_{k_n} f_n\| \le M$, so $m$ is not strongly rich.

We have now shown that $m_a$ is strongly rich if $m$ is strongly rich. We can use the same proof for the weak case. ∎

In this paper, we consider a uniform algebra $A$, and attempt to show that $A$ has certain properties. Many of the properties we study involve showing that another uniform algebra, arising from $A$, is all continuous functions. The property of being weakly or strongly rich is no exception. Given a measure $m \in M(K)$, and a closed subspace $X$ of $C(K)$, define $(X,m)_{wr}$ and $(X,m)_{sr}$ to be the sets of those $g \in C(K)$ such that

$$S_g(f_n) = f_n g + X \xrightarrow{w} 0 \quad \text{and} \quad \|S_g(f_n)\| = \|f_n g + X\| \to 0$$

respectively, whenever $\{f_n\}$ is a bounded sequence in $X$ such that

$$\int |f_n|\,d|m| \to 0.$$

PROPOSITION 3. *Let $X$ be a closed subspace of $C(K)$. Then $(X,m)_{wr}$ and $(X,m)_{sr}$ are closed subalgebras of $C(K)$ with $X_b$ containing $(X,m)_{sr}$.*

Proof. It is easy to see that $(X,m)_{wr}$ and $(X,m)_{sr}$ are closed. Now, suppose $g, h \in (X,m)_{sr}$, and $\{f_n\}$ is a bounded sequence in $X$ such that $\int |f_n|\,d|m| \to 0$. Then we may find $j_n \in X$ and $k_n \in C(K)$ with $\|k_n\| \to 0$ and $gf_n + j_n = k_n$. Since $\{j_n\}$ is bounded and $\int |j_n|\,d|m| \to 0$, we may similarly write $hj_n + \widetilde{j}_n = \widetilde{k}_n$, so $ghf_n - \widetilde{j}_n = hk_n - \widetilde{k}_n$. Therefore, $\|ghf_n + X\| \to 0$, and so $gh \in (X,m)_{sr}$. Hence, $(X,m)_{sr}$ is an algebra. Since weakly null sequences in $X$ are those bounded sequences tending to zero pointwise on $K$, $S_g$ is completely continuous whenever $g \in (X,m)_{sr}$. In other words, $(X,m)_{sr} \subseteq X_b$.

Let $g, h \in (X,m)_{wr}$, and assume that $gh \notin (X,m)_{wr}$, so we may find a bounded sequence $\{f_n\}$ in $X$ with $\int |f_n|\,d|m| \to 0$, and a $\mu \in X^\perp$ such that $|\int ghf_n\,d\mu| \ge \varepsilon$ for some $\varepsilon > 0$ and for all $n$. After multiplying by constants of modulus 1, we may assume $\int ghf_n\,d\mu \ge \varepsilon$. Since $gf_n + X \xrightarrow{w} 0$, we may find functions $F_n$, convex combinations of functions from $\{f_k\}_{k=n}^\infty$, such that $\|gF_n + X\| \to 0$. Hence, $\{F_n\}$ is bounded in $X$, $\int hgF_n\,d\mu \ge \varepsilon$, and

$$\int |F_n|\,d|m| \le \sup_{k\ge n} \int |f_k|\,d|m|,$$

so $\int |F_n|\,d|m| \to 0$. Now, since $\|gF_n + X\| \to 0$, there exist functions $j_n \in X$ and $k_n \in C(K)$ with $\|k_n\| \to 0$ such that $gF_n + j_n = k_n$. Since $\{j_n\}$ is bounded and $\int |j_n|\,d|m| \to 0$, it follows that $hj_n + X \xrightarrow{w} 0$, so $\int hj_n\,d\mu \to 0$. Since $hgF_n + hj_n = hk_n$, and $\|hk_n\| \to 0$, we have $\int hgF_n\,d\mu \to 0$, a contradiction. Hence, $(X,m)_{wr}$ is an algebra. ∎



By definition, $(X,m)_{wr}$ or $(X,m)_{sr}$ equal $C(K)$ if and only if $X$ is weakly or strongly rich, respectively. Hence, to exhibit richness, it suffices to consider enough functions to generate $C(K)$ as a Banach algebra. Note that if $A$ is a uniform algebra, then $(A,m)_{wr}$ and $(A,m)_{sr}$ are uniform algebras containing $A$. Note also that it now follows from Theorem 1 that $X$ has the Dunford–Pettis property when $X$ is a strongly rich subspace. Moreover, we have the following.

THEOREM 2 (Bourgain, [2]). *Let $K$ be a compact space and let $X$ be a closed subspace of $C(K)$. If $X$ is a strongly rich subspace, then $X$ has the Pełczyński property and $X^*$ is weakly sequentially complete.*

The following technical result, which is not difficult to deduce from the proof of Theorem 2 (see Proposition 1 in [2]), will also be used.

THEOREM 2′ (Bourgain, [2]). *Let $K$ be a compact space, and let $X$ be a closed subspace of $C(K)$. Suppose that for every $g \in C(K)$ there exists a probability measure $m_g$ with the property that when $\{f_n\}$ is a bounded sequence in $X$ and*

$$\int |f_n|\,dm_g \to 0,$$

*then*

$$\|f_n g + X\| \to 0.$$

*Then $X$ has the Pełczyński property, and $X^*$ is weakly sequentially complete.*

The notion of a rich subspace first appeared in Bourgain's paper [2] (also, see [30]), where it was shown that certain spaces of analytic and smooth functions have the Pełczyński property. Recall that the weak sequential completeness of $X^*$ follows from the Pełczyński property.

Let $A$ be a uniform algebra, and consider the space $\mathscr{B}_{A^\perp}/A^\perp$. In many of the examples we consider in this paper, we show that $\mathscr{B}_{A^\perp}/A^\perp$ is separable. The significance of $A$ having this property is noted in Section 1, and we now show this has applications to the weakly rich property.

PROPOSITION 4. *Let $A$ be a uniform algebra on a compact space $K$. If $\mathscr{B}_{A^\perp}/A^\perp$ is separable, then there exists a weakly rich measure for $A$.*

Proof. Let $\{\nu_n + A^\perp\}$ be a norm dense sequence in $\mathscr{B}_{A^\perp}/A^\perp$, and define a measure $m$ by

$$m = \sum_{n=1}^{\infty} \frac{|\nu_n|}{\|\nu_n\| 2^n}.$$

Then, if $\{f_n\}$ is a bounded sequence in $A$, and $\int |f_n|\, dm \to 0$, it follows that $\int f_n\, d\nu \to 0$ for every $\nu \in \mathscr{B}_{A^\perp}$. Hence, if $g \in C$, and $\mu \in A^\perp$, then $\int f_n g\, d\mu \to 0$. Since the dual of $C/A$ is $A^\perp$, this implies $f_n g + A \xrightarrow{w} 0$ in $C/A$. ■

**4. Tight uniform algebras.** Although many of the ideas and results in this section and the sections that follow can be applied to closed subspaces of $C(K)$, we now consider only uniform algebras.

Let $A$ be a uniform algebra on a compact space $K$. We say $A$ is *tight* (resp. *strongly tight*) on $K$ if for every $g \in C(K)$, the operator $S_g$ defined in Section 2 is weakly compact (resp. compact). Let $A_{cg}$ and $A_{CG}$ be the sets of those $g \in C(K)$ such that $S_g$ is weakly compact and compact, respectively. Then $A$ is tight or strongly tight if and only if $A_{cg}$ or $A_{CG}$ equal $C(K)$, respectively.

The notion of tightness was introduced in [8], where it was shown that $A_{cg}$ is a closed subalgebra of $C(K)$. The same is true of $A_{CG}$. To see that $A_{CG}$ is closed, note that the map $C \to L(A, C/A)$ given by $g \mapsto S_g$ satisfies $\|S_g\| \le \|g\|$ and is therefore a bounded linear operator. Since the compact operators are closed, $A_{CG}$ is closed (similarly for $A_{cg}$). To see that $A_{CG}$ is an algebra, let $g, h \in A_{CG}$, and let $\{f_n\}$ be a bounded sequence in $A$. After passing to a subsequence we may find $j_n \in A$ and $k_n \in C(K)$ with $\|k_n\| \to 0$ such that $gf_n - H + j_n = k_n$ for some $H \in C(K)$, and similarly after another subsequence $hj_n - \widetilde{H} + \widetilde{j}_n = \widetilde{k}_n$, so $ghf_n - (hH - \widetilde{H}) - \widetilde{j}_n = hk_n - \widetilde{k}_n$, and $ghf_n + A \xrightarrow{\text{norm}} (hH - \widetilde{H}) + A$. Hence, $gh \in A_{CG}$. Therefore, $A_{cg}$ and $A_{CG}$ are uniform algebras containing $A$ with $A_{CG} \subseteq A_{cg}$. Hence, to show $A$ is tight or strongly tight, it suffices to consider enough functions to generate $C(K)$ as a Banach algebra.

If $A$ is a uniform algebra on $K$, it is not difficult to show that $A^{**} + C(K)$ is a closed subspace of $C(K)^{**}$. In [8], Cole and Gamelin prove the following theorem, which illustrates the central application of tight uniform algebras.

THEOREM 3 (Cole–Gamelin, [8]). *Let $K$ be a compact space, and let $A$ be a uniform algebra on $K$. Then the following are equivalent:*

(a) *$A$ is a tight uniform algebra on $K$.*
(b) *$A^{**} + C(K)$ is a (closed) subalgebra of $C(K)^{**}$.*



The following is an analog of a result in [8] on tight uniform algebras.

PROPOSITION 5. *Suppose $A$ is a uniform algebra on $K$ and $E$ is a closed subset of $K$ with the property that $A|_E$ is closed in $C(E)$. If $A$ is strongly tight on $K$, then $A|_E$ is strongly tight on $E$.*

Proof. Let $g \in C(E)$, and let $\overline{g}$ be any extension of $g$ to $C(K)$. Suppose $\{f_n|_E\}$ is a bounded sequence in $A|_E$. By the open mapping theorem, we may assume $\{f_n\}$ is a bounded sequence in $A$. Since $A$ is strongly tight on $K$, we may find an $h \in C(K)$ such that, after passing to a subsequence, $\overline{g} f_n + A \xrightarrow{\text{norm}} h + A$. We may therefore find functions $j_n \in A$, $k_n \in C(K)$, with $\|k_n\| \to 0$, such that $\overline{g} f_n - h + j_n = k_n$. Restricting to $E$, we have $gf_n|_E - h|_E + j_n|_E = k_n|_E$, and it follows that

$$gf_n|_E + A|_E \xrightarrow{\text{norm}} h|_E + A|_E.$$

Hence, the operator

$$S_g : A|_E \to C(E)/A|_E$$

is compact. ■

We would now like to make some remarks on the relationships between tightness and the other properties we have considered so far. If $S_g$ is compact, then $(S_g)^{**}$ is compact and therefore completely continuous. Hence $A_{CG} \subseteq A_B$ for any uniform algebra $A$. Therefore, if $A$ is strongly tight, it follows from Theorem 1 that $A^*$ has the Dunford–Pettis property. On the other hand, if $A$ has the Dunford–Pettis property, then weakly compact operators from $A$ are completely continuous, so $A_{cg} \subseteq A_b$. Similarly, if $A^{**}$ has the Dunford–Pettis property, then $A_{cg} \subseteq A_B$. Recall that completely continuous operators are unconditionally converging operators. Therefore, if $A$ has the Pełczyński property, then completely continuous operators on $A$ are weakly compact, which implies $A_b \subseteq A_{cg}$. Now, since $(A, m)_{sr} \subseteq A_b$ for any measure $m$, it follows from Theorem 2 that any strongly rich uniform algebra is tight.

As we noted above, the fact that $A^*$ has the Dunford–Pettis property when $A$ is a strongly tight uniform algebra follows immediately from Bourgain's result in Theorem 1. We will now show that Bourgain's work on the Pełczyński property can be applied to these algebras as well.

THEOREM 4. *Let $A$ be a uniform algebra on a compact space $K$. If $A$ is strongly tight, then $A$ and $A^*$ have the Dunford–Pettis property, $A$ has the Pełczyński property, and $A^*$ is weakly sequentially complete. In particular, if $K$ is metrizable, then $A$ is a strongly rich uniform algebra.*

Proof. Assume $A$ is strongly tight on $K$. It follows immediately from Theorem 1 and the remarks above that $A^*$, and hence $A$, has the Dunford–Pettis property.

CLAIM. *If $B$ is a separable subspace of $C(K)$, then there exists a probability measure $m$ with the property that when $\{f_n\}$ is a bounded sequence in $A$ such that $\int |f_n|\,dm \to 0$, then $\|f_n g + A\| \to 0$ for every $g$ in $B$.*

The fact that $A$ has the Pełczyński property, which implies $A^*$ is weakly sequentially complete, now follows from Theorem 2′.

Proof of the Claim. We may write

$$A^* \cong \frac{\mathscr{B}_{A^\perp}}{A^\perp} \oplus_{l^1} \mathscr{S}_{A^\perp}.$$

Note that $S_g^*$, the adjoint of $S_g$, maps $A^\perp$ into $A^*$ and satisfies $S_g^*(A^\perp) \subseteq \mathscr{B}_{A^\perp}/A^\perp$. Let $M \subseteq \mathscr{B}_{A^\perp}/A^\perp$ be defined by

$$M = \overline{\mathrm{sp}} \bigcup_{g\in B} S_g^*(A^\perp).$$

If $g \in C(K)$, then $S_g^*$ is compact, which implies $S_g^*(A^\perp)$ is separable. Therefore, since $B$ is separable, $M$ is separable. Let $\{\nu_n + A^\perp\}$ be a dense subset of $M$, and define

$$m = \sum_{n=1}^{\infty} \frac{|\nu_n|}{\|\nu_n\| 2^n}.$$

If $\{f_n\}$ is a bounded sequence in $A$, and $\int |f_n|\,dm \to 0$, then $\int f_n\,d\nu \to 0$ whenever $\nu + A^\perp \in M$. In particular, if $\mu \in A^\perp$ and $g \in B$, then $\int f_n g\,d\mu \to 0$. Hence, $S_g(f_n) = f_n g + A \xrightarrow{w} 0$ in $C/A$. Since $S_g$ is compact, $\|f_n g + A\| \to 0$, proving the claim.

When $K$ is metrizable, we can take $B = C(K)$, and $m$ will be a strongly rich measure. ■

It is not difficult to show that in general,

$$\mathscr{B}_{A^\perp}/A^\perp = \overline{\mathrm{sp}} \bigcup_{g\in C(K)} S_g^*(A^\perp).$$

The above proof, along with the notes in Section 1, now yields the following.

COROLLARY. *If $A$ is strongly tight and $K$ is metrizable, then $\mathscr{B}_{A^\perp}/A^\perp$ is separable, and $A^*$ is a separable distortion of an $L^1$-space.*

When $A$ is the ball-algebra, this recovers G. M. Henkin's result in [20] (also, see [27]) that $A^*$ is a separable distortion of an $L^1$-space. Henkin's proof involved the concept of an analytic measure, but it is not difficult to see from Henkin's proof that the ball-algebras are strongly tight. The fact that the ball-algebras are strongly tight will also follow from Theorem 7 in this paper.



**5. The Davie property.** Let $A$ be a uniform algebra on $K$. If $m \in M(K)$, let $m_a + m_s$ be the Lebesgue decomposition of $m$ with respect to $\mathscr{B}_{A^\perp}$, the band generated by the annihilating measures. Let $H^\infty(\mathscr{B}_{A^\perp})$ and $H^\infty(m_a)$ denote the weak-star closure of $A$ in $L^\infty(\mathscr{B}_{A^\perp})$ and $L^\infty(m_a)$ respectively. We will now show the weakly rich property is equivalent to some ideas from pointwise bounded approximation theory.

PROPOSITION 6. *Let $A$ be a uniform algebra on a compact space $K$, and let $m$ be an element of $M(K)$. Then the following are equivalent.*

(a) *The natural projection $H^\infty(\mathscr{B}_{A^\perp}) \to H^\infty(m_a)$ is one-to-one.*

(b) *If $\{f_n\}$ is a bounded sequence in $A$ such that $\int |f_n|\,d|m| \to 0$, then $f_n \xrightarrow{w^*} 0$ in $L^\infty(\mu)$ for every $\mu \in A^\perp$.*

(c) *$m$ is a weakly rich measure for $A$.*

*Furthermore, if the above hold, and $K$ is metrizable, then $\mathscr{B}_{A^\perp}/A^\perp$ is separable and $A^*$ is a separable distortion of an $L^1$-space.*

The above ideas are related to Davie's work in [11] (see also [10]) where it was shown that if $A = R(K)$ for a compact planar set $K$, and $\lambda_Q$ is planar measure restricted to the non-peak points of $A$, then the natural projection is an isometry between $H^\infty(\mathscr{B}_{A^\perp})$ and $H^\infty(\lambda_Q)$, and a weak-star homeomorphism. This can be shown (see [10]) to be equivalent to saying that when $f \in H^\infty(\lambda_Q)$, there exists a sequence $\{f_n\}$ in $A$ with $\|f_n\| \le \|f\|$ such that $f_n \to f$ pointwise a.e. $[\lambda_Q]$. The following lemma is elementary.

LEMMA 2. *If $\{f_n\}$ is a bounded sequence in $A$, then the following are equivalent.*

(a) *$f_n \xrightarrow{w^*} 0$ in $L^\infty(\mu)$ for all $\mu \in A^\perp$.*

(b) *$f_n g + A \xrightarrow{w} 0$ for all $g \in C(K)$.*

Proof of Proposition 6. To show (a) implies (b), we suppose $\{f_n\}$ is a bounded sequence in $A$ such that $\int |f_n|\,d|m| \to 0$. Then $\int |f_n|\,d|m_a| \to 0$, so $f_n \xrightarrow{w^*} 0$ in $L^\infty(m_a)$. Let $\mu \in A^\perp$, and assume $f_n$ fails to tend to zero weak-star in $L^\infty(\mu)$. After passing to a subsequence, we may assume 0 is not in the weak-star closure of $\{f_n\}$ in $L^\infty(\mu)$. Let $F$ be a weak-star cluster point of $\{f_n\}$ in $H^\infty(\mathscr{B}_{A^\perp})$. Then $F_{m_a}$ is a weak-star cluster point of $\{f_n\}$ in $L^\infty(m_a)$, which implies $F_{m_a} = 0$. Since the natural projection is one-to-one, $F_\mu = 0$. Since $F_\mu$ is a weak-star cluster point of $\{f_n\}$ in $L^\infty(\mu)$, this implies 0 is in the weak-star closure of $\{f_n\}$ in $L^\infty(\mu)$, a contradiction. Hence, (a) implies (b).

Now, (b) implies (c) follows from the lemma, so assume $m$ is weakly rich. Proposition 2 implies $m_a$ is weakly rich. Let $F \in H^\infty(\mathscr{B}_{A^\perp})$, and suppose $F_{m_a} = 0$.

CLAIM. *$F_\mu = 0$ for every $\mu \in \mathscr{B}_{A^\perp}$.*

Proof. Suppose there exists some $\mu \in \mathscr{B}_{A^\perp}$ with $F_\mu \neq 0$. Since every measure in $\mathscr{B}_{A^\perp}$ is absolutely continuous with respect to some measure in $A^\perp$ (see [10], V.17.11), we may assume $\mu \in A^\perp$. Since $A^{**}$ is isometrically isomorphic to $H^\infty(\mathscr{B}_{A^\perp}) \oplus_{l^\infty} L^\infty(\mathscr{S}_{A^\perp})$, we may find a net $\{f_\alpha\}$ in $A$ with $\|f_\alpha\| \le \|F\|$ such that $f_\alpha \xrightarrow{w^*} F$ in $H^\infty(\mathscr{B}_{A^\perp})$. Then $f_\alpha \xrightarrow{w^*} 0$ in $L^\infty(m_a)$, and $f_\alpha \xrightarrow{w^*} F_\mu$ in $L^\infty(\mu)$. Let $\nu = |m_a| + |\mu|$. Then $f_\alpha \xrightarrow{w^*} G$ in $L^\infty(\nu)$ for a $G \in L^\infty(\nu)$ such that $G = F_\mu$ a.e. $[\mu]$, and $G = 0$ a.e. $[m_a]$. Since $f_\alpha \xrightarrow{w} G$ in $L^1(\nu)$, we may find a sequence $\{f_n\}$ in $A$, each $f_n$ a convex combination of the functions in $\{f_\alpha\}$, such that $\int |f_n - G|\,d\nu \to 0$. Then $\|f_n\| \le \|F\|$, and $\int |f_n|\,d|m_a| \to 0$, so $f_n \xrightarrow{w^*} 0$ in $L^\infty(\mu)$. Also, since $f_n \xrightarrow{\text{norm}} G$ in $L^1(\mu)$, $G = 0$ in $L^\infty(\mu)$, a contradiction, which proves the claim.

We have now shown the natural projection is one-to-one, proving that (c) implies (a).

The natural projection described above is the adjoint of the injection

$$\frac{L^1(m_a)}{L^1(m_a) \cap A^\perp} \to \frac{\mathscr{B}_{A^\perp}}{A^\perp}.$$

If $K$ is metrizable, then $L^1(m_a)$ is separable, and so $L^1/L^1 \cap A^\perp$ is separable. If the projection is injective, then the injection has dense range, which implies $\mathscr{B}_{A^\perp}/A^\perp$ is separable. The fact that $A^*$ is a separable distortion of an $L^1$-space now follows from the material in Section 1. ∎

Combining Propositions 4 and 6, we have the following.

COROLLARY. *If $K$ is metrizable, then the following are equivalent.*

(a) *There exists a weakly rich measure for $A$.*

(b) *$\mathscr{B}_{A^\perp}/A^\perp$ is separable.*

**6. Analytic measures.** Let $A$ be a uniform algebra on $K$, and let $S$ be any subset of $K$. We say a bounded sequence $\{f_n\}$ in $A$ is an *$S$-Montel sequence* if $f_n(z) \to 0$ for every $z \in S$. A measure $\mu \in M(K)$ is called an *$S$-analytic measure* if $\int f_n d\mu \to 0$ whenever $\{f_n\}$ is an $S$-Montel sequence in $A$. Let $AM(A,S)$ denote the set of all $S$-analytic measures. It is easily seen that $AM(A,S)$ is a closed subspace of $M(K)$ containing $A^\perp$ and all complex representing measures of points in $S$. Let $A_H(S)$ be the set of all $g \in C(K)$ with the property that $g d\mu \in AM(A,S)$ whenever $\mu \in AM(A,S)$. Since $AM(A,S)$ is closed, $A_H(S)$ is closed, and is a uniform algebra containing $A$. The following proposition is elementary.

PROPOSITION 7. *Let $A$ be a uniform algebra on $K$, and let $S$ be a subset of $K$. Then the following are equivalent.*



(a) *$A_H(S) = C(K)$.*

(b) *$AM(A,S)$ is a band.*

(c) *If $\mu \in AM(A,S)$, and $\{f_n\}$ is an $S$-Montel sequence in $A$, then $f_n \xrightarrow{w^*} 0$ in $L^\infty(\mu)$.*

The notion of an analytic measure originated in Henkin's work in [20], a version of which is in [27]. When $A = A(\mathbb{B}_n)$, and $S$ is the open ball, it is shown in [27] that $AM(A,S)$ is a band, and that $AM(A,S)/A^\perp$ is separable. It now follows that $A^*$ is a separable distortion of an $L^1$-space. One of the main results in Henkin's work is that the dual of $A(\mathbb{T}^m)$ is not isomorphic to a separable distortion of an $L^1$-space when $m$ is greater than 1, so $A(\mathbb{B}_n)$ is not isomorphic to $A(\mathbb{T}^m)$. It will follow from Proposition 8 below that, for the ball-algebras, in order to show $AM(A,S)$ is a band, it suffices to demonstrate that $A(\mathbb{B}_n)$ is tight on $\overline{\mathbb{B}}_n$. It is not difficult to extract the fact that $A(\mathbb{B}_n)$ is strongly tight from the proof in [27].

If $S \subseteq K$, we say $A$ has *property* $(P_S)$ if the following holds: if $\{f_n\}$ is a bounded sequence in $A$, and $g \in C(K)$, and $f_n(z) \to g(z)$ for every $z$ lying in S, then $g \in A$. For example $A(D)$, the uniform algebra of continuous functions on $\overline{D}$ that are analytic in $D$, has property $(P_D)$ for any open bounded domain $D$ in $\mathbb{C}^n$.

PROPOSITION 8. *Suppose $A$ is a uniform algebra on $K$, and $S$ is a subset of $K$ such that $A$ has property $(P_S)$. Then $A_{cg} \subseteq A_H(S)$. In particular, if $A$ is tight, then $AM(A,S)$ is a band.*

Proof. Assume $A$ has property $(P_S)$, and let $g \in A_{cg}$. Suppose $g \notin A_H(S)$. Then there exists a $\mu$ in $AM(A,S)$, an $S$-Montel sequence $\{f_n\}$ in $A$, and some $\varepsilon > 0$ such that, after passing to a subsequence and multiplying the $\{f_n\}$ by constants of modulus 1 if necessary, $\int f_n g\,d\mu \ge \varepsilon$ for $n = 1, 2, 3, \ldots$ Since $S_g$ is weakly compact, after taking convex combinations we may assume $f_n g + A \xrightarrow{\text{norm}} h + A$ for some $h \in C(K)$. Let $j_n \in A$ and $k_n \in C(K)$ with $\|k_n\| \to 0$ such that $f_n g - h + j_n = k_n$. Note that this implies the sequence $\{j_n\}$ is bounded in $A$. Since $\{f_n\}$ is $S$-Montel, $j_n \to h$ pointwise on $S$, which implies by property $(P_S)$ that $h \in A$, and so $\{j_n - h\}$ is an $S$-Montel sequence. This implies $\int f_n g\,d\mu \to 0$, a contradiction. ∎

It now follows that $AM(A(D), D)$ is a band whenever $A(D)$ is tight on $\overline{D}$.

**7. T-invariant uniform algebras.** Let $K$ be a compact subset of $\mathbb{C}$, and define $P(K)$ and $R(K)$ to be the closure in $C(K)$ of the polynomials on $K$ and the rational functions on $K$ with poles off $K$, respectively. Define $A(K)$ to be the continuous functions on $K$ that are analytic in the interior of $K$.

Let $g \in C_c^1(\mathbb{C})$, and let $K \subset \mathbb{C}$ be compact. For $f \in C(K)$, let $f = 0$ off $K$, and define, for $z \in K$,

$$(T_g f)(z) = f(z)g(z) + \frac{1}{\pi} \iint \frac{1}{w - z} \frac{\partial g}{\partial \bar{z}}(w) f(w)\, dx\, dy(w).$$

Then $T_g : C(K) \to C(K)$ is a bounded linear operator. We say a uniform algebra $A$ on $K$ is *T-invariant* if $A$ contains $R(K)$ and $A$ is invariant under $T_g$ for all $g \in C_c^1(\mathbb{C})$. For example, $R(K)$ and $A(K)$ are T-invariant for any compact $K$. For more information on T-invariant planar algebras, see [8], [10], or [18].

The following is essentially Theorem 6.5 of [8].

PROPOSITION 9. *If $A$ is T-invariant on $K$, then $A$ is a strongly tight uniform algebra on $K$.*

Proof. Let $g \in C_c^1$, and let $V_g : A \to C(K)$ be the operator defined by $(T_g f)(z) = (fg)(z) + (V_g f)(z)$. It is shown in [8] that $V_g$ is a compact operator. It now follows from the commutative diagram

$$\begin{array}{ccc} & A & \\ {}^{-V_g}\swarrow & & \searrow^{S_g} \\ C(K) & \xrightarrow[q]{} & C(K)/A \end{array}$$

that $S_g$ is compact. Hence, $C_c^1 \subseteq A_{CG}$, which implies $A_{CG} = C(K)$ since $A_{CG}$ is a uniform algebra. ∎

COROLLARY. *$P(K)$ is strongly tight on $K$.*

Proof. Let $\widehat{K}$ be the polynomial convex hull of $K$. By Proposition 9, $R(\widehat{K})$ is strongly tight on $\widehat{K}$. Since $R(\widehat{K}) = P(\widehat{K})$, and $P(\widehat{K})|_K = P(K)$, the corollary follows from Proposition 5. ∎

Theorem 4 and Proposition 9 now yield the following.

THEOREM 5. *Let $K$ be a compact subset of $\mathbb{C}$, and let $A$ be a T-invariant uniform algebra on $K$. Then $A$ and $A^*$ have the Dunford–Pettis property, $A$ has the Pełczyński property, and $A^*$ is weakly sequentially complete.*

If $A$ is T-invariant and $Q$ is the set of non-peak points of $A$, then Theorem 13 of [17] implies that $A$ has property $(P_Q)$ (the proof, by B. Cole, uses Davie's theorem). It now follows from Proposition 8 that $AM(A,Q)$ is a band.

Note that Theorem 4 tells us that strongly rich measures exist for $A$. Consider the measure $\lambda_Q$, the restriction of planar measure to $Q$. Then $\lambda_Q \in \mathscr{B}_{A^\perp}$ (see [10]), and Davie's theorem (see [11] or [10]) says that the natural projection $H^\infty(\mathscr{B}_{A^\perp}) \to H^\infty(\lambda_Q)$ is an isometry onto $H^\infty(\lambda_Q)$ and is a weak-star homeomorphism. In particular, it is one-to-one, so by



Proposition 6, $\lambda_Q$ is weakly rich. We do not, however, need the power of Davie's theorem to show that $\lambda_Q$ is a weakly rich measure for a T-invariant algebra.

If $\mu$ is a compactly supported measure in $\mathbb{C}$, let $\widehat{\mu}$ be the Cauchy transform of $\mu$, and define

$$E_\mu = \left\{ z \in \mathbb{C} \;\middle|\; \int \frac{d|\mu|(w)}{|w - z|} < \infty \right\},$$

so that $E_\mu$ has full planar measure in $\mathbb{C}$. It is well known (see [18]) that if $\widehat{\mu} = 0$ a.e. $[dxdy]$ then $\mu$ is the zero measure.

LEMMA 3. *Suppose $A$ is T-invariant on $K$, $\mu \in A^\perp$, and $f \in A$. Then $\widehat{f d\mu} = f\widehat{\mu}$ a.e. $[dxdy]$.*

Proof. Since $A$ contains $R(K)$, $\widehat{f d\mu}(z) = f(z)\widehat{\mu}(z) = 0$ whenever $z \in \mathbb{C} \setminus K$. If $z \in E_\mu \cap K$, and $f$ extends to be analytic in a neighborhood of $z$, then $(f(\cdot) - f(z))/((\cdot) - z) \in A$ (see [10]), so

$$\int \frac{f(w) - f(z)}{w - z}\, d\mu(w) = 0,$$

which implies $\widehat{f d\mu}(z) = f(z)\widehat{\mu}(z)$. If $f \in A$ is arbitrary, and $z \in E_\mu \cap K$, then $f$ may be approximated uniformly by $\{f_n\} \subseteq A$ such that $f_n$ extends to be analytic in a neighborhood of $z$ for all $n$ (again, see [10]), so $\widehat{f d\mu}(z) = f(z)\widehat{\mu}(z)$. Hence, $\widehat{f d\mu} = f\widehat{\mu}$ a.e. $[dxdy]$ for all $f \in A$. ∎

It is not difficult to show, using Lemma 3, that when $\mu \in A^\perp$, and $z$ is a peak point of $A$ in $E_\mu$, then $\widehat{\mu}(z) = 0$.

PROPOSITION 10. *If $A$ is T-invariant on $K$, then $\lambda_Q$ is a strongly rich measure for $A$.*

Proof. We will first show $\lambda_Q$ has the property of Proposition 6(b), which implies $\lambda_Q$ is weakly rich. Suppose $\{f_n\}$ is a bounded sequence in $A$, and $\int |f_n|\, d\lambda_Q \to 0$. Let $\mu \in A^\perp$, and suppose $f_n$ fails to tend to zero weak-star in $L^\infty(\mu)$. Then, after passing to a subsequence, we may assume $f_n \xrightarrow{w^*} h$ in $L^\infty(\mu)$ for some non-zero $h \in L^\infty(\mu)$, and $f_n \to 0$ pointwise a.e. $[dxdy]$ in $Q$.

Now, for any $z \in E_\mu$, $\widehat{f_n d\mu}(z) \to \widehat{h d\mu}(z)$. If $z \in \mathbb{C} \setminus K$, then $\widehat{f_n d\mu}(z) = 0$, so $\widehat{h d\mu}(z) = 0$. If $z \in E_\mu \cap (K \setminus Q)$, then $z$ is a peak point for $A$, so by the remarks above, $\widehat{f_n d\mu}(z) = 0$, and hence $\widehat{h d\mu}(z) = 0$. Since $\widehat{f_n d\mu} = f_n\widehat{\mu}$ a.e. $[dxdy]$, and $f_n \to 0$ pointwise a.e. $[dxdy]$ in $Q$, it follows that $\widehat{h d\mu} = 0$ a.e. $[dxdy]$ in $Q$. Hence, $\widehat{h d\mu} = 0$ a.e. $[dxdy]$ in $\mathbb{C}$, so $h d\mu$ is the zero measure, contradicting the assumption that $h \neq 0$ in $L^\infty(\mu)$.

Hence, $\lambda_Q$ is a weakly rich measure for $A$. Since $A$ is strongly tight, $\lambda_Q$ is strongly rich. ∎

**8. Strictly pseudoconvex domains.** In [8], it was shown that for domains $D$ in $\mathbb{C}^n$, the property of $A(D)$ being tight is closely related to the solvability of a certain $\overline{\partial}$-problem. This idea can be carried to domains on manifolds.

We say an open, bounded domain $D \subset \mathbb{C}^n$ is *strictly pseudoconvex with smooth, $C^k$ boundary* if there exists a neighborhood $U$ of $\partial D$ and a $C^k$ function $\lambda : U \to \mathbb{R}$ such that

(a) $D \cap U = \{z \in U \mid \lambda(z) < 0\}$.

(b) $\lambda$ is strictly plurisubharmonic. I.e.,

$$\sum_{i,j=1}^{n} \frac{\partial^2 \lambda}{\partial z_i \partial \overline{z}_j}(z) \mu_i \overline{\mu}_j > 0$$

for $z \in U$ and $\mu \in \mathbb{C}^n$.

(c) $\nabla \lambda(z) \neq 0$ for $z \in \partial D$.

Since property (b) is invariant under holomorphic changes of coordinates, we can use the same definition when $D$ is a relatively compact domain in a complex manifold $M$.

We say $M$ is a *Stein manifold* if there exists a $C^\infty$ function $\varphi : M \to \mathbb{R}$ such that

(a) $\varphi$ is strictly plurisubharmonic.

(b) $\{z \in M \mid \varphi(z) \le c\}$ is compact in $M$ for all $c \in \mathbb{R}$.

Let $D$ be a relatively compact domain in a complex manifold, and let $K^\infty_{(0,1)}(D)$ be the space of bounded, $C^\infty$, $\overline{\partial}$-closed (0,1)-forms in $D$. We may define a norm $\|\cdot\|_\infty$ on $K^\infty_{(0,1)}(D)$ by taking a finite covering of $D$ with coordinate charts. Any two norms defined this way will be equivalent and make $K^\infty_{(0,1)}(D)$ into a normed linear space.

PROPOSITION 11. *Let $M$ be a complex manifold, and let $D$ be an open, relatively compact domain in $M$. Let $A = A(D)$ be the uniform algebra of continuous functions on $\overline{D}$ that are analytic in $D$. Assume there exists a weakly compact (resp. compact) linear operator $R : K^\infty_{(0,1)}(D) \to C(\overline{D})$ with the property that $\overline{\partial} \circ R = \mathrm{id}$. Then $AM(A, D)$ is a band, and $A$ is a tight (resp. strongly tight), weakly (resp. strongly) rich uniform algebra on $\overline{D}$.*

Proof. If $g \in C^\infty(\overline{D})$, define $T_g : A \to K^\infty_{(0,1)}(D)$ by $T_g(f) = \overline{\partial}(fg) = f\overline{\partial}g$, so $\|T_g\| \le \|\overline{\partial}g\|$. Then the diagram



$$\begin{array}{ccc} A & \xrightarrow{S_g} & C(\overline{D})/A \\ {\scriptstyle T_g}\downarrow & & \uparrow{\scriptstyle q} \\ K^\infty_{(0,1)}(D) & \xrightarrow{R} & C(\overline{D}) \end{array}$$

commutes, which implies $S_g$ is weakly compact (resp. compact). Therefore,

$$C^\infty(\overline{D}) \subseteq A_{cg} \quad (\text{resp. } A_{CG}).$$

Since $A_{cg}$ and $A_{CG}$ are uniform algebras, it follows that $A$ is tight (resp. strongly tight) on $\overline{D}$.

Since $A$ clearly has property $(P_D)$, it follows from Proposition 8 that $AM(A, D)$ is a band. Now, let $\{z_n\}$ be a dense sequence in $D$, and define

$$m = \sum_{n=1}^{\infty} \frac{\delta_n}{2^n},$$

where $\delta_n$ is the point mass at $z_n$.

CLAIM. *$m$ is weakly rich.*

Suppose $\{f_n\}$ is a bounded sequence in $A$, and $\int |f_n|\, dm \to 0$. Then $\{f_n\}$ is a $D$-Montel sequence. If $g \in C$ and $\mu \in A^\perp$, then $g d\mu \in AM(A, D)$, which implies $\int f_n g\, d\mu \to 0$. This implies $S_g(f_n) = f_n g + A \xrightarrow{w} 0$ in $C/A$, proving the claim.

When $A$ is strongly tight, $S_g$ is compact. Therefore, the weak convergence above can be replaced with norm convergence. Thus, $m$ will be strongly rich in this case. ∎

The following is implicit in [22].

THEOREM 6 (Kerzman, [22]). *Let $D$ be an open, relatively compact, strictly pseudoconvex domain with smooth, $C^4$ boundary in some Stein manifold. Then there exists a compact operator $R$ having the property mentioned in Proposition 11.*

It follows from Theorem V.2.7 in [28] that, in Kerzman's theorem, when $D$ is a domain in $\mathbb{C}^n$, we need only assume $D$ has a $C^2$ boundary. Combining these results with Proposition 11 and Theorem 4, we have the following.

THEOREM 7. *Let $D$ be as in Kerzman's theorem, where we need only assume a $C^2$ boundary when $D$ is a domain in $\mathbb{C}^n$, and let $A = A(D)$. Then $AM(A, D)$ is a band, and $A$ is a strongly tight, strongly rich uniform algebra on $\overline{D}$. Hence, $A$ and $A^*$ have the Dunford–Pettis property, $A$ has the Pełczyński property, and $A^*$ is weakly sequentially complete.*

It is noted in [9] that even when we are working on a Stein manifold, Kerzman's results are still valid when we only assume the boundary is $C^2$.

We have exhibited some strongly rich measures for $A(D)$ for the domains $D$ considered in Proposition 11. It also follows from work done in [9] that for these domains, if $m$ is the surface-area measure on $\partial D$ induced by some Kaehler metric, then $m$ is a weakly rich measure. Since $A(D)$ is strongly tight, $m$ is a strongly rich measure.

It is shown in [27] (based on the work of Henkin in [20]) that when $m$ is greater than one, the dual of the polydisc algebra $A(\mathbb{T}^m)$ is not isomorphic to a separable distortion of an $L^1$-space. It may also be shown directly that when $A = A(\mathbb{T}^m)$, and $m > 1$, $\mathscr{B}_{A^\perp}/A^\perp$ is not separable. It now follows from Proposition 6 that $A(\mathbb{T}^m)$ can never be weakly rich when $m$ is greater than one.

It was shown in [21] that $AM(A,D)$ is a band for $C^3$-smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^n$. The same result was extended to domains on manifolds in [9], where, as in our paper, the results of Kerzman are applied. Part of the method in [9] is to observe from Kerzman's work that the operator $R$ has a continuous extension $K^2_{(0,1)}(D) \to L^2(v)$ where $v$ is the volume measure on $D$. Our method of showing $AM(A,D)$ is a band only requires $R$ to exist and to be weakly compact.

Some examples of domains in $\mathbb{C}^n$ which satisfy the assumptions of Proposition 11 with a compact operator but are not strictly pseudoconvex are given in [8] and [25].

We have recently received a preprint by S. Li and B. Russo [24], where they have apparently shown that if $D$ is a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, or $D$ is pseudoconvex in $\mathbb{C}^2$ of finite type, then $A(D)$ and $A(D)^*$ have the Dunford–Pettis property. More precisely, they show $A(D)_B = C(\overline{D})$.

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# The Cauchy problem and self-similar solutions for a nonlinear parabolic equation

by

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**Abstract.** The existence of solutions to the Cauchy problem for a nonlinear parabolic equation describing the gravitational interaction of particles is studied under minimal regularity assumptions on the initial conditions. Self-similar solutions are constructed for some homogeneous initial data.

**1. Introduction.** Our aim in this paper is to construct local and global-in-time solutions to the Cauchy problem for the parabolic equation

$$u_t = \Delta u + \nabla \cdot (u \nabla \varphi), \tag{1}$$

in $\mathbb{R}^n \times \mathbb{R}^+$, where the coefficient $\nabla\varphi$ is determined from $u$ via the potential

$$\varphi = E_n * u, \tag{2}$$

$E_n$ being the fundamental solution of the Laplacian in $\mathbb{R}^n$. Since $\Delta\varphi = u$, the equation (1) can be rewritten as a parabolic equation with a nonlocal coefficient $\nabla\varphi$:

$$u_t = \Delta u + u^2 + \nabla u \cdot \nabla \varphi. \tag{1'}$$

The physical interpretations of the equation (1) with an initial (nonnegative) condition

$$u(x,0) = u_0(x) \tag{3}$$

come from nonequilibrium statistical mechanics. In particular, (1)–(3) is an evolution version of the Chandrasekhar equation for the gravitational equilibrium of polytropic stars. Here $u$ is the density of particles in $\mathbb{R}^n$ interacting with themselves through the gravitational potential $\varphi$. Another motivation for studying the above system is presented in the introduction

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